Probabilistic Methods in Combinatorics
Lecture 7

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Ramsey numbers

Theorem (Spencer, 1975)

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- For $T \in \binom{[n]}{k}$, let B_T be the event that T is an independent set of G; $\Pr(B_t) = (1-p)^{\binom{k}{2}}$.
- Dependence graph: $d_{SS} \leq 3n$, $d_{ST} \leq 3\binom{n}{k-2}$, $d_{TS} \leq \binom{k}{2}n$, and $d_{TT} \leq \binom{k}{2}\binom{n}{k-2}$.

Proof

By LLL, we only require

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\begin{aligned}
p^3 &\le x(1-x)^{3n}(1-y)^{3\binom{n}{k-2}} \\
(1-p)^{\binom{k}{2}} &\le y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{k}{2}\binom{n}{k-2}}.
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We can choose $p = c_1 n^{-1/2}$, $k = c_2 n^{1/2} \log n$, $x = c_3 n^{-3/2}$, and $y = c_4 / \binom{n}{k}$.

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This gives $R(3,k) > c_5 k^2 / \log^2 k$. $\square$

$R(4,k)$

Best bounds for $R(r,k)$ (for fixed r and k large),

$$c\left(\frac{k}{\log k}\right)^{(r+1)/2} < R(r,k) < (1+o(1))\frac{k^{r-1}}{\log^{r-2} k}.$$

Erdős conjecture \$250: Prove

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The best lower bound is using LLL; $R(4,k) > c'\frac{k^{2.5}}{\log^{2.5} k}$.

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Theorem [Alon and Linial (1989) If $e(\Delta\delta + 1)(1 - 1/k)^{\delta} < 1$, then D contains a (directed, simple) cycle of length $0 \mod k$.

Proof: First we can assume every out-degree is δ by deleting some edges if necessary. Consider $f: V \to \mathbb{Z}_k$. Bad event A_v: no $u \in \Gamma^+(v)$ with $f(u) = f(v) + 1$.

$$\Pr(A_v) = (1 - 1/k)^{\delta}.$$

Each event depends on at most $\delta\Delta$ others. Apply LLL. $\square$

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If the conjecture is true, then it is tight.

$$\mathrm{la}(G) \geq \frac{nd}{2(n-1)} > \frac{d}{2}.$$

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- G is d-regular if $d^+(v) = d^-(v) = d$ for any vertex v.
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DLA conjecture for d implies LA conjecture for $2d$.

A proposition

Proposition: *Let* $H = (V, E)$ *be a graph with maximum degree* d*, and let* $V = V_1 \cup V_2 \cup \cdots \cup V_r$ *be a partition of* V*. If* $|V_i| \geq 2ed$*, then there is an independent set of vertices* W *that contains a vertex from each* V_i*.*

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Proposition: *Let $H = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of V. If $|V_i| \geq 2ed$, then there is an independent set of vertices W that contains a vertex from each V_i.*

Proof: WLOG, we assume

$$|V_1| = |V_2| = \cdots = |V_r| = \lceil 2ed \rceil = g.$$

Pick from each V_i a vertex randomly and independently. Let W be the random set of the vertices picked. For each edge f, let A_f be the event that both ends in W. The maximum degree in the dependence graph is at most $2gd - 1$. We have $e \cdot 2gd \cdot \frac{1}{g^2} = \frac{2ed}{g} < 1$. Apply LLL. □

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Theorem Let $G = (U, F)$ be a d-regular digraph with directed girth $g \geq 8ed$. Then

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Proof: Using Hall's matching theorem, we can partition F into d pairwise disjoint 1-regular spanning subgraphs $F_1, \ldots, F_d$ of G.

Continue

Each F_i is a union of vertex disjoint directed cycles. Let $V_1, \ldots, V_r$ are the sets of edges of all cycles. Then

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Apply the proposition to the line-graph H of G. Note H is $4d - 2$-regular.

There exists an independent set M_1 of H. Now $M_1, F_1 \setminus M_1, \ldots, F_d \setminus M_1$ forms $d + 1$ linear directed forests. $\square$

General d-regular graphs

Theorem [Alon 1988] There is an absolute constant $c > 0$ such that for every d-regular directed graph G

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The error terms can be improved to $cd^{2/3} \log^{1/3} d$.

Proof

Pick a prime p. Color each vertex randomly and uniformly into p colors. I.e., consider a random map

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Define for $i \in \mathbb{Z}_p$,

$$E_i = \{(u, v) \in E : f(v) = f(u) + i\}.$$

Let $G_i = (V, E_i)$ and

- Δ_i^+: the maximum out-degree of G_i.
- Δ_i^-: the maximum in-degree of G_i.
- Δ_i: the maximum of Δ_i^+ and Δ_i^-.

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There exists a f satisfying

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- All G_i can be completed to a Δ_i-regular directed graph without deceasing the girth.

$$\mathrm{dla}(G) \leq 2\Delta_0 + \sum_{i=1}^{p-1}(\Delta_i + 1) \leq d + d/p + p + C\sqrt{dp\log d}.$$

Now choose $p \sim d^{1/2}$.

Chernoff Inequality

Suppose $X = \sum_{i=1}^{n} X_i$, where X_i are independent 0-1 random variables. Then we have

$$Pr(X < E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2E(X)}}$$
$$Pr(X > E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}}$$