Laplacian of Random Hypergraphs

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Selected Topics on Spectral Graph Theory (V)
Nankai University, Tianjin, June 12, 2014
Five talks

Selected Topics on Spectral Graph Theory

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.

2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.

3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.

4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.

5. Laplacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.
Backgrounds

I: Spectral Graph Theory
II: Random Graph Theory
III: Random Matrix Theory
$G = (V, E)$: a weighted graph; each edge $xy$ is associated with a positive integer weight $w(x, y)$. ($w(x, y) = 0$ if $xy \notin E(G)$.)
The Laplacians of graphs

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- $A$: adjacency matrix, $A(x, y) = w(x, y)$.
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  \( (w(x, y) = 0 \) if \( xy \notin E(G) \).)
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- \( d_u = \sum_v w(u, v) \): the degree of \( u \).
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- $L = I - T^{-1/2}A T^{-1/2}$: the (normalized) Laplacian.
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- $d_u = \sum_v w(u, v)$: the degree of $u$.
- $T = \text{diag}(d_1, \ldots, d_n)$: the diagonal matrix of degrees.
- $\mathcal{L} = I - T^{-1/2} AT^{-1/2}$: the (normalized) Laplacian.
- Laplacian spectrum: $LSP(G) := \{\lambda_0, \ldots, \lambda_{n-1}\}$
  \[0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2.\]
An example

\[ A = \begin{pmatrix}
0 & 2 & 1 & 1 \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \]
An example

\[ A = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

\[ \mathcal{L} = I - T^{-1/2}AT^{-1/2} = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \]
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Laplacian eigenvalues: \( \lambda_0 = 0, \lambda_1 = \lambda_2 = 1, \lambda_3 = 2 \)
Some properties

- $\lambda_1 > 0$ if and only if $G$ is connected.
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- $\lambda_1 = \lambda_{n-1}$ if and only if $G$ is a complete graph (with the same weight).
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- \( \lambda_1 = \lambda_{n-1} \) if and only if \( G \) is a complete graph (with the same weight).
- Rayleigh quotients:

\[
\lambda_1 = \inf_{f \perp T_1} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x, y)}{\sum_x f(x)^2 d_x},
\]

\[
\lambda_{n-1} = \sup_{f \perp T_1} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x, y)}{\sum_x f(x)^2 d_x}.
\]
An important parameter

$\lambda_1$ is related to

- the mixing rate of random walks
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- diameter
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- neighborhood/edge expansion
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- quasi-randomness
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- diameter
- neighborhood/edge expansion
- conductance
- Cheeger’s constant
- quasi-randomness
- many other applications.
A walk on a graph is a sequence of vertices together a sequence of edges:

\[ v_0, v_1, v_2, v_3, \ldots, v_k, v_{k+1}, \ldots \]

\[ v_0 v_1, v_1 v_2, v_2 v_3, \ldots, v_k v_{k+1}, \ldots \]
Random walks

A walk on a graph is a sequence of vertices together a sequence of edges:

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Random walks on a graph \( G \):

\[ f_{k+1} = f_k T^{-1} A. \]

\[ \| (f_k - \pi) T^{-1/2} \| \leq \bar{\lambda}^k \| (f_0 - \pi) T^{-1/2} \|. \]

\[ T^{-1} A \sim T^{-1/2} A T^{-1/2} = I - \mathcal{L}. \]

\( \bar{\lambda} \) determines the mixing rate of random walks.
For $0 \leq \alpha \leq 1$, at time $t$, with probability $\alpha$, stay at the current vertex; with probability $1 - \alpha$, move to a neighbor vertex randomly.
\(\alpha\)-lazy random walks

For \(0 \leq \alpha \leq 1\), at time \(t\), with probability \(\alpha\), stay at the current vertex; with probability \(1 - \alpha\), move to a neighbor vertex randomly.

- Transition matrix
  \[ P_\alpha := \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2}\mathcal{L}_\alpha T^{1/2}. \]
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  \[
  P_\alpha := \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2}L_\alpha T^{1/2}.
  \]
- \( L_\alpha := I - (1 - \alpha)L \), its second largest eigenvalue is

\[
\bar{\lambda}_\alpha = \max\{|1 - (1 - \alpha)\lambda_1|, |1 - (1 - \alpha)\lambda_{n-1}|\}.
\]
For $0 \leq \alpha \leq 1$, at time $t$, with probability $\alpha$, stay at the current vertex; with probability $1 - \alpha$, move to a neighbor vertex randomly.

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  \[ P_\alpha := \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2}\mathcal{L}_\alpha T^{1/2}. \]

- **$\mathcal{L}_\alpha$** := $I - (1 - \alpha)\mathcal{L}$, its second largest eigenvalue is
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- **Stationary distribution** $\pi := \frac{1}{\text{vol}(G)}(d_1, d_2, \ldots, d_n)$. 

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- Stationary distribution \(\pi := \frac{1}{\text{vol}(G)}(d_1, d_2, \ldots, d_n)\).
- Let \(f_k\) be the distribution at time \(k\).
Theorem:

\[ \| (f_k - \pi)T^{-1/2} \| \leq \bar{\lambda}^k \alpha \| (f_0 - \pi)T^{-1/2} \|. \]

Proof:

\[ \| (f_k - \pi)T^{-1/2} \| = \| (f_0 P^k_\alpha - \pi P^k_\alpha)T^{-1/2} \| \]
Proof

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Theorem:

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$$= \| (f_0 - \pi) P^k T^{-1/2} \|$$
$$= \| (f_0 - \pi) T^{-1/2} \mathcal{L}_\alpha \|$$
Theorem:

\[ \| (f_k - \pi) T^{-1/2} \| \leq \bar{\lambda}_\alpha^k \| (f_0 - \pi) T^{-1/2} \|. \]

Proof:

\[
\begin{align*}
\| (f_k - \pi) T^{-1/2} \| &= \| (f_0 P^k_\alpha - \pi P^k_\alpha) T^{-1/2} \| \\
&= \| (f_0 - \pi) P^k_\alpha T^{-1/2} \| \\
&= \| (f_0 - \pi) T^{-1/2} \mathcal{L}^k_\alpha \| \\
&\leq \bar{\lambda}_\alpha^k \| (f_0 - \pi) T^{-1/2} \|. 
\end{align*}
\]

\[\square\]
Theorem [Chung (1989)]

If $G$ is not a complete weighted graph, then we have

$$\text{diam}(G) \leq \left\lceil \frac{\log(\text{vol}(G)/\delta)}{\log(\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1})} \right\rceil,$$

where $\delta$ is the minimum degree of $G$. 
For any two subsets $X$ and $Y$, we have
\[
\left| \left| E(X, Y) \right| - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| \leq \bar{\lambda} \sqrt{\frac{\text{vol}(X)\text{vol}(Y)\text{vol}(\bar{X})\text{vol}(\bar{Y})}{\text{vol}(G')}}.
\]

where
\[
\text{vol}(X) = \sum_{x \in X} d_x,
\]
\[
\text{vol}(G') = \sum_{x \in V(G')} d_x,
\]
\[
\text{vol}(\bar{X}) = \text{vol}(G) - \text{vol}(X),
\]
\[
\bar{\lambda} = \max\{\left|1 - \lambda_1\right|, \left|\lambda_{n-1} - 1\right|\}.
\]
Cheeger’s Constant

$$h(S) := \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}.$$ 

$$h_G := \min_{S \subset V(G)} h(S).$$

Cheeger’s inequality

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2}.$$
A directed graph $D$ is **Eulerian** if the in-degree equals the out-degree at any vertex $x$. \((d^+_x = d^-_x = d_x)\)

- Any weak connected component in $D$ is also a strongly connected component.
A directed graph $D$ is *Eulerian* if the in-degree equals the out-degree at any vertex $x$. ($d^+_x = d^-_x = d_x$)

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**Chung [2005]** defined the Laplacian of Eulerian directed graphs.

\[
\mathcal{L} = \frac{\vec{L} + \vec{L}'}{2}.
\]
\(\alpha\)-lazy random walks on \(D\)

- Transition matrix

\[ P_\alpha := \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2}\vec{\mathcal{L}}_\alpha T^{1/2}. \]
\( \alpha \)-lazy random walks on \( D \)

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\]

