Probabilistic Methods in Combinatorics
Lecture 8

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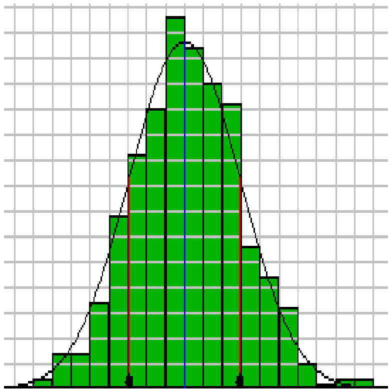
Binomial distribution $B(n, p)$

- n is the number of variables.
- $0 < p < 1$ is the probability.



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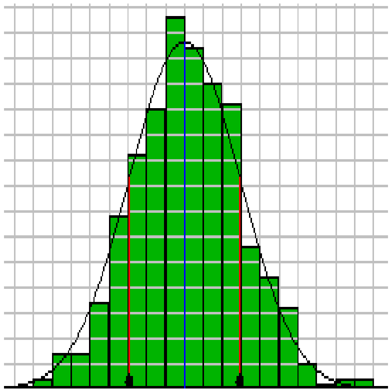

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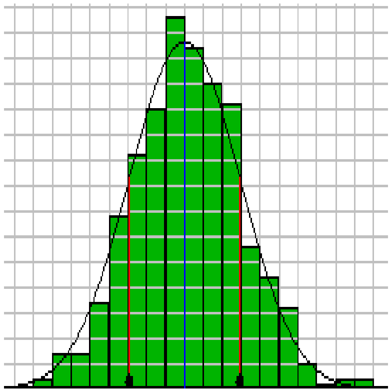

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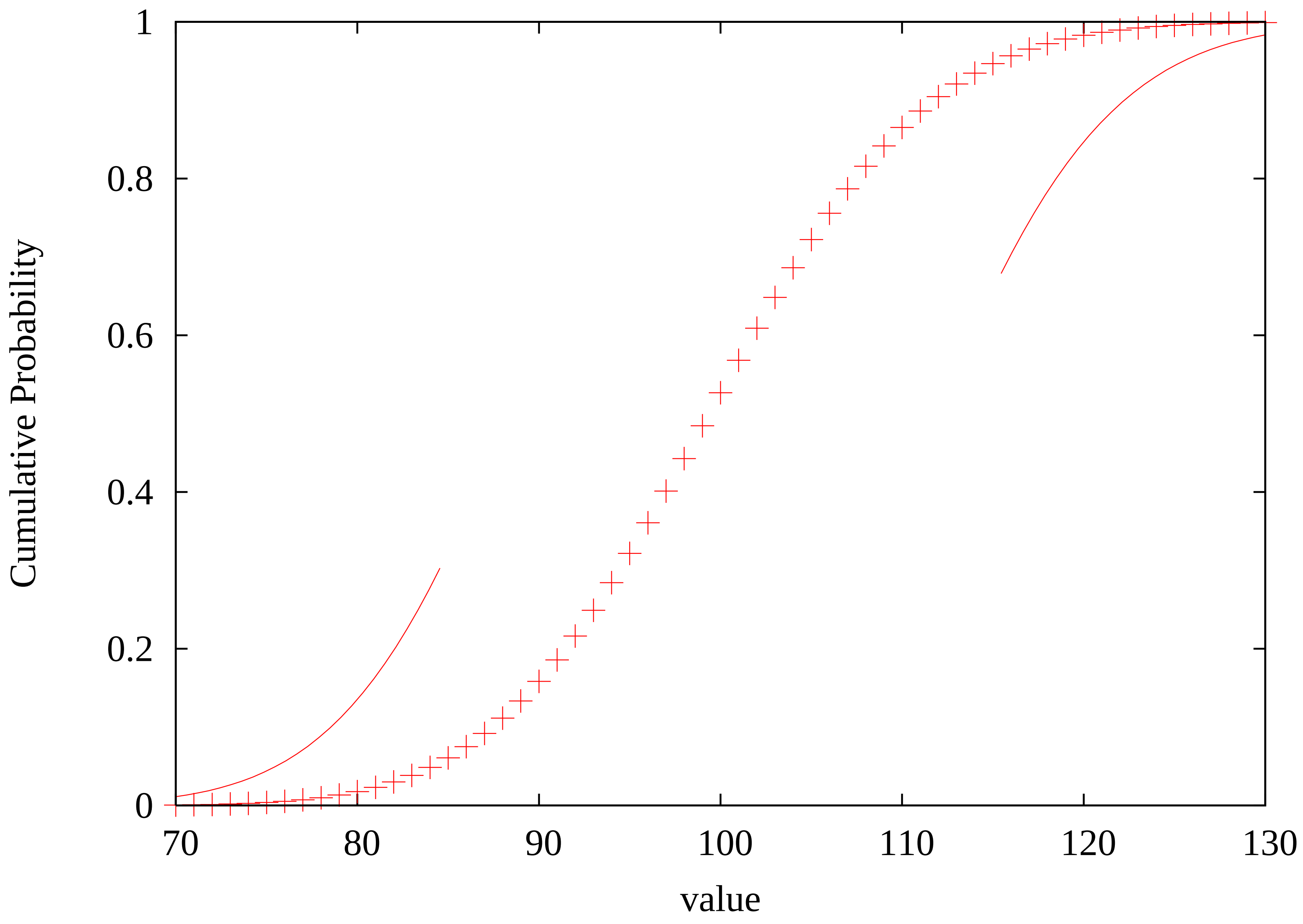
$$B(n, p) \sim N(\mu, \sigma)$$

