Probabilistic Methods in Combinatorics
Lecture 9
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(\Omega, \mathcal{F}, P): a probability space.

A, B: two events.

A and B are independent if

\[ \Pr(AB) = \Pr(A)\Pr(B). \]

A and B are \textbf{positively correlated} if

\[ \Pr(AB) \geq \Pr(A)\Pr(B). \]

A and B are \textbf{negatively correlated} if

\[ \Pr(AB) \leq \Pr(A)\Pr(B). \]
Four Functions Theorem

- \( N := \{1, 2, 3 \ldots, n\} \)
- \( P(N) \): the power set of \( N \).
- \( \alpha, \beta, \gamma, \delta: P(N) \rightarrow \mathbb{R}^+ \)
- For \( \mathcal{A} \subset P(N) \), and \( \phi \in \{\alpha, \beta, \gamma, \delta\} \), let \( \phi(\mathcal{A}) = \sum_{A \in \mathcal{A}} \phi(A) \).
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**Theorem [Ahlswede, Daykin (1978)]:** If for any \( A, B \subset N \),

\[
\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B),
\]

then for any \( \mathcal{A}, \mathcal{B} \subset P(N) \),

\[
\alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \cup \mathcal{B})\delta(\mathcal{A} \cap \mathcal{B}),
\]
Simplification:

- Modifying \( \alpha \) so that \( \alpha(A) = 0 \) for all \( A \notin A \).
- Modifying \( \beta \) so that \( \beta(B) = 0 \) for all \( B \notin B \).
- Modifying \( \gamma \) so that \( \gamma(C) = 0 \) for all \( C \notin A \cup B \).
- Modifying \( \delta \) so that \( \delta(D) = 0 \) for all \( D \notin A \cap B \).
Simplification:

- Modifying $\alpha$ so that $\alpha(A) = 0$ for all $A \notin A$.
- Modifying $\beta$ so that $\beta(B) = 0$ for all $B \notin B$.
- Modifying $\gamma$ so that $\gamma(C) = 0$ for all $C \notin A \cup B$.
- Modifying $\delta$ so that $\delta(D) = 0$ for all $D \notin A \cap B$.

$$\alpha(A)\alpha(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

still holds. It is sufficient to prove for $A = B = P(N)$. 
**Induction on** $n$

Initial case $n = 1$: $P(N) = \{\emptyset, N\}$. Use index 0 for $\emptyset$ and 1 for $N$. We have

\[
\begin{align*}
\alpha_0 \beta_0 & \leq \gamma_0 \delta_0 \\
\alpha_0 \beta_1 & \leq \gamma_1 \delta_0 \\
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\alpha_1 \beta_1 & \leq \gamma_1 \delta_1.
\end{align*}
\]

We need prove

\[
(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \leq (\gamma_0 + \gamma_1)(\delta_0 + \delta_1).
\]

It can be directly verified.
Inductive step

Suppose it holds for $n - 1$ and let us prove it for $n \geq 2$. Let $N' = N \setminus \{n\}$ and for each $\phi \in \{\alpha, \beta, \gamma, \delta\}$ and $A \in N'$ define

$$\phi'(A) = \phi(A) + \phi(A \cup \{n\}).$$

Note that $\phi(P(N)) = \phi'(P(N'))$. Apply inductive hypothesis for functions $\alpha'$, $\beta'$, $\gamma'$, and $\delta'$. It suffices to check

$$\alpha'(A)\alpha'(B) \leq \gamma'(A \cup B)\delta'(A \cap B).$$

This is similar to the case $n = 1$.  \[\square\]
(L, ∨, ∧) is a lattice if it satisfies

- Commutative laws: \( a \lor b = b \lor a \), \( a \land b = b \land a \).
$(L, \lor, \land)$ is a lattice if it satisfies

- **Commutative laws:** $a \lor b = b \lor a$, $a \land b = b \land a$.

- **Associative laws:** $a \lor (b \lor c) = (a \lor b) \lor c$, $a \land (b \land c) = (a \land b) \land c$.
Distributive lattice

$(L, \lor, \land)$ is a lattice if it satisfies

- **Commutative laws:** $a \lor b = b \lor a$, $a \land b = b \land a$.

- **Associative laws:** $a \lor (b \lor c) = (a \lor b) \lor c$, $a \land (b \land c) = (a \land b) \land c$

- **Absorption laws:** $a \lor (a \land b) = a$, $a \land (a \lor b) = a$.

