Explicit Construction of Small Folkman Graphs

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on Discrete Mathematics and Algorithms
Ramsey number $R(3, 3) = 6$

If edges of $K_6$ are 2-colored then there exists a monochromatic triangle.
Ramsey number \( R(3, 3) = 6 \)

- If edges of \( K_6 \) are 2-colored then there exists a monochromatic triangle.

- There exists a 2-coloring of edges of \( K_5 \) with no monochromatic triangle.
Rado’s arrow notation

\[ G \to (H) : \text{if the edges of } G \text{ are 2-colored then there exists a monochromatic subgraph of } G \text{ isomorphic to } H. \]
Rado’s arrow notation

\[ G \rightarrow (H) \]: if the edges of \( G \) are 2-colored then there exists a monochromatic subgraph of \( G \) isomorphic to \( H \).

Fact: If \( K_6 \subset G \), then \( G \rightarrow (K_3) \).
Is there a $K_6$-free graph $G$ with $G \rightarrow (K_3)$?
A question of Erdős and Hajnal

Is there a $K_6$-free graph $G$ with $G \rightarrow (K_3)$?

Graham (1968): Yes!

$K_8 \setminus C_5$
Suppose $G$ has no monochromatic triangle.
Graham’s graph \[ K_8 \setminus C_5 = K_3 \ast C_5 \]

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Suppose \( G \) has no monochromatic triangle.

Label the vertices of \( C_5 \) by either \((r, b)\) or \((b, r)\).

A red triangle is unavoidable since \( \chi(C_5) = 3 \).
**$K_5$-free graphs $G$ with $G \to (K_3)$**

| Year  | Authors                     | $|G|$ |
|-------|-----------------------------|-----|
| 1969  | Schäuble                    | 42  |
| 1971  | Graham, Spencer             | 23  |
| 1973  | Irving                      | 18  |
| 1979  | Hadziivanov, Nenov          | 16  |
| 1981  | Nenov                       | 15  |
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In 1998, Piwakowski, Radziszowski and Urbański used a computer-aided exhaustive search to rule out all possible graphs on less than 15 vertices.
General results

Folkman’s theorem (1970): For any $k_2 > k_1 \geq 3$, there exists a $K_{k_2}$-free graph $G$ with $G \rightarrow (K_{k_1})$.

These graphs are called Folkman Graphs.
**General results**

**Folkman’s theorem (1970):** For any \( k_2 > k_1 \geq 3 \), there exists a \( K_{k_2} \)-free graph \( G \) with \( G \rightarrow (K_{k_1}) \).

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**Nešetřil-Rödl’s theorem (1976):** For \( p \geq 2 \) and any \( k_2 > k_1 \geq 3 \), there exists a \( K_{k_2} \)-free graph \( G \) with \( G \rightarrow (K_{k_1})_p \).

Here \( G \rightarrow (H)_p \): if the edges of \( G \) are \( p \)-colored then there exists a monochromatic subgraph of \( G \) isomorphic to \( H \).
Let $f(p, k_1, k_2)$ denote the smallest integer $n$ such that there exists a $K_{k_2}$-free graph $G$ on $n$ vertices with $G \rightarrow (K_{k_1})_p$.

Graham

$$f(2, 3, 6) = 8.$$
Let $f(p, k_1, k_2)$ denote the smallest integer $n$ such that there exists a $K_{k_2}$-free graph $G$ on $n$ vertices with $G \rightarrow (K_{k_1})^p$.

- **Graham**
  
  $f(2, 3, 6) = 8$.

- **Nenov, Piwakowski, Radziszowski and Urbański**
  
  $f(2, 3, 5) = 15$. 

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f(2, 3, 5) = 15.
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- **What about \( f(2, 3, 4) \)?**
Upper bound of $f(2, 3, 4)$

Folkman, Nešetřil-Rödl’s upper bound is huge.
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- Erdős re-set a prize of $100$ for the new challenge
  
  \[ f(2, 3, 4) \leq 10^6. \]
The most wanted Folkman Graph
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Problem on triangle-free subgraphs in graphs containing no $K_4$
(proposed by Erdős)

Let $f(p, k_1, k_2)$ denote the smallest integer $n$ such that there is a graph $G$ with $n$ vertices satisfying the properties:
(1) any edge coloring in $p$ colors contains a monochromatic $K_{k_1}$;
(2) $G$ contains no $K_{k_2}$.
Prove or disprove:

$$f(2, 3, 4) < 10^6.$$
Difficulty

There is no efficient algorithm to test whether
\[ G \rightarrow (K_3) . \]
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For moderate $n$, Folkman graphs are very rare among all $K_4$-free graphs on $n$ vertices.
Difficulty

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- For moderate $n$, Folkman graphs are very rare among all $K_4$-free graphs on $n$ vertices.