- \( \vec{\mathcal{L}}_\alpha := I - (1 - \alpha)\vec{\mathcal{L}}, \phi_0 := \frac{1}{\text{vol}(G)}(\sqrt{d_1}, \ldots, \sqrt{d_n}) \).
\( \alpha \)-lazy random walks on \( D \)

- **Transition matrix**
  \[ P_\alpha := \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2} \vec{L}_\alpha T^{1/2}. \]

- **\( \vec{L}_\alpha \)**:
  \[ \vec{L}_\alpha := I - (1 - \alpha)\vec{L}, \quad \phi_0 := \frac{1}{\text{vol}(G)}(\sqrt{d_1}, \ldots, \sqrt{d_n}). \]

- **Stationary distribution** \( \pi := \frac{1}{\text{vol}(G)}(d_1, d_2, \ldots, d_n) \)
\( \alpha \)-lazy random walks on \( D \)

- Transition matrix
\[
P_\alpha := \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2} \tilde{\mathcal{L}}_\alpha T^{1/2}.
\]

- \( \tilde{\mathcal{L}}_\alpha := I - (1 - \alpha)\mathcal{L}, \) \( \phi_0 := \frac{1}{\text{vol}(G)}(\sqrt{d_1}, \ldots, \sqrt{d_n}) \).

- Stationary distribution \( \pi := \frac{1}{\text{vol}(G)}(d_1, d_2, \ldots, d_n) \)

- Let \( f_k \) be the distribution at time \( k \).
\(\alpha\)-lazy random walks on \(D\)

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- Stationary distribution \(\pi := \frac{1}{\text{vol}(G)}(d_1, d_2, \ldots, d_n)\).

- Let \(f_k\) be the distribution at time \(k\).

**Theorem:** \(\| (f_k - \pi)T^{-1/2} \| \leq \sigma_\alpha^k \| (f_0 - \pi)T^{-1/2} \|.\)

Here \(\sigma_\alpha := \max_{f \perp \phi_0} \frac{\| \vec{\mathcal{L}}_\alpha f \|}{\| f \|}\) is the second largest singular value of \(\vec{\mathcal{L}}_\alpha\).
The estimation of $\sigma_\alpha$

Lemma:

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Lemma:

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- \(\sigma_\alpha^2 \leq \alpha^2 + 2\alpha(1 - \alpha)\lambda_1 + (1 - \alpha)^2\sigma_0^2\).

Choosing \(\alpha\) to minimize \(\sigma_\alpha\), we get

\[
\min_{0 \leq \alpha < 1} \{\sigma_\alpha\} \leq \begin{cases} 
\sigma_0 & \text{if } \lambda_1 \leq 1 - \sigma_0^2; \\
\sqrt{1 - \frac{\lambda_1^2}{2\lambda_1 + \sigma_0^2 - 1}} & \text{otherwise}.
\end{cases}
\]
Lemma:
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\end{cases}
\]

In particular, \( \min_{0 \leq \alpha < 1} \{ \sigma_\alpha \} \leq \sqrt{1 - \frac{\lambda_1}{2}} \).
Theorem [Chung 2005]:

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diam(D) \leq \left\lfloor \frac{2 \log(\text{vol}(G)/\delta)}{\log \frac{2}{2-\lambda_1}} \right\rfloor + 1,
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We improved it into

Theorem [Lu-Peng 2011]:

\[ \text{diam}(D) \leq \left\lceil \frac{\log(\text{vol}(D)/\delta)}{\log \sigma_\alpha} \right\rceil, \]

for any \( 0 < \alpha < 1. \)
Theorem [Lu-Peng 2011]: Let $D$ be a Eulerian directed graph. If $X$ and $Y$ are two subsets of $V(D)$, then we have

$$\left| \frac{|E(X,Y)| + |E(Y,X)|}{2} - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(D)} \right| \leq \bar{\lambda} \sqrt{\frac{\text{vol}(X)\text{vol}(Y)\text{vol}(\bar{X})\text{vol}(\bar{Y})}{\text{vol}(D)}}.$$
$H = (V, E)$ is an $r$-uniform hypergraph.

- $V$: the set of vertices
- $E$: the set of edges, each edge has cardinality $r$. 
Hypergraphs

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A 3-uniform loose cycle  
A 3-uniform tight cycle
Notations on hypergraphs

- $s: 1 \leq s \leq r - 1$
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- $V^s$: the set of all $s$-tuples with distinct elements from $V$. 
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- $S$: an $s$-subset of $V$
- Degree $d_S$: the number of edges passing through $S$.

$$\sum_{S \in \binom{V}{s}} d_S = \binom{r}{s}|E(H)|.$$
For $1 \leq s \leq r - 1$, an $s$-walk on $H$ consists of

- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying

$$F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\} \text{ for } 1 \leq i \leq k.$$
$s$-walks on hypergraphs

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  $F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$ for $1 \leq i \leq k$.

\[ |F_i \cap F_{i+1}| = s \]

A 1-walk in a 3-graph
For $1 \leq s \leq r - 1$, an $s$-walk on $H$ consists of

- a vertex sequence: $v_1, v_2, \ldots, v_{(k-1)(r-s)+r}$
- an edge sequence: $F_1, F_2, \ldots, F_k$ satisfying
  \[ F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\} \text{ for } 1 \leq i \leq k. \]

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For $1 \leq s \leq r - 1$, an $s$-walk on $H$ consists of

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$$|F_i \cap F_{i+1}| = s$$

A 2-walk in a 4-graph
Loose walk: \( 1 \leq s \leq \frac{r}{2} \).
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Observation: an $s$-th random walk on $H$ is essentially a random walk on an auxiliary weighted graph $G^{(s)}$. 
Loose walk: \( 1 \leq s \leq \frac{r}{2} \).

Observation: an \( s \)-th random walk on \( H \) is essentially a random walk on an auxiliary weighted graph \( G^{(s)} \).

- Vertex set \( V(G^{(s)}) = V^{s} \)
- Weight function \( w: V^{s} \times V^{s} \rightarrow \mathbb{Z} \):

\[
w(S, T) = \begin{cases} 
0 & \text{if } [S] \cap [T] \neq \emptyset \\
\text{d}_{[S] \cup [T]} & \text{if } [S] \cap [T] = \emptyset.
\end{cases}
\]
For $1 \leq s \leq r/2$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $G^{(s)}$. 
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$\mathcal{L}^{(1)}$ is the same as the Laplacian of hypergraph introduced by Rodríguez [2009].
Tight walk: \( \frac{r}{2} < s \leq r - 1 \).
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Observation: an $s$-th random walk on $H$ is “essentially” a random walk on an auxiliary directed graph $D^{(s)}$. 
Tight random walks

Tight walk: \( \frac{r}{2} < s \leq r - 1 \).

Observation: an \( s \)-th random walk on \( H \) is “essentially” a random walk on an auxiliary directed graph \( D^{(s)} \).

- Vertex set \( V(G^{(s)}) = V^s \)
- For \( x = (x_1, \ldots, x_s) \) and \( y = (y_1, \ldots, y_s) \), \( xy \) is a directed edge if
  - \( x_{r-s+j} = y_j \) for \( 1 \leq j \leq 2s - r \).
  - \( \{x_1, \ldots, x_s, y_{2s-r+1}, y_s\} \) is an edge of \( H \).
For \( r/2 < s \leq r - 1 \), the \( s \)-th Laplacian of \( H \), denoted by \( \mathcal{L}^{(s)} \), is defined as the Laplacian of \( D^{(s)} \).
For \( r/2 < s \leq r - 1 \), the \( s \)-th Laplacian of \( H \), denoted by \( L^{(s)} \), is defined as the Laplacian of \( D^{(s)} \).

- \( D^{(s)} \) is Eulerian, i.e., indegree=outdegree at any vertex.
For $r/2 < s \leq r - 1$, the $s$-th Laplacian of $H$, denoted by $L^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

- $D^{(s)}$ is Eulerian, i.e., indegree=outdegree at any vertex.
- **Chung [2005]** defined the Laplacian of directed graphs. In the case of Eulerian directed graph, we have

$$L = T^{-1/2} \frac{A + A'}{2} T^{-1/2}.$$
Laplacians of hypergraph (II)

For $r/2 < s \leq r - 1$, the $s$-th Laplacian of $H$, denoted by $\mathcal{L}^{(s)}$, is defined as the Laplacian of $D^{(s)}$.

