Probabilistic Methods in Combinatorics
Lecture 9

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Correlation Inequalities

- $(\Omega, \mathcal{F}, P)$: a probability space.
- A, B: two events.
- A and B are independent if

$$\Pr(AB) = \Pr(A)\Pr(B).$$

- A and B are **positively correlated** if

$$\Pr(AB) \geq \Pr(A)\Pr(B).$$

- A and B are **negatively correlated** if

$$\Pr(AB) \leq \Pr(A)\Pr(B).$$

Four Functions Theorem

- $N := \{1, 2, 3 \ldots, n\}$
- $P(N)$: the power set of N.
- $\alpha, \beta, \gamma, \delta \colon P(N) \to \mathbb{R}^+$
- For $\mathcal{A} \subset P(N)$, and $\phi \in \{\alpha, \beta, \gamma, \delta\}$, let $\phi(\mathcal{A}) = \sum_{A \in \mathcal{A}} \phi(A)$.

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Theorem [Ahlswede, Daykin (1978)]: If for any $A, B \subset N$,

$$\alpha(A)\beta(B) \le \gamma(A \cup B)\delta(A \cap B),$$

then for any $\mathcal{A}, \mathcal{B} \subset P(N)$,

$$\alpha(\mathcal{A})\beta(\mathcal{B}) \le \gamma(\mathcal{A} \cup \mathcal{B})\delta(\mathcal{A} \cap \mathcal{B}),$$

Proof

Simplification:

- Modifying α so that $\alpha(A) = 0$ for all $A \notin \mathcal{A}$.
- Modifying β so that $\beta(B) = 0$ for all $B \notin \mathcal{B}$.
- Modifying γ so that $\gamma(C) = 0$ for all $C \notin \mathcal{A} \cup \mathcal{B}$.
- Modifying δ so that $\delta(D) = 0$ for all $D \notin \mathcal{A} \cap \mathcal{B}$.

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$$\alpha(A)\alpha(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

still holds. It is sufficient to prove for $\mathcal{A} = \mathcal{B} = P(N)$.

Induction on n

Initial case $n = 1$: $P(N) = \{\emptyset, N\}$. Use index 0 for $\emptyset$ and 1 for N. We have

$$\begin{aligned} \alpha_0\beta_0 &\leq \gamma_0\delta_0 \\ \alpha_0\beta_1 &\leq \gamma_1\delta_0 \\ \alpha_1\beta_0 &\leq \gamma_1\delta_0 \\ \alpha_1\beta_1 &\leq \gamma_1\delta_1. \end{aligned}$$

We need prove

$$(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \leq (\gamma_0 + \gamma_1)(\delta_0 + \delta_1).$$

It can be directly verified.

Inductive step

Suppose it holds for $n-1$ and let us prove it for $n \geq 2$. Let $N' = N \setminus \{n\}$ and for each $\phi \in \{\alpha, \beta, \gamma, \delta\}$ and $A \in N'$define

$$\phi'(A) = \phi(A) + \phi(A \cup \{n\}).$$

Note that $\phi(P(N)) = \phi'(P(N'))$. Apply inductive hypothesis for functions α', β', γ', and δ'. It suffices to check

$$\alpha'(A)\alpha'(B) \leq \gamma'(A \cup B)\delta'(A \cap B).$$

This is similar to the case $n = 1$. □

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- Absorption laws: $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.

It is distributive if it further satisfies the distributive laws:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

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4FT on distributive lattice

Theorem [Ahlswede, Daykin (1978)]: Let L be a distributive lattice and $\alpha, \beta, \gamma, \delta \colon L \to \mathbb{R}^+$. If for any $x, y \in L$,

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Note any distributive lattice can be embedded into $P([n])$. This is a corollary of the previous theorem.

FKG inequalities

- A function $\mu\colon L \to \mathbb{R}^+$ is log-supermodular if $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$ for all x, y.

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The FKG Inequality [Fortuin-Kasteleyn-Ginibre 1971]:
If μ is log-supermodular and f, g are increasing, then

$$\sum_{x\in L} f(x)\mu(x) \sum_{x\in L} g(x)\mu(x) \le \sum_{x\in L} f(x)g(x)\mu(x) \sum_{x\in L} \mu(x).$$

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If one is increasing and the other is decreasing, then

$$\sum_{x\in L} f(x)\mu(x) \sum_{x\in L} g(x)\mu(x) \ge \sum_{x\in L} f(x)g(x)\mu(x) \sum_{x\in L} \mu(x).$$

A probabilistic view

- $(P(N), \mu)$: a probability space where μ is log-supermodular.
- An event $\mathcal{A}$ is monotone increasing if $A \in \mathcal{A}$ and $A \subset B$ implies $B \in \mathcal{A}$.

Proposition: If both A and B are monotone increasing or monotone decreasing, then

$$\mathrm{Pr}(\mathcal{A}\mathcal{B}) \geq \mathrm{Pr}(\mathcal{A})\mathrm{Pr}(\mathcal{B}).$$

If one is monotone increasing and the other one is monotone decreasing, then

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Applying to $G(n,p)$

In $G(n,p)$, for any graph H,

$$\mu(H) = \Pr(H) = p^{|E(H)|}(1-p)^{|E(\bar{H})|}.$$

Observe that this μ is log-supermodular. We get a lot of correlation inequalities on monotone events.

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Example of monotone events:

- Triangle-free.
- Planarity.
- k-connected.
- Hamiltonian.
- H-free.
- Diameter less than k.