It is distributive if it further satisfies the distributive laws:

\[
\begin{align*}
  a \land (b \lor c) & = (a \land b) \lor (a \land c), \\
  a \lor (b \land c) & = (a \lor b) \land (a \lor c).
\end{align*}
\]
Theorem [Ahlsweede, Daykin (1978)]: Let $L$ be a distributive lattice and $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}^+$. If for any $x, y \in L$,

$$\alpha(x) \alpha(y) \leq \gamma(x \lor y) \delta(x \land y),$$

then for any $X, Y \subset L$,

$$\alpha(X) \alpha(Y) \leq \gamma(X \lor Y) \delta(X \land Y),$$
Theorem [Ahlswede, Daykin (1978)]: Let $L$ be a distributive lattice and $\alpha, \beta, \gamma, \delta: L \rightarrow \mathbb{R}^+$. If for any $x, y \in L$,

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Note any distributive lattice can be embedded into $P([n])$. This is a corollary of the previous theorem.
A function $\mu : L \rightarrow \mathbb{R}^+$ is log-supermodular if

$$\mu(x) \mu(y) \leq \mu(x \lor y) \mu(x \land y)$$

for all $x, y$. 
FKG inequalities

- A function $\mu: L \rightarrow \mathbb{R}^+$ is log-supermodular if $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$ for all $x, y$.

- $f: L \rightarrow \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is decreasing if $f(x) \geq f(y)$ whenever $x \leq y$. 
FKG inequalities

- A function $\mu : L \rightarrow \mathbb{R}^+$ is log-supermodular if $\mu(x) \cdot \mu(y) \leq \mu(x \vee y) \cdot \mu(x \wedge y)$ for all $x, y$.

- $f : L \rightarrow \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is decreasing if $f(x) \geq f(y)$ whenever $x \leq y$.

The FKG Inequality [Fortunin-Kasteleyn-Ginibre 1971]:

If $\mu$ is log-supermodular and $f, g$ are increasing, then

$$\sum_{x \in L} f(x) \cdot \mu(x) \sum_{x \in L} g(x) \cdot \mu(x) \leq \sum_{x \in L} f(x) g(x) \cdot \mu(x) \sum_{x \in L} \mu(x).$$
A function $\mu : L \rightarrow \mathbb{R}^+$ is log-supermodular if $\mu(x)\mu(y) \leq \mu(x \lor y)\mu(x \land y)$ for all $x, y$.

$f : L \rightarrow \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is decreasing if $f(x) \geq f(y)$ whenever $x \leq y$.

The FKG Inequality [Fortuin-Kasteleyn-Ginibre 1971]: If $\mu$ is log-supermodular and $f, g$ are increasing, then

$$\sum_{x \in L} f(x)\mu(x) \leq \sum_{x \in L} g(x)\mu(x) \leq \sum_{x \in L} f(x)g(x)\mu(x) \sum_{x \in L} \mu(x).$$

If one is increasing and the other is decreasing, then

$$\sum_{x \in L} f(x)\mu(x) \geq \sum_{x \in L} g(x)\mu(x) \geq \sum_{x \in L} f(x)g(x)\mu(x) \sum_{x \in L} \mu(x).$$
A probabilistic view

- \((P(N), \mu)\): a probability space where \(\mu\) is log-supermodular.
- An event \(\mathcal{A}\) is monotone increasing if \(A \in \mathcal{A}\) and \(A \subset B\) implies \(B \in \mathcal{A}\).

**Proposition:** If both \(A\) and \(B\) are monotone increasing or monotone decreasing, then

\[
\Pr(\mathcal{A}\mathcal{B}) \geq \Pr(\mathcal{A})\Pr(\mathcal{B}).
\]

If one is monotone increasing and the other one is monotone decreasing, then

\[
\Pr(\mathcal{A}\mathcal{B}) \leq \Pr(\mathcal{A})\Pr(\mathcal{B}).
\]
Applying to $G(n, p)$

In $G(n, p)$, for any graph $H$,

$$\mu(H) = \Pr(H) = p^{|E(H)|} (1 - p)^{|E(\bar{H})|}.$$ 

Observe that this $\mu$ is log-supermodular. We get a lot of correlation inequalities on monotone events.
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Example of monotone events:

- Triangle-free.
- Planarity.
- $k$-connected.
- Hamiltonian.
- $H$-free.
- Diameter less than $k$.  