$N(\mu, \sigma)$: normal distribution with $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

Large deviations



Large deviation inequality I

Chernoff inequalities: Suppose $X = \sum_{i=1}^{n} X_i$, where X_i are independent 0-1 random variables with

$$\Pr(X_i = 1) = p_i, \quad \Pr(X_i = 0) = 1 - p_i.$$

Then we have

$$\begin{aligned} Pr(X < E(X) - \lambda) &\leq e^{-\frac{\lambda^2}{2E(X)}} \\ Pr(X > E(X) + \lambda) &\leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}} \end{aligned}$$

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A weighted version of Chernoff's inequality:

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Theorem [Chung,Lu] We have

$$Pr(X < E(X) - \lambda) \leq e^{-\lambda^2/2\nu} \quad (1)$$

$$Pr(X > E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\nu + M\lambda/3)}}. \quad (2)$$

Large deviation inequality III

Theorem [McDiarmid]: Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i - E(X_i) \leq M$ for a positive constant M. Let $X = \sum_{i=1}^n X_i$. Then

$$Pr(X - E(X) > \lambda) \leq e^{-\frac{\lambda^2}{2(Var(X) + M\lambda/3)}}.$$

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Note: If $Pr(X_i = a_i) = p_i$ and $Pr(X_i = 0) = 1 - p_i$, then $\mathrm{Var}(X) = a_i^2 p_i (1 - p_i) \leq \nu$. Thus

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This theorem implies inequality of upper tail in previous Theorem.

Large deviation inequality IV

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \leq M$, for $1 \leq i \leq n$. Let $X = \sum_{i=1}^n X_i$ and $\|X\| = \sqrt{\sum_{i=1}^n \mathrm{E}(X_i^2)}$. Then we have

$$\Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\|X\|^2 + M\lambda/3)}}.$$

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This theorem implies McDiarmid's Theorem.

$$\text{Let } X_i' = X_i - \mathrm{E}(X_i), \text{ and } X' = X - \mathrm{E}(X).$$

$$X - \mathrm{E}(X) = X' - \mathrm{E}(X')$$

$$\|X'\|^2 = \sum_{i=1}^{n} \mathrm{E}(X_i'^2) = \mathrm{Var}(X).$$

Lower tail

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \geq 0$, for $1 \leq i \leq n$. Let $X = \sum_{i=1}^{n} X_i$ and $\|X\| = \sqrt{\sum_{i=1}^{n} \mathrm{E}(X_i^2)}$. Then we have

$$\Pr(X \leq E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2\|X\|^2}}.$$

Lower tail

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$$\Pr(X \leq E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2\|X\|^2}}.$$