- $D^{(s)}$ is Eulerian, i.e., indegree=outdegree at any vertex.
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$$\mathcal{L} = T^{-1/2} \left( \frac{A + A'}{2} T^{-1/2} \right).$$

- $\mathcal{L}^{(r-1)}$ is close related to the Laplacian of a regular hypergraph introduced by Chung [1993].
For \( r/2 < s \leq r - 1 \), the \( s \)-th Laplacian of \( H \), denoted by \( \mathcal{L}^{(s)} \), is defined as the Laplacian of \( D^{(s)} \).

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- **Chung [2005]** defined the Laplacian of directed graphs. In the case of Eulerian directed graph, we have

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\]

- \( \mathcal{L}^{(r-1)} \) is close related to the Laplacian of a regular hypergraph introduced by **Chung [1993]**.

\( \lambda_1^{(s)} \), \( \lambda_{max}^{(s)} \), and \( \bar{\lambda}^{(s)} \) are defined in the same way.
Examples

Some eigenvalues of Laplacians of complete hypergraph $K^r_n$:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\lambda^{(4)}_1$</th>
<th>$\lambda^{(3)}_1$</th>
<th>$\lambda^{(2)}_1$</th>
<th>$\lambda^{(1)}_1$</th>
<th>$\lambda^{(1)}_{\text{max}}$</th>
<th>$\lambda^{(2)}_{\text{max}}$</th>
<th>$\lambda^{(3)}_{\text{max}}$</th>
<th>$\lambda^{(4)}_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^3_6$</td>
<td>3/4</td>
<td>6/5</td>
<td>6/5</td>
<td>3/2</td>
<td>6/5</td>
<td>3/2</td>
<td>6/5</td>
<td>3/2</td>
</tr>
<tr>
<td>$K^3_7$</td>
<td>7/10</td>
<td>7/6</td>
<td>7/6</td>
<td>3/2</td>
<td>7/6</td>
<td>3/2</td>
<td>7/6</td>
<td>3/2</td>
</tr>
<tr>
<td>$K^4_6$</td>
<td>1/3</td>
<td>5/6</td>
<td>6/5</td>
<td>3/2</td>
<td>6/5</td>
<td>3/2</td>
<td>1.76759</td>
<td>6/5</td>
</tr>
<tr>
<td>$K^4_7$</td>
<td>3/8</td>
<td>9/10</td>
<td>7/6</td>
<td>7/4</td>
<td>7/6</td>
<td>7/5</td>
<td>7/4</td>
<td>7/5</td>
</tr>
<tr>
<td>$K^5_6$</td>
<td>0.1464</td>
<td>1/2</td>
<td>5/6</td>
<td>6/5</td>
<td>3/2</td>
<td>3/2</td>
<td>6/5</td>
<td>3/2</td>
</tr>
<tr>
<td>$K^5_7$</td>
<td>0.1977</td>
<td>5/8</td>
<td>9/10</td>
<td>7/6</td>
<td>3/2</td>
<td>3/2</td>
<td>3/2</td>
<td>1.809</td>
</tr>
</tbody>
</table>
### Examples

Some eigenvalues of Laplacians of complete hypergraph $K_n^r$:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\lambda_1^{(4)}$</th>
<th>$\lambda_1^{(3)}$</th>
<th>$\lambda_1^{(2)}$</th>
<th>$\lambda_1^{(1)}$</th>
<th>$\lambda_{\text{max}}^{(1)}$</th>
<th>$\lambda_{\text{max}}^{(2)}$</th>
<th>$\lambda_{\text{max}}^{(3)}$</th>
<th>$\lambda_{\text{max}}^{(4)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_6^3$</td>
<td>3/4</td>
<td>6/5</td>
<td>6/5</td>
<td>6/5</td>
<td>3/2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_7^3$</td>
<td>7/10</td>
<td>7/6</td>
<td>7/6</td>
<td>7/6</td>
<td>3/2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_6^4$</td>
<td>1/3</td>
<td>5/6</td>
<td>6/5</td>
<td>6/5</td>
<td>3/2</td>
<td>1.76759</td>
<td></td>
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<tr>
<td>$K_7^4$</td>
<td>3/8</td>
<td>9/10</td>
<td>7/6</td>
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<td>7/5</td>
<td>7/4</td>
<td></td>
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</tr>
<tr>
<td>$K_6^5$</td>
<td>0.1464</td>
<td>1/2</td>
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<td>1.809</td>
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</tr>
</tbody>
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Applications

$\lambda_1^{(s)}$ (and/or) $\bar{\lambda}_{max}^{(s)}$ is related to

- the mixing rate of random $s$-walk
- $s$-diameter
- neighborhood/edge expansion
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\(\lambda_1^{(s)} \) (and/or) \(\bar{\lambda}_{\text{max}}^{(s)}\) is related to

- the mixing rate of random \(s\)-walk
- \(s\)-diameter
- neighborhood/edge expansion

Each application is divided into the loose case and the tight case.
Theorem [Lu-Peng 2011]: For $1 \leq s \leq r/2$, suppose that $H$ is an $s$-connected $r$ uniform hypergraph. For $0 \leq \alpha < 1$, the joint distribution $f_k$ at the $k$-th stop of the $\alpha$-lazy random walk at time $k$ converges to the stationary distribution $\pi$ in probability. In particular, we have

$$\| (f_k - \pi) T^{-1/2} \| \leq (\bar{\lambda}_\alpha^{(s)})^k \| (f_0 - \pi) T^{-1/2} \|,$$

where $\bar{\lambda}_\alpha^{(s)} = \max\{|1 - (1 - \alpha)\lambda_1^{(s)}|, |(1 - \alpha)\lambda_{\max}^{(s)} - 1|$, and $f_0$ is the probability distribution at the initial stop.
Theorem [Lu-Peng 2011]: For $r/2 < s \leq r - 1$, suppose that $H$ is an $s$-connected $r$ uniform hypergraph. For $0 < \alpha < 1$, the joint distribution $f_k$ at the $k$-th stop of the $\alpha$-lazy random walk at time $k$ converges to the stationary distribution $\pi$ in probability. In particular, we have

$$\|(f_k - \pi)T^{-1/2}\| \leq (\sigma^{(s)}_\alpha)^k \|(f_0 - \pi)T^{-1/2}\|,$$

where $\sigma^{(s)}_\alpha \leq \sqrt{1 - 2\alpha(1 - \alpha)\lambda_1^{(s)}}$, and $f_0$ is the probability distribution at the initial stop.
**$s$-Diameter (I)**

**Theorem [Lu-Peng 2011]:** Let $H$ be a $r$-uniform hypergraph. For $1 \leq s \leq \frac{r}{2}$, if $\lambda_{\text{max}}(s) > \lambda_1(s) > 0$, then the $s$-diameter of an $r$-uniform hypergraph $H$ satisfies

$$\text{diam}^{(s)}(H) \leq \left\lfloor \log \frac{\text{vol}(V^s)}{\delta^{(s)}} \frac{\lambda_{\text{max}} + \lambda_1^{(s)}}{\lambda_{\text{max}} - \lambda_1^{(s)}} \right\rfloor.$$

Here $\text{vol}(V^s) = \sum_{x \in V^s} d_x = |E(H)| \frac{r!}{(r-2s)!}$ and $\delta^{(s)}$ is the minimum degree in $G^{(s)}$. 
Theorem [Lu-Peng 2011]: Let $H$ be a $r$-uniform hypergraph. For $r/2 < s \leq r - 1$, if $\lambda_1^{(s)} > 0$, then the $s$-diameter of $H$ satisfies

\[
\text{diam}^{(s)}(H) \leq \left\lceil \frac{2 \log \frac{\text{vol}(V^s)}{\delta^{(s)}}}{\log \frac{2}{2 - \lambda_1^{(s)}}} \right\rceil.
\]

Here $\text{vol}(V^s) = \sum_{x \in V^s} d_x = |E(H)|^r r!$ and $\delta^{(s)}$ is the minimum degree in $D^{(s)}$. 
Theorem [Lu-Peng 2011]: For $1 \leq t \leq s \leq r - t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let

$$E(S, T) = \{ F \in E(H) : \exists x \in S, \exists y \in T, x \cap y = \emptyset, \text{ and } x \cup y \subseteq \}$$