- Probabilistic methods are generally good choices for asymptotic results. However, it is not good for moderate size $n$. 
Our approach

Find a simple and sufficient condition for $G \rightarrow (K_3)$, and an efficient algorithm to verify this condition.
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- Localization and $\delta$-fairness.
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- Localization and $\delta$-fairness.

- Circulant graphs and $L(m, s)$. 
Our result

We claim the reward by proving

**Theorem 1 (Lu, 2007)** \( f(2, 3, 4) \leq 9697. \)
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We explicitly constructed 4 Folkman graphs with orders

9697, 30193, 33121, 57401.
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We explicitly constructed 4 Folkman graphs with orders

\[ 9697, \quad 30193, \quad 33121, \quad 57401. \]

**Recent update:** Dudek and Rödl (2008) proved

\[ f(2, 3, 4) \leq 941. \]
Spencer’s Lemma

Notations:
- $G_v$: the induced graph on the neighborhood of $v$.
- $b(H)$: the maximum size of edge-cuts for $H$. 
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For $0 < \delta < \frac{1}{2}$, a graph $H$ is $\delta$-fair if

$$b(H) < \left(\frac{1}{2} + \delta\right)|E(H)|.$$
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$G$ is a Folkman graph if for each $v$

- $G_v$ is $\frac{1}{6}$-fair.
- $G_v$ is $K_3$-free.
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For vertex transitive graph $G$, all $G_v$'s are isomorphic.
Spectral lemma

- $H$: a graph on $n$ vertices
- $A$: the adjacency matrix of $H$
- $d = (d_1, d_2, \ldots, d_n)$: degrees of $H$
- $\text{Vol}(S) = \sum_{v \in S} d_v$: the volume of $S$
- $\bar{d} = \frac{\text{Vol}(H)}{n}$: the average degree
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Lemma (Lu) If the smallest eigenvalue of $M = A - \frac{1}{\text{Vol}(H)} d \cdot d'$ is greater than $-2\delta \bar{d}$, then $H$ is $\delta$-fair.
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**Lemma (Lu)** *If the smallest eigenvalue of $M = A - \frac{1}{\text{Vol}(H)}d \cdot d'$ is greater than $-2\delta \bar{d}$, then $H$ is $\delta$-fair.*

Similar results hold for $A$ and $L$. However, they are weaker than using $M$ in experiments.
Corollary

Suppose $H$ is a $d$-regular graph and the smallest eigenvalue of its adjacency matrix $A$ is greater than $-2\delta d$. Then $H$ is $\delta$-fair.
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Proof: We can replace $M$ by $A$ in the previous lemma.

- 1 is an eigenvector of $A$ with respect to $d$.
- $M$ is the projection of $A$ to the hyperspace $1^\perp$.
- $M$ and $A$ have the same smallest eigenvalues.
The proof of the Lemma

\[ V(H) = X \cup Y: \text{a partition of the vertex-set.} \]
The proof of the Lemma

- $V(H) = X \cup Y$: a partition of the vertex-set.
- $1_X, 1_Y$: indicated functions of $X$ and $Y$.

\[ 1_X + 1_Y = 1. \]
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$$1_X + 1_Y = 1.$$ 

- We observe $M1 = 0$.

- For each $t \in (0, 1)$, let $\alpha(t) = (1 - t)1_X - t1_Y$. We have

$$\alpha(t)' \cdot M \cdot \alpha(t) = -e(X, Y) + \frac{1}{\Vol(H) \Vol(X) \Vol(Y)}.$$
The proof of the Lemma

Let $\rho$ be the smallest eigenvalue of $M$. We have

$$e(X, Y) - \frac{\text{Vol}(X)\text{Vol}(Y)}{\text{Vol}(H)} \leq -\alpha(t)' \cdot M \cdot \alpha(t) \leq -\rho \|\alpha_t\|^2.$$
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Choose $t = \frac{|X|}{n}$ so that $\|\alpha(t)\|^2$ reaches its minimum $\frac{|X||Y|}{n}$.
The proof of the Lemma

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Choose $t = \frac{|X|}{n}$ so that $\|\alpha(t)\|^2$ reaches its minimum $\frac{|X||Y|}{n}$. We have

$$e(X, Y) \leq \frac{\text{Vol}(X)\text{Vol}(Y)}{\text{Vol}(H)} + \rho \frac{|X||Y|}{n}.$$

$$\leq \frac{\text{Vol}(H)}{4} - \rho \frac{n}{4}$$

$$< \left(\frac{1}{2} + \delta\right)|E(H)|, \text{ since } \rho > -2\delta \bar{d}. \quad \Box$$
Circulant graphs

- \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \)
- \( S \): a subset of \( \mathbb{Z}_n \) satisfying \(-S = S\) and \( 0 \notin S \).

