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- Dependence graph: \( d_{SS} \leq 3n \), \( d_{ST} \leq 3\binom{n}{k-2} \), \( d_{TS} \leq \binom{k}{2}n \), and \( d_{TT} \leq \binom{k}{2}\binom{n}{k-2} \).
Proof

By LLL, we only require

\[ p^3 \leq x(1 - x)^3n(1 - y)^3\binom{n}{k-2} \]

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\]

We can choose \( p = c_1n^{-1/2} \), \( k = c_2n^{1/2}\log n \), \( x = c_3n^{-3/2} \), and \( y = c_4/\binom{n}{k} \).
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This gives \( R(3, k) > c_5 k^2 / \log^2 k. \) □
Best bounds for $R(r, k)$ (for fixed $r$ and $k$ large),

$$c \left( \frac{k}{\log k} \right)^{(r+1)/2} < R(r, k) < (1 + o(1)) \frac{k^{r-1}}{\log^{r-2} k}.$$

**Erdős conjecture $\$250$: Prove**

$$R(4, k) > c' \frac{k^3}{\log^c k}$$

for some constants $c', c > 0$. 
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**Erdős conjecture $250$:** Prove

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The best lower bound is using LLL; $R(4, k) > c' \frac{k^{2.5}}{\log^{2.5} k}$.
Directed cycles

- \( D = (V, E) \): a simple directed graph.
- \( \delta \): minimum outdegree.
- \( \Delta \): maximum indegree.
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**Theorem [Alon and Linial (1989)]**

If $e(\Delta \delta + 1)(1 - 1/k)^\delta < 1$, then $D$ contains a (directed, simple) cycle of length $0 \mod k$. 

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**Proof:** First we can assume every out-degree is $\delta$ by deleting some edges if necessary. Consider $f : V \to \mathbb{Z}_k$. Bad event $A_v$: no $u \in \Gamma^+(v)$ with $f(u) = f(v) + 1$.

$$\Pr(A_v) = (1 - 1/k)^\delta.$$

Each event depends on at most $\delta \Delta$ others. Apply LLL. □
Linear Arboricity

- Linear forest: disjoint union of paths.
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The Linear Arboricity Conjecture (Akiyama, Exoo, Harary [1981]): For every \( d \)-regular graph \( G \),

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If the conjecture is true, then it is tight.

$$\text{la}(G) \geq \frac{nd}{2(n - 1)} > \frac{d}{2}.$$
Directed graphs

- $G = (V, E)$: a directed graph.
- $G$ is $d$-regular if $d^+(v) = d^-(v) = d$ for any vertex $v$.
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DLA conjecture for $d$ implies LA conjecture for $2d$. 
Proposition: Let $H = (V, E)$ be a graph with maximum degree $d$, and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of $V$. If $|V_i| \geq 2ed$, then there is an independent set of vertices $W$ that contains a vertex from each $V_i$. 
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**Proposition:** Let $H = (V, E)$ be a graph with maximum degree $d$, and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of $V$. If $|V_i| \geq 2ed$, then there is an independent set of vertices $W$ that contains a vertex from each $V_i$.

**Proof:** WLOG, we assume

\[
|V_1| = |V_2| = \cdots = |V_r| = \lceil 2ed \rceil = g.
\]

Pick from each $V_i$ a vertex randomly and independently. Let $W$ be the random set of the vertices picked. For each edge $f$, let $A_f$ be the event that both ends in $W$. The maximum degree in the dependence graph is at most $2gd - 1$. We have $e \cdot 2gd \cdot \frac{1}{g^2} = \frac{2ed}{g} < 1$. Apply LLL. \qed
With large girth

The directed girth of a digraph is the minimum length of a directed cycle in it.
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**Theorem** Let $G = (U, F)$ be a $d$-regular digraph with directed girth $g \geq 8ed$. Then

$$\text{dla}(G) = d + 1.$$
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**Theorem** Let $G = (U, F)$ be a $d$-regular digraph with directed girth $g \geq 8ed$. Then

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**Proof:** Using Hall’s matching theorem, we can partition $F$ into $d$ pairwise disjoint 1-regular spanning subgraphs $F_1, \ldots, F_d$ of $G$. 

**With large girth**
Each $F_i$ is a union of vertex disjoint directed cycles. Let $V_1, \ldots, V_r$ are the sets of edges of all cycles. Then

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Apply the proposition to the line-graph $H$ of $G$. Note $H$ is $4d - 2$-regular.
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Apply the proposition to the line-graph $H$ of $G$. Note $H$ is $4d - 2$-regular.

There exists an independent set $M_1$ of $H$. Now $M_1, F_1 \setminus M_1, \ldots, F_d \setminus M_1$ forms $d + 1$ linear directed forests. □
Theorem [Alon 1988] There is an absolute constant $c > 0$ such that for every $d$-regular directed graph $G$

$$\text{dla}(G) \leq d + cd^{3/4} \log^{1/2} d.$$
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The error terms can be improved to $cd^{2/3} \log^{1/3} d$. 

Pick a prime $p$. Color each vertex randomly and uniformly into $p$ colors. I.e., consider a random map

$$f : V \rightarrow \mathbb{Z}_p.$$
Proof

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$$f : V \rightarrow \mathbb{Z}_p.$$  

Define for $i \in \mathbb{Z}_p$,

$$E_i = \{(u, v) \in E : f(v) = f(u) + i\}.$$  

Let $G_i = (V, E_i)$ and

- $\Delta_i^+$: the maximum out-degree of $G_i$.
- $\Delta_i^-$: the maximum in-degree of $G_i$.
- $\Delta_i$: the maximum of $\Delta_i^+$ and $\Delta_i^-$. 
There exists a $f$ satisfying

- All $G_i$ are almost regular: $\Delta_i \leq \frac{d}{p} + 3\sqrt{d/p}\sqrt{\log d}$. 
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$$\text{dla}(G) \leq 2\Delta_0 + \sum_{i=1}^{p-1} (\Delta_i + 1) \leq d + d/p + p + C\sqrt{dp\log d}.$$ 

Now choose $p \sim d^{1/2}$. 
Suppose $X = \sum_{i=1}^{n} X_i$, where $X_i$ are independent 0-1 random variables. Then we have

$$Pr(X < E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2E(X)}}$$

$$Pr(X > E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}}$$