Let $e(S) = \frac{\text{vol}(S)}{\text{vol}(\binom{V}{s})}$ and $e(S, T) = \frac{|E(S, T)|}{|E(\binom{V}{s}, \binom{V}{t})|}$. 

$$|e(S, T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$
**Theorem [Lu-Peng 2011]:** For $1 \leq t < \frac{r}{2} < s < s + t \leq r$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $e(S, T) = \frac{|E(S, T)|}{|E(\binom{V}{s}, \binom{V}{t})|}$. If $|x \cap y| \neq \min\{t, 2s - r\}$ for any $x \in S$ and $y \in T$, then we have

$$|\frac{1}{2}e(S, T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$
Theorem [Lu-Peng 2011]: Suppose $\frac{r}{2} < s \leq r - 1$. For $S, T \subseteq \binom{V}{s}$, let

$$E'(S, T) = \{ F \in E(H) \mid \exists x \in S, \exists y \in T, F = x \cup y \}$$

and $e'(S, T) = \frac{|E'(S, T)|}{|E'(\binom{V}{s}, \binom{V}{s})|}$. We have

$$|e'(S, T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$
Connections of different $\mathcal{L}^{(s)}$

**Theorem [Lu, Peng 2011]** We have the following inequalities for the “loose” Laplacian eigenvalues.

$$
\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \ldots \geq \lambda_{1}^{(\lfloor r/2 \rfloor)}; \\
\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \ldots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}.
$$
Reduced Laplacian (I)

For $1 \leq s \leq r/2$, let $G^{(s)'}$ be the weighted graph defined as

- **Vertex set** $V(G^{(s)'}) = \binom{V}{s}$
- **Weight function** $w : \binom{V}{s} \times \binom{V}{s} \to \mathbb{Z}$:

$$w(S, T) = \begin{cases} 
0 & \text{if } S \cap T \neq \emptyset \\
\quad d_{S \cup T} & \text{if } S \cap T = \emptyset.
\end{cases}$$
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Since $G^{(s)}$ is a blow-up of $G^{(s)'}$, we have

$$LSP(G^{(s)}) = LSP(G^{(s)'}) \cup \{1 \text{ with multi. } \binom{n}{s}(s! - 1) \}.$$
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Therefore, two graphs has the same $\lambda_1$, $\lambda_{max}$, and $\bar{\lambda}$. 
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Therefore, two graphs has the same $\lambda_1$, $\lambda_{max}$, and $\bar{\lambda}$. The Laplacian of $G^{(s)'}$ is called the $s$-th reduced Laplacian of $H$. 

Laplacian of Random Hypergraphs
Theorem: For $1 \leq s \leq r/2$, the reduced $s$-th Laplacian eigenvalues of $K^r_n$ is the eigenvalues of $s$-th reduced Lapacian of $K^r_n$ are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}}$$

with multiplicity $\binom{n}{i} - \binom{n}{i-1}$

for $0 \leq i \leq s$. 
The Kneser graph $K(n, s)$ is a graph over the vertex set $\binom{[n]}{s}$; two $s$-sets $S$ and $T$ form an edge of $K(n, s)$ if and only if $S \cap T = 0$. 
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The eigenvalues of the adjacency matrix of $K(n, s)$ are $(-1)^i \binom{n-s+i}{s-i}$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$ for $0 \leq i \leq s$. 
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The eigenvalues of the adjacency matrix of $K(n, s)$ are $(-1)^i \binom{n-s-i}{s-i}$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$ for $0 \leq i \leq s$.

Note $K(n, s)$ is a regular graph; so the Laplacian eigenvalues can be determined from the eigenvalues of its adjacency matrix.
Proof

We observe that $G^{(s')}(K^r_n)$ is essentially the Kneser graph $K(n, s)$ with each edge associated with a weight $\binom{n-2s}{r-2s}$. Note the multiplicative factor $\binom{n-2s}{r-2s}$ is canceled after normalization. The $\mathcal{L}^{(s)}$ (for $K^r_n$) is exactly the Laplacian of Kneser graph. Thus, the eigenvalues of $s$-th Lapacian of $K^r_n$ are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1}$$

for $0 \leq i \leq s$. 
Erdős-Ko-Rado Theorem  If the $n \geq 2s$, then the size of the maximum intersecting family of $s$-sets in $[n]$ is at most $\binom{n-1}{s-1}$.
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The simplest proof is due to Katona [1972].
An application

**Erdős-Ko-Rado Theorem** If the $n \geq 2s$, then the size of the maximum intersecting family of $s$-sets in $[n]$ is at most $\binom{n-1}{s-1}$.

The simplest proof is due to Katona [1972].

Here we present a “new” proof using the $s$-th Laplacian eigenvalues of $K^r_n$. 
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The simplest proof is due to Katona [1972].

Here we present a “new” proof using the $s$-th Laplacian eigenvalues of $K^r_n$.(Actually it is due to Calderbank-Frankl [1992].)
It suffices to any intersecting family $U$ has size at most $\binom{n-1}{s-1}$.

Note $U$ is an independent set in $G^{(s)'}(K^r_n)$. Let $\mathcal{L}$ be the Laplacian of $G^{(s)'}(K^r_n)$. We have $\mathcal{L}|_U = I$. By Cauchy’s interlace theorem, we have

$$\lambda_k^{(s)} \leq 1 \leq \lambda_{\binom{n}{s} - |U| + k}^{(s)}$$

for $0 \leq k \leq |U| - 1$. 
It suffices to any intersecting family \( U \) has size at most
\[
\binom{n-1}{s-1}.
\]
Note \( U \) is an independent set in \( G^{(s)'}(K_r^n) \). Let \( \mathcal{L} \) be the Laplacian of \( G^{(s)'}(K_r^n) \). We have \( \mathcal{L}|_U = I \). By Cauchy’s interlace theorem, we have

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\lambda_k^{(s)} \leq 1 \leq \lambda_{\binom{n}{s} - |U| + k}^{(s)},
\]
for \( 0 \leq k \leq |U| - 1 \).

Let \( N^+ \) (or \( N^- \)) be the number of eigenvalues of \( \mathcal{L}^{(s)} \) which is \( \geq 1 \) (or \( \leq 1 \)) respectively. We have \( |U| \leq N^+ \) and \( |U| \leq N^- \).
The eigenvalues of $\mathcal{L}$ are

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}}$$

with multiplicity $\binom{n}{i} - \binom{n}{i-1}$

for $0 \leq i \leq s$.

$$N^+ = \sum_{i=0}^{\lfloor (s-1)/2 \rfloor} \left( \binom{n}{2i+1} - \binom{n}{2i} \right)$$

$$N^- = \sum_{i=0}^{\lfloor s/2 \rfloor} \left( \binom{n}{2i} - \binom{n}{2i-1} \right).$$
We have

\[ |U| \leq \min\{N^+, N^-\} \]

\[
= \sum_{i=0}^{s-1} (-1)^{s-1-i} \binom{n}{i} \\
= \binom{n}{s-1} - \binom{n}{s-2} + \binom{n}{s-3} - \binom{n}{s-4} + \cdots \\
= \binom{n-1}{s-1}.
\]
A random graph is a set of graphs together with a probability distribution on that set.
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**Example:** A random graph on 3 vertices and 2 edges with the uniform distribution on it.

Probability $\frac{1}{3}$

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A random graph is a set of graphs together with a probability distribution on that set.