We define a circulant graph \( H \) by

- \( V(H) = \mathbb{Z}_n \)
- \( E(H) = \{xy \mid x - y \in S\} \).

**Example:** A circulant graph with \( n = 8 \) and \( S = \{\pm1, \pm3\} \).
**Lemma:** The eigenvalues of the adjacency matrix for the circulant graph generated by $S \subset \mathbb{Z}_n$ are

$$
\sum_{s \in S} \cos \frac{2\pi is}{n}
$$

for $i = 0, \ldots, n - 1$. 

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for $i = 0, \ldots, n - 1$.

Proof: Note $A = g(J)$, where

$$g(x) = \sum_{s \in S} x^s.$$

$$J = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$
Proof continues...

Let \( \phi = e^{\frac{2\pi\sqrt{-1}}{n}} \) denote the primitive \( n \)-th unit root. 
\( J \) has eigenvalues

\[ 1, \phi, \phi^2, \ldots, \phi^{n-1}. \]
Proof continues...

Let $\phi = e^{\frac{2\pi\sqrt{-1}}{n}}$ denote the primitive $n$-th unit root. $J$ has eigenvalues

$$1, \phi, \phi^2, \ldots, \phi^{n-1}.$$ 

Thus, the eigenvalues of $A = g(J)$ are

$$g(1), g(\phi), \ldots, g(\phi^{n-1}).$$
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Thus, the eigenvalues of \( A = g(J) \) are

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For \( i = 0, 1, 2, \ldots, n - 1 \), we have

\[ g(\phi^i) = \Re(g(\phi^i)) = \sum_{s \in S} \cos \frac{2\pi i s}{n}. \]
Graph $L(m, s)$

Suppose $s$ and $m$ are relatively prime to each other. Let $n$ be the least positive integer satisfying

$$s^n \equiv 1 \mod m.$$
Graph $L(m, s)$

Suppose $s$ and $m$ are relatively prime to each other. Let $n$ be the least positive integer satisfying

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We define the graph $L(m, s)$ to be a circulant graph on $m$ vertices with

$$S = \{ s^i \mod m \mid i = 0, 1, 2, \ldots, n - 1 \}.$$  

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**Proposition:** The local graph $G_v$ of $L(m, s)$ is also a circulant graph.
Algorithm

For each $L(m, s)$, compute the local graph $G_v$. 
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- If $G_v$ is not triangle-free, reject it and try a new graph $L(m, s)$. 
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- If $G_v$ is not triangle-free, reject it and try a new graph $L(m, s)$.
- If the ratio the smallest eigenvalue verse the largest eigenvalue of $G_v$ is less than $-\frac{1}{3}$, reject it and try a new graph $L(m, s)$.
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- Output a Folkman graph $L(m, s)$. 
Computational results

<table>
<thead>
<tr>
<th>$L(m, s)$</th>
<th>$\sigma$</th>
</tr>
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<tbody>
<tr>
<td>$L(127, 5)$</td>
<td>$-0.6363\ldots$</td>
</tr>
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<td>$L(761, 3)$</td>
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$\sigma$ is the ratio of the smallest eigenvalue to the largest eigenvalue in the local graph.
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<td>$L(1801, 125)$</td>
<td>$-0.4912 \cdots$</td>
</tr>
<tr>
<td>$L(2641, 2)$</td>
<td>$-0.4275 \cdots$</td>
</tr>
<tr>
<td>$L(9697, 4)$</td>
<td>$-0.3307 \cdots$</td>
</tr>
<tr>
<td>$L(30193, 53)$</td>
<td>$-0.3094 \cdots$</td>
</tr>
<tr>
<td>$L(33121, 2)$</td>
<td>$-0.2665 \cdots$</td>
</tr>
<tr>
<td>$L(57401, 7)$</td>
<td>$-0.3289 \cdots$</td>
</tr>
</tbody>
</table>

- $\sigma$ is the ratio of the smallest eigenvalue to the largest eigenvalue in the local graph.
- All graphs on the left are $K_4$-free.
- Graphs in red are Folkman graphs.
- Graphs in black are good candidates.
Open questions

Exoo conjectured $L(127, 5)$ is a Folkman graph.
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- A new challenge: prove or disprove $f(2, 3, 4) \leq 100$. 

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