Proof: Let $X_i' = -X_i$ and $X' = -X$. Applying the upper tail to X' with $M = 0$, we get

$$\begin{aligned}\Pr(X \leq E(X) - \lambda) &= \Pr(X' \geq E(X') + \lambda) \\ &\leq e^{-\frac{\lambda^2}{2\|X'\|^2}} = e^{-\frac{\lambda^2}{2\|X\|^2}}.\end{aligned}$$

A special function

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Facts:

- $g(0) = 1.$
- $g(y) \leq 1$, for $y < 0.$
- $g(y)$ is monotone increasing, for $y \geq 0.$
- For $y < 3$, we have

$$g(y) = 2\sum_{k=2}^{\infty} \frac{y^{k-2}}{k!} \leq \sum_{k=2}^{\infty} \frac{y^{k-2}}{3^{k-2}} = \frac{1}{1 - y/3}.$$

Proof of upper tail

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$$\begin{aligned}
\mathrm{E}(e^{tX}) &= \prod_{i=1}^{n} \mathrm{E}(e^{tX_i}) \\
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\end{aligned}$$

Proof of upper tail

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&= \prod_{i=1}^{n} \mathrm{E}\Big(1 + t\mathrm{E}(X_i) + \frac{1}{2} t^2 X_i^2 g(tX_i)\Big) \\
&\leq \prod_{i=1}^{n} \Big(1 + t\mathrm{E}(X_i) + \frac{1}{2} t^2 \mathrm{E}(X_i^2) g(tM)\Big)
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&\leq \prod_{i=1}^{n} (1 + t\mathrm{E}(X_i) + \frac{1}{2} t^2 \mathrm{E}(X_i^2) g(tM)) \\
&\leq \prod_{i=1}^{n} e^{t\mathrm{E}(X_i) + \frac{1}{2} t^2 \mathrm{E}(X_i^2) g(tM)}
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Hence, for t satisfying $tM < 3$, we have

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□

Large deviation inequality V

Theorem [Chung, Lu] Let X_i denote independent random variables satisfying $X_i \leq \mathrm{E}(X_i) + a_i + M$, for $1 \leq i \leq n$. For, $X = \sum_{i=1}^{n} X_i$, we have

$$\Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\mathrm{Var}(X) + \sum_{i=1}^{n} a_i^2 + M\lambda/3)}}.$$

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continue

$$\Pr(X \geq E(X) + \lambda) \;=\; \Pr(X' \geq E(X') + \lambda)$$

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It remains to verify

$$X' - \mathrm{E}(X') = X - \mathrm{E}(X).$$

$$\|X'\|^2 = \mathrm{Var}(X) + \sum_{i=1}^n a_i^2.$$

□

Large deviation inequality VI

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \leq \mathrm{E}(X_i) + M_i$, for $0 \leq i \leq n$. We order X_i's so that M_i are in an increasing order. Let $X = \sum_{i=1}^n X_i$. Then for any $1 \leq k \leq n$, we have

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Compared with McDiarmid's inequality

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- McDiarmid's inequality is a special case with $k = n$.

Proof

For fixed k, we choose $M = M_k$ and

$$a_i = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ M_i - M_k & \text{if } k \leq i \leq n \end{cases}$$

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Apply previous theorem with these a_i's. □

An application

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Expectation and Variance

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\begin{aligned}
\mathrm{E}(X) &= \sum_{i=1}^{n} \mathrm{E}(X_i) \\
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In particular, for constant $p \in (0,1)$ and $\lambda = \Theta(n^{\frac{1}{2}+\epsilon})$, we have

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