**Example:** A random graph on 3 vertices and 2 edges with the uniform distribution on it.

A random graph $G$ *almost surely* satisfies a property $P$, if

$$\Pr(G \text{ satisfies } P) \to 1 \text{ as } |V(G)| \to \infty.$$
Erdős-Rényi model $G(n, p)$

For each pair of vertices in the vertex set of size $n$, an edge is created independently with probability $p$. 
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![Diagram of a graph with 4 vertices and 4 edges]
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\[ \begin{array}{c}
\bullet & & \bullet \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & & \bullet \\
\end{array} \]

$1 - p$ $1 - p$ $p$ $p$ $p$ $p$
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The probability of this graph is

$$p^4(1 - p)^2.$$
ON THE EVOLUTION OF RANDOM GRAPHS

by

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1. Definition of a random graph

Let $E_{n,N}$ denote the set of all graphs having $n$ given labelled vertices $V_1, V_2, \ldots, V_n$ and $N$ edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing $N$ out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \ldots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$. There is however an other slightly
Evolution of $G(n, p)$

$p$

- $0$: the empty graph.
- $\frac{c}{n}$: disjoint union of trees.
- $\frac{1}{n}$: cycles of any size.
- $\frac{c'}{n}$: the double jumps.
- $\frac{\log n}{n}$: one giant component, others are trees.
- $\Omega(\frac{\log n}{n})$: $G(n, p)$ is connected.
- $\Omega(n^{\epsilon-1})$: connected and almost regular.
- $\Theta(1)$: finite diameter.
- $1$: dense graphs, diameter is 2.
- $1$: the complete graph.
Füredi and Komlós (1981): If $np(1 - p) \gg \log^6 n$, then almost surely

$$\mu_n(G(n, p)) = (1 + o(1))np$$

$$\max_{1 \leq i \leq n-1} \{|\mu_i(G(n, p))|\} \leq (2 + o(1)) \sqrt{np(1 - p)}.$$
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\max_{1 \leq i \leq n-1} \{ |\mu_i(G(n, p))| \} \leq (2 + o(1)) \sqrt{np(1 - p)}.
\]

What about the Laplacian eigenvalues of \( G(n, p) \)?

---

Laplacian of Random Hypergraphs
Random graph with expected degree sequence:

- Each vertex $i$ is associated with a weight $w_i$.
- The probability that $ij$ is an edge is $w_i w_j \frac{1}{\sum_{k=1}^{n} w_k}$.
- The expected degree of $i$ is $w_i$. \

\[
G(w_1, w_2, \ldots, w_n)
\]
Random graph with expected degree sequence:
- Each vertex $i$ is associated with a weight $w_i$.
- The probability that $ij$ is an edge is $w_i w_j \frac{1}{\sum_{k=1}^{n} w_k}$.
- The expected degree of $i$ is $w_i$.

**Chung-Lu-Van (2003):**

$$\bar{\lambda} \leq (1 + o(1)) \frac{4}{\sqrt{\bar{w}}} + \frac{g(n) \log^2 n}{w_{\min}} ,$$

where $w_{\min}$ is the minimum weight and $\bar{w}$ is the average weight.
$G(n, p)$ is a special case of $G(w_1, w_2, \ldots, w_n)$ with $w_1 = w_2 = \cdots = w_n = np$. 
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Applying Chung-Lu-Van’s result to $G(n, p)$, we have

**Chung-Lu-Van (2003):** For $1 - \epsilon > p \gg \frac{\log^6 n}{n}$,

$$\bar{\lambda}(G(n, p)) \leq (4 + o(1)) \frac{1}{\sqrt{np}}.$$
Random $d$-regular graphs $G_{n,d}$

- The space is the set of all $d$-regular graphs on $n$ vertices.
- Each graph has an equal probability.
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**Friedman (1989)** For random $2d$-regular graph, almost surely

\[
\max_{1 \leq i \leq n-1} \{|\mu_i(G_{n,d})|\} \leq 2\sqrt{2d - 1} + 2 \log d + O(1).
\]
Random $d$-regular graphs $G_{n,d}$

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**Friedman (2002)** For random $d$-regular graph with $d \geq 4$, almost surely

$$\max_{1 \leq i \leq n-1} \{|\mu_i(G_{n,d})|\} = (2 + o(1))\sqrt{d-1}.$$
Random hypergraphs

Random \( r \)-uniform hypergraph \( H^r(n, p) \):

- \( n \): the number of vertices
- \( p \): probability, \( 0 < p < 1 \).
  
  For any \( F \in \binom{[n]}{r} \), \( F \) is an edge with probability \( p \) independently.
Random $r$-uniform hypergraph $H^r(n, p)$:

- $n$: the number of vertices
- $p$: probability, $0 < p < 1$.
  For any $F \in \binom{[n]}{r}$, $F$ is an edge with probability $p$ independently.

**Question:** What is Laplacian eigenvalues of $H^r(n, p)$?
Our result (I)

**Theorem [Lu, Peng 2011]** Let $H^r(n, p)$ be a random $r$-uniform hypergraph. For $1 \leq s \leq r/2$, if $p(1 - p) \gg \frac{\log^4 n}{nr - s}$ and $1 - p \gg \frac{\log n}{n^2}$, then almost surely

$$\bar{\lambda}^{(s)}(H^r(n, p)) \leq \frac{s}{n-s} + \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{(n-s)p}}.$$

Theorem [Lu, Peng 2011] Let \( H^r(n, p) \) be a random \( r \)-uniform hypergraph. For \( 1 \leq s \leq r/2 \), if \( p(1 - p) \gg \frac{\log^4 n}{n^{r-s}} \) and \( 1 - p \gg \frac{\log n}{n^2} \), then almost surely

\[
\overline{\lambda}^{(s)}(H^r(n, p)) \leq \frac{s}{n-s} + \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}.
\]

Moreover, for \( 1 \leq k \leq \binom{n}{s} - 1 \) almost surely we have

\[
|\lambda^{(s)}_k(H^r(n, p)) - \lambda^{(s)}_k(K^r_n)| \leq \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}.
\]
Our result: If \( p(1 - p) \gg \frac{\log^4 n}{n} \), then

\[
\bar{\lambda}(G(n, p)) \leq (3 + o(1)) \frac{1}{\sqrt{np}}.
\]
Our result: If $p(1 - p) \gg \frac{\log^4 n}{n}$, then

$$\bar{\lambda}(G(n, p)) \leq (3 + o(1)) \frac{1}{\sqrt{np}}.$$ 

Chung-Lu-Van (2003): If $1 - \epsilon > p \gg \frac{\log^6 n}{n}$, then

$$\bar{\lambda}(G(n, p)) \leq (4 + o(1)) \frac{1}{\sqrt{np}}.$$
Lemma 1: Given any two \((N \times N)\)-Hermitian matrices \(A\) and \(B\), for \(1 \leq k \leq N\), let \(\mu_k(A)\) (or \(\mu_k(B)\)) be the \(k\)-th eigenvalues of \(A\) (or \(B\)) in the increasing order. We have

\[
|\mu_k(A) - \mu_k(B)| \leq \|A - B\|.
\]
Lemma 1: Given any two \((N \times N)\)-Hermitian matrices \(A\) and \(B\), for \(1 \leq k \leq N\), let \(\mu_k(A)\) (or \(\mu_k(B)\)) be the \(k\)-th eigenvalues of \(A\) (or \(B\)) in the increasing order. We have

\[
|\mu_k(A) - \mu_k(B)| \leq \|A - B\|.
\]

Proof: By the Min-Max Theorem,

\[
\mu_k(A) = \min_{S_k} \max_{x \in S_k, \|x\| = 1} x'Ax
\]

\[
= \min_{S_k} \max_{x \in S_k, \|x\| = 1} (x'Bx + x'(A - B)x)
\]

\[
\leq \min_{S_k} \max_{x \in S_k, \|x\| = 1} (x'Bx + \|A - B\|)
\]

\[
= \mu_k(B) + \|A - B\|.
\]
Lemma 1: Given any two \((N \times N)\)-Hermitian matrices \(A\) and \(B\), for \(1 \leq k \leq N\), let \(\mu_k(A)\) (or \(\mu_k(B)\)) be the \(k\)-th eigenvalues of \(A\) (or \(B\)) in the increasing order. We have

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Proof: Thus,

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\mu_k(A) \leq \mu_k(B) + \|A - B\|.
\]

Similarly we have

\[
\mu_k(A) \geq \mu_k(B) - \|A - B\|.
\]
Write \( \mathcal{L}^{(s)}(K^r_n) - \mathcal{L}^{(s)}(H^r(n, p)) = M_1 + M_2 + M_3 + M_4 \), where

\[
M_1 = \frac{1}{\binom{r-s}{s}} \left( D^{-1/2}(W - E(W))D^{-1/2} - d^{-1}(W - E(W)) \right),
\]

\[
M_2 = \frac{1}{\binom{r-s}{s}} d (W - E(W)),
\]

\[
M_3 = \frac{1}{\binom{r-s}{s}} D^{-1/2}E(W)D^{-1/2} - \frac{d}{\binom{n}{s}} D^{-1/2} JD^{-1/2}
\]

\[
- \frac{1}{\binom{n-s}{s}} K + \frac{1}{\binom{n}{s}} J,
\]

\[
M_4 = \frac{1}{\binom{n}{s}} (dD^{-1/2} JD^{-1/2} - J).
\]
\[ \|M_1\| = O \left( \frac{\sqrt{(1 - p) \log N}}{d} \right), \quad \text{easy!} \]

\[ \|M_2\| \leq \frac{(2 + o(1)) \sqrt{1 - p}}{\sqrt{\binom{r-s}{s}} d}, \quad \text{hard!} \]

\[ \|M_3\| = O \left( \frac{\sqrt{\log N}}{n \sqrt{d}} \right), \quad \text{easy!} \]

\[ \|M_4\| \leq (1 + o(1)) \sqrt{\frac{1 - p}{d}}. \quad \text{tricky!} \]

Putting together, \[ \|M\| \leq \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}. \]
**Chenoff’s inequality:** Let $X_1, \ldots, X_n$ be independent 0-1 random variables with We consider the sum $X = \sum_{i=1}^{n} X_i$. Then we have

(Lower tail) \[ \Pr(X \leq E(X) - \lambda) \leq e^{-\lambda^2/2E(X)}, \]

(Upper tail) \[ \Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}}. \]
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(Upper tail) \[ \Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}}. \]

Lemma: If $d := \binom{n}{s} p \geq \log N$, then with probability at least $1 - \frac{1}{N^3}$, for any $S \in \binom{V}{s}$, we have

\[ d_S \in (d - 3\sqrt{d \log N}, d + 3\sqrt{d \log N}). \]
Let $C = W - E(W)$. One of the major task is to estimate $\|C\|$. We have the following Lemma.
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**Lemma 2:** Suppose $p(1 - p) \gg \frac{\log^4 n}{n^{r-s}}$. Almost surely, we have $\|C\| \leq (2 + o(1)) \sqrt{\binom{r-s}{s}} d(1 - p)$. 
Let $C = W - \mathbb{E}(W)$. One of the major tasks is to estimate $\|C\|$. We have the following Lemma.

**Lemma 2:** Suppose $p(1 - p) \gg \frac{\log^4 n}{n^{r-s}}$. Almost surely, we have $\|C\| \leq (2 + o(1)) \sqrt{\binom{r-s}{s} d (1 - p)}$.

Recall $M_2 = \frac{1}{\binom{r-s}{s} d} C$. We get

$$\|M_2\| \leq \frac{(2 + o(1)) \sqrt{1 - p}}{\sqrt{\binom{r-s}{s} d}}$$
Lemma 3: For any $k \ll (n^{r-s}p(1-p))^{1/4}$, we have

$$E(\text{Trace}(C^{2k})) = (1 + o(1)) \frac{n^{s+k(r-s)} \binom{2k}{k} p^k (1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k},$$

$$E(\text{Trace}(C^{2k+1})) = O \left( \frac{k(2k+1)n^{s+k(r-s)} \binom{2k}{k} p^k (1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k} \right).$$
Proof of Lemma 2

Let \( U := \frac{n^{s+k(r-s)}}{(k+1)(s!)^k + ((r-2s)!)^k} \binom{2k}{k} p^k (1 - p)^k \). By Markov’s inequality,

\[
\Pr \left( \|C\| \geq (1 + \epsilon)^{\frac{2k}{\sqrt{U}}} \right) = \Pr \left( \|C\|^{2k} \geq (1 + \epsilon)^{2kU} \right) \\
\leq \frac{\mathbb{E}(\|C\|^{2k})}{(1 + \epsilon)^{2kU}} \\
\leq \frac{\mathbb{E} \left( \text{Trace}(C^{2k}) \right)}{(1 + \epsilon)^{2kU}} \\
= 1 + o(1). \\
\]
Choose $k = \sigma g(n) \log n$ and $\epsilon = 1/g(n)$.

\[
\| \| C \| \leq (1 + o(1))^{2k} \sqrt{U} \\
= (1 + o(1)) \left( \frac{n^{\sigma + k(r-s)} \binom{2k}{k} p^k (1 - p)^k}{(k + 1)(s!)^{k+1}((r - 2s)!)^k} \right)^{\frac{1}{2k}} \\
< n^{\frac{s}{2k}} 2 \sqrt{\frac{n^{r-s} p(1 - p)}{s!(r - 2s)!}} \\
= (2 + o(1)) \sqrt{\binom{r-s}{s}} d(1 - p).
\]
Wigner (1958)

- $A$ is a real symmetric $N \times N$ matrix.
- Entries $a_{ij}$ are independent random variables.
- $E(a_{ij}^{2k+1}) = 0$.
- $E(a_{ij}^2) = m^2$.
- $E(a_{ij}^{2k}) < M$.

The distribution of eigenvalues of $A$ converges into a semicircle distribution of radius $2m\sqrt{N}$. 
Wigner’s semicircle law

Wigner (1958)
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Füredi and Komlós (1981): The eigenvalues of $G(n, p)$ follows Wigner’s semicircle law.
Definition

Let $A$ be a Hermitian matrix of dimension $N \times N$. The *empirical distribution* of the eigenvalues of $A$ is

$$F(A, x) := \frac{1}{N} \left| \left\{ \text{eigenvalues of } A \text{ less than } x \right\} \right|.$$

We say, the empirical distribution of the eigenvalues of $A$ follows the Semicircle Law centered at $c$ with radius $R$ if

$$F\left( \frac{1}{R} (A - cI), x \right) \overset{p}{\to} F(x).$$
Theorem [Lu, Peng 2011] For $1 \leq s \leq r/2$, if $p(1 - p)n^{r-s} \gg \log n$, then almost surely the empirical distribution of eigenvalues of the $s$-th Laplacian of $H^r(n, p)$ follows the Semicircle Law centered at 1 and with radius \[(2 + o(1))\sqrt{\frac{1-p}{(r-s)(n-s)p}}.\]
Our result (II)

**Theorem [Lu, Peng 2011]** For $1 \leq s \leq r/2$, if $p(1 - p)n^{r-s} \gg \log n$, then almost surely the empirical distribution of eigenvalues of the $s$-th Laplacian of $H^r(n, p)$ follows the Semicircle Law centered at 1 and with radius

$$(2 + o(1)) \sqrt{\frac{1-p}{n}} \sqrt{\frac{r-s}{r-s}p}.$$

**Corollary:** If $p(1 - p)n^{r-s} \gg \log n$, then

$$\max_{1 \leq i \leq \binom{n}{s}-1} |\lambda_k^{(s)}(H^r(n, p)) - \lambda_k^{(s)}(K^r_n)|$$

$$\geq \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + o(1)\right) \sqrt{\frac{1-p}{n}} \sqrt{\frac{r-s}{r-s}p}.$$
Theorem: If \( n^{r-s}p(1-p) \to \infty \), then the empirical distribution of the eigenvalues of \( C \) follows the semicircle law centered at 0 with radius
\[
R := 2\sqrt{\binom{r-s}{s} \binom{n-s}{r-s} p(1-p)}.
\]
Theorem: If $nr^{-s}p(1 - p) \to \infty$, then the empirical distribution of the eigenvalues of $C$ follows the semicircle law centered at 0 with radius $R := 2\sqrt{(r-s)(n-s)p(1-p)}$.

Proof: Let $C_{nor} := \frac{1}{R}C$. For any $k$, we have

\[
E(\text{Trace}(C_{nor}^{2k})) = (1 + o(1))\frac{(2k)!}{2^{2k}k!(k + 1)!}
\]

\[
E(\text{Trace}(C_{nor}^{2k+1})) = o(1).
\]

It converges to the $2k$-th (and $2k + 1$-th) moment of the Semicircle distribution. \qed
Lemma 4: If

- $A$: an $(N \times N)$-Hermitian matrices satisfying the Semicircle Law centered at $c$ with radius $R$,
- $B$: an $(N \times N)$-Hermitian matrices either $\|B\| = o(R)$ or $\text{rank}(B) = o(N)$,

then $A + B$ satisfies the Semicircle Law centered at $c$ with radius $R$. 

Case \( \|B\| = o(R) \)

\[
\left| \mu_k \left( \frac{1}{R}(A + B - cI) \right) - \mu_k \left( \frac{1}{R}(A - cI) \right) \right| \leq \frac{\|B\|}{R} = o(1).
\]
Case $\| B \| = o(R)$

$$\left| \mu_k \left( \frac{1}{R} (A + B - cI) \right) - \mu_k \left( \frac{1}{R} (A - cI) \right) \right| \leq \frac{\| B \|}{R} = o(1).$$

Thus, we have

$$F \left( \frac{1}{R} (A - cI), x - \frac{\| B \|}{R} \right) \leq F \left( \frac{1}{R} (A + B - cI), x \right) \leq F \left( \frac{1}{R} (A - cI), x + \frac{\| B \|}{R} \right).$$
Case $\|B\| = o(R)$

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Thus, we have

$$F \left( \frac{1}{R} (A - cI), x - \frac{\|B\|}{R} \right) \leq F \left( \frac{1}{R} (A + B - cI), x \right) \leq F \left( \frac{1}{R} (A - cI), x + \frac{\|B\|}{R} \right).$$

Since

$$F \left( \frac{1}{R} (A - cI), x - \frac{\|B\|}{R} \right) \xrightarrow{p} F(x) \quad \text{and}$$

$$F \left( \frac{1}{R} (A - cI), x + \frac{\|B\|}{R} \right) \xrightarrow{p} F(x).$$

By the Squeeze theorem, we have

$$F \left( \frac{1}{R} (A + B - cI), x \right) \xrightarrow{p} F(x).$$
Let $U$ be the kernel of $B$ (i.e. $B|_U = 0$) and $Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$. 
Case $\text{rank}(B) = o(N)$

Let $U$ be the kernel of $B$ (i.e. $B|_U = 0$) and 
$Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$. By Cauchy’s interlace theorem, for $1 \leq j \leq N - \text{rank}(B)$, we have

$$
\mu_j \left( \frac{1}{R}(A - cI) \right) \leq \mu_j(Z) \leq \mu_{j+\text{rank}(B)} \left( \frac{1}{R}(A - cI) \right),
$$

$$
\mu_j \left( \frac{1}{R}(A + B - cI) \right) \leq \mu_j(Z) \leq \mu_{j+\text{rank}(B)} \left( \frac{1}{R}(A + B - cI) \right).
$$
Let $U$ be the kernel of $B$ (i.e. $B|_U = 0$) and $Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$. By Cauchy’s interlace theorem, for $1 \leq j \leq N - \text{rank}(B)$, we have

$$
\mu_j \left( \frac{1}{R}(A - cI) \right) \leq \mu_j(Z) \leq \mu_{j+\text{rank}(B)} \left( \frac{1}{R}(A - cI) \right),
$$

$$
\mu_j \left( \frac{1}{R}(A + B - cI) \right) \leq \mu_j(Z) \leq \mu_{j+\text{rank}(B)} \left( \frac{1}{R}(A + B - cI) \right).
$$

Thus, for $\text{rank}(B) + 1 \leq j \leq N - \text{rank}(B)$, we have

$$
\mu_{j-\text{rank}(B)} \left( \frac{1}{R}(A - cI) \right) \leq \mu_j \left( \frac{1}{R}(A + B - cI) \right) \leq \mu_{j+\text{rank}(B)} \left( \frac{1}{R}(A - cI) \right).
$$
It implies

\[
F\left(\frac{1}{R}(A + B - cI), x\right) \geq F\left(\frac{1}{R}(A - cI), x\right) - \frac{\text{rank}(B)}{N},
\]
\[
F\left(\frac{1}{R}(A + B - cI), x\right) \leq F\left(\frac{1}{R}(A - cI), x\right) + \frac{\text{rank}(B)}{N}.
\]

Since \(\text{rank}(B) = o(N)\), we have

\[
F\left(\frac{1}{R}(A - cI), x\right) \pm \frac{\text{rank}(B)}{N} \xrightarrow{p} F(x). \quad \text{By the Squeeze theorem, we have}
\]

\[
F\left(\frac{1}{R}(A + B - cI), x\right) \xrightarrow{p} F(x). \quad \square
\]
From $C$ to $\mathcal{L}^{(s)}(H^r(n, p))$

Recall $\mathcal{L}^{(s)}(K^r_n) - \mathcal{L}^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4$.

Let $c := 1 - \frac{(-1)^s}{\binom{n}{s}}$ and $R := (2 + o(1)) \sqrt{\frac{1-p}{\binom{r-s}{s} \binom{n-s}{r-s} p}}$. 
Recall $L^{(s)}(K_r^n) - L^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4$.

Let $c := 1 - \frac{(-1)^s}{n \choose s}$ and $R := (2 + o(1)) \sqrt{\frac{1-p}{r-s} \frac{n-s}{r-s} p}$.

$\|M_1\| = O \left( \frac{\sqrt{(1-p) \log N}}{d} \right) = o(R)$. 
Recall \( \mathcal{L}^{(s)}(K^r_n) - \mathcal{L}^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4 \).

Let \( c := 1 - \frac{(-1)^s}{\binom{n}{s}} \) and \( R := (2 + o(1)) \sqrt{\frac{1-p}{\binom{r-s}{s} \binom{n-s}{r-s} p}} \).

- \( \|M_1\| = O\left(\frac{\sqrt{(1-p) \log N}}{d}\right) = o(R) \).
- \( M_2 \) satisfies the Semicircle Law centered at 0 with radius \( R \).
Recall $L^{(s)}(K^r_n) - L^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4$.

Let $c := 1 - \binom{-1}{s}^{n/s}$ and $R := (2 + o(1)) \sqrt{\frac{1-p}{(r-s)(n-s)p}}$.

- $\|M_1\| = O\left(\frac{\sqrt{(1-p)\log N}}{d}\right) = o(R)$.
- $M_2$ satisfies the Semicircle Law centered at 0 with radius $R$.
- $\|M_3\| = O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) = o(R)$. 
From $C$ to $\mathcal{L}^{(s)}(H^r(n, p))$

Recall $\mathcal{L}^{(s)}(K^r_n) - \mathcal{L}^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4$.
Let $c := 1 - \frac{(-1)^s}{\binom{n}{s}}$ and $R := (2 + o(1)) \sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}}}p$.

- $\|M_1\| = O \left(\frac{\sqrt{(1-p)\log N}}{d}\right) = o(R)$.
- $M_2$ satisfies the Semicircle Law centered at 0 with radius $R$.
- $\|M_3\| = O \left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) = o(R)$.
- $\text{rank}(M_4) \leq 4$. 
From $C$ to $\mathcal{L}^{(s)}(H^r(n, p))$

Recall $\mathcal{L}^{(s)}(K^r_n) - \mathcal{L}^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4$.

Let $c := 1 - \frac{(-1)^s}{\binom{n}{s}}$ and $R := (2 + o(1)) \sqrt{\frac{1-p}{(r-s)(n-s)p}}$.

- $\|M_1\| = O\left(\frac{\sqrt{(1-p) \log N}}{d}\right) = o(R)$.
- $M_2$ satisfies the Semicircle Law centered at 0 with radius $R$.
- $\|M_3\| = O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) = o(R)$.
- $\text{rank}(M_4) \leq 4$.
- $\text{rank}(\mathcal{L}^{(s)}(K^r_n) - cI) = \binom{n}{s-1} = o(N)$. 

Laplacian of Random Hypergraphs
Recall $\mathcal{L}^{(s)}(K^n_r) - \mathcal{L}^{(s)}(H^r(n, p)) = M_1 + M_2 + M_2 + M_4$.

Let $c := 1 - \frac{(-1)^s}{\binom{n}{s}}$ and $R := (2 + o(1)) \sqrt{\frac{1-p}{\binom{r-s}{s} \binom{n-s}{r-s} p}}$.

- $\|M_1\| = O\left(\frac{\sqrt{(1-p)\log N}}{d}\right) = o(R)$.
- $M_2$ satisfies the Semicircle Law centered at 0 with radius $R$.
- $\|M_3\| = O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) = o(R)$.
- $\text{rank}(M_4) \leq 4$.
- $\text{rank}(\mathcal{L}^{(s)}(K^n_r) - cI) = \binom{n}{s-1} = o(N)$.

Hence $\mathcal{L}^{(s)}(K^n_r)$ satisfies the Semicircle Law centered at 1 with radius $R$. □
It remains to prove the following Lemma.

**Lemma 3:** For any $k \ll (n^{r-s} p(1-p))^{1/4}$, we have

\[
E(\text{Trace}(C^{2k})) = (1 + o(1)) \frac{n^{s+k(r-s)} \binom{2k}{k} p^k (1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k},
\]

\[
E(\text{Trace}(C^{2k+1})) = O \left( \frac{k(2k+1)n^{s+k(r-s)} \binom{2k}{k} p^k (1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k} \right).
\]
Estimating the Trace

\[ E(\text{Trace}(C^t)) = \sum \text{E}(c_{S_1S_2}^{F_1} c_{S_2S_3}^{F_2} \cdots c_{S_tS_1}^{F_t}), \]

The sum is over all closed \( s \)-walks \( S_1F_1S_2F_2\cdots S_tF_tS_1 \).

Here \( C_{ST}^F = X_F - E(X_F) \) if \( S \cap T = \emptyset \) and \( S \cup T \subset F \); and \( C_{ST}^F = 0 \) otherwise.
\[ \mathbb{E}(\text{Trace}(C^t)) = \sum_{\text{closed } s\text{-walks}} \mathbb{E}(c_{S_1S_2}^F c_{S_2S_3}^F \cdots c_{S_tS_1}^F), \]

The sum is over all closed \( s \)-walks \( S_1F_1S_2F_2 \cdots S_tF_tS_1 \).

Here \( C_{ST}^F = X_F - \mathbb{E}(X_F) \) if \( S \cap T = \emptyset \) and \( S \cup T \subset F \); and \( C_{ST}^F = 0 \) otherwise. Group factors with same \( F \) together.

Different groups are mutually independent. In any non-zero product, every \( F \) appears at least twice.
Estimating the Trace

\[
E(\text{Trace}(C^t)) = \sum_{\text{closed } s\text{-walks}} E(c_{S_1S_2}^F c_{S_2S_3}^F \cdots c_{S_tS_1}^F),
\]

The sum is over all closed \( s \)-walks \( S_1F_1S_2F_2 \cdots S_tF_tS_1 \).

Here \( C_{ST}^F = X_F - E(X_F) \) if \( S \cap T = \emptyset \) and \( S \cup T \subset F \); and \( C_{ST}^F = 0 \) otherwise. Group factors with same \( F \) together.

Different groups are mutually independent. In any non-zero product, every \( F \) appears at least twice. Those closed walks are called “good”.

Laplacian of Random Hypergraphs
For $1 \leq i \leq \left\lfloor \frac{t}{2} \right\rfloor$, let $\mathcal{G}_i^j$ be the set of good closed walks with exactly $i$ distinct edges and $j$ distinct vertices; and let $\mathcal{G}_i := \bigcup_j \mathcal{G}_i^j$. 
For $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$, let $G^j_i$ be the set of good closed walks with exactly $i$ distinct edges and $j$ distinct vertices; and let $G_i := \bigcup_j G^j_i$. 

- If $w := S_1 F_1 S_2 F_2 \cdots S_t F_t S_1 \in G^j_i$, then 

$$E(c_{S_1 S_2}^{F_1} c_{S_2 S_3}^{F_2} \cdots c_{S_t S_1}^{F_t}) \leq p^i (1 - p)^i.$$
For $1 \leq i \leq \left\lfloor \frac{t}{2} \right\rfloor$, let $G_i^j$ be the set of good closed walks with exactly $i$ distinct edges and $j$ distinct vertices; and let $G_i := \bigcup_j G_i^j$.

- If $w := S_1F_1S_2F_2\cdots S_tF_tS_1 \in G_i^j$, then
  \[
  \mathbb{E}(c_{S_1S_2}^F c_{S_2S_3}^F \cdots c_{S_tS_1}^F) \leq p^i(1 - p)^i.
  \]

- The maximum $j$ such that $G_i^j \neq \emptyset$ is $m_i := s + i(r - s)$. 
For $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$, let $G_i^j$ be the set of good closed walks with exactly $i$ distinct edges and $j$ distinct vertices; and let $G_i := \bigcup_j G_i^j$.

- If $w := S_1 F_1 S_2 F_2 \cdots S_t F_t S_1 \in G_i^j$, then
  \[ E(c_{S_1 S_2}^{F_1} c_{S_2 S_3}^{F_2} \cdots c_{S_t S_1}^{F_t}) \leq p^i (1 - p)^i. \]

- The maximum $j$ such that $G_i^j \neq \emptyset$ is $m_i := s + i(r - s)$.

- $|G_i| = (1 + o(1)) |G_i^{m_i} |$. 

Counting good walks
Mapping every walk in $G_{i}^{m}$ into a triple $(S, E, C)$ where

- $S := \{S_1, S_2, \ldots, S_i\}$, each $S_l$ is a $s$-set.
- $E := \{E_1, E_2, \ldots, E_{i-1}\}$, each $E_l$ is a $r - 2s$-set.
- The sets in $S \cup E$ are pairwise disjoint.
- $C$ is a valid string consists of $i$ pairs of parentheses and $t - 2i$ *’s. For example,

$$(((*)()) \ast (*)* )$$
Mapping every walk in $\mathcal{G}_{i}^{m_{i}}$ into a triple $(\mathcal{S}, \mathcal{E}, \mathcal{C})$ where

- $\mathcal{S} := \{S_{1}, S_{2}, \ldots, S_{i}\}$, each $S_{l}$ is a $s$-set.
- $\mathcal{E} := \{E_{1}, E_{2}, \ldots, E_{i-1}\}$, each $E_{l}$ is a $r - 2s$-set.
- The sets in $\mathcal{S} \cup \mathcal{E}$ are pairwise disjoint.
- $\mathcal{C}$ is a valid string consists of $i$ pairs of parentheses and $t - 2i$ ‘s. For example,

\[
( (\ast ( )) \ast (\ast )\ast )
\]

\[
|\mathcal{G}_{i}^{m_{i}}| \leq \frac{N!}{(N-m_{i})!(s!)^{r}((r-2s)!)^{r-1}} \left( \begin{array}{c} t \\ 2i \end{array} \right) \frac{1}{i+1} \left( \begin{array}{c} 2i \\ i \end{array} \right) \left( \begin{array}{c} i(r-s) \\ s \end{array} \right)^{t-2i}.
\]
When $t = 2k$, the major contribution is from the walks in $\mathcal{G}_{k}^{m,k}$ which can be encoded by $(S, E, C)$ where

- $S := \{S_1, S_2, \ldots, S_k\}$, each $S_l$ is a $s$-set.
- $E := \{E_1, E_2, \ldots, E_{k-1}\}$, each $E_l$ is a $r - 2s$-set.
- The sets in $S \cup E$ are pairwise disjoint.
- $C$ is a valid string consists of $k$ pairs of parentheses.

For example, $((()))$ is corresponding to the walk

$$S_1F_1S_2F_2S_3F_2S_2F_1S_1F_3S_4F_3S_1$$

where $F_1 = S_1 \cup S_2 \cup E_1$, $F_2 = S_2 \cup S_3 \cup E_2$, and $F_3 = S_4 \cup S_1 \cup E_3$. 
Estimating $\mathbb{E}(\text{Trace}(C^{2k}))$

Let $a_i = |G_i^{m_i}| p^i (1 - p)^i$.

\[
\frac{a_i}{a_k} \leq \frac{(2k+1)(2i+1)}{2i+1} \left( \frac{i^2}{s!(r - 2s)! n^{r-s} p(1 - p)} \right)^{k-i} \leq \epsilon^{k-i},
\]

where $\epsilon := \frac{9k^4}{s!(r-2s)! n^{r-s} p(1-p)} = o(1)$, since

$n^{r-s} p(1 - p) \gg k^4$.

\[
\mathbb{E}(\text{Trace}(C^{2k})) \approx \sum_{i=1}^{k} a_i = (1 + o(1)) a_k.
\]

Done!
References


Homepage: http://www.math.sc.edu/~lu/

Thank You