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- $H$ is called balanced if for any subgraph $H'$,
  \[ \rho(H') \leq \rho(H). \]
- $H$ is called strictly balanced if for any proper subgraph $H'$,
  \[ \rho(H') < \rho(H). \]
**Theorem:** Let $H$ be a balanced graph with $v$ vertices and $e$ edges. Let $A(G)$ be the event that $H$ is a subgraph (not necessarily induced) of $G$. Then $p = n^{-v/e}$ is the threshold function for $A$. 
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If $p \ll n^{-v/e}$, then $E(X) = o(1)$; $X = 0$ almost surely.

If $p \gg n^{-v/e}$, then $E(X) \to \infty$. We have

$$\Delta^* = O\left(\sum_{i=2}^{v} n^{v-i} p^{e-(ie/v)}\right) = o(E(X)).$$
Two other results

**Theorem:** Let $H$ be a strictly balanced graph with $v$ vertices and $e$ edges and $\alpha$ automorphisms. Let $X$ be the copies of $H$ in $G(n, p)$. Assume $p \gg n^{-v/s}$. Then almost always

$$X \sim \frac{n^v p^e}{\alpha}.$$
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**Theorem:** Let $H$ be any fixed graph. For every subgraph $H'$ of $H$ (including $H$ itself) let $X_{H'}$ denote the number of copies of $H'$ in $G(n, p)$. Assume $p$ is such that $E(X_{H'}) \to \infty$ for every $H'$. Then almost surely

$$X_H \sim E(X_H).$$
Clique number of $G(n, 1/2)$

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**Proof:** For each $k$-set $S$, let $X_S$ be the indicator random variable that $S$ is a clique and $X = \sum_{|S|=k} X_S$.

$$E(X) = \binom{n}{k} 2^{\binom{k}{2}} = f(k).$$
\[ \Delta^* = \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{(i)} - \binom{k}{2}. \]

\[ \frac{\Delta^*}{E(|X|)} = \sum_{i=2}^{k-1} g(i), \]

where \( g(i) = \frac{\binom{k}{i} \binom{n-k}{k-i}}{\binom{n}{k}} 2^{(i)} \).
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where \( g(i) = \frac{\binom{k}{i} \binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}. \) Then

\[ g(i) \leq \max\{g(2), g(k - 1)\} = o(n^{-1}). \]

Thus, \( \Delta^* = o(E(X)) \).

\[ \square \]
Remark

\[ \frac{f(k+1)}{f(k)} = \frac{n-k}{k+1} 2^{-k}. \]

For \( k \sim 2 \log_2 n \), then

\[ \frac{f(k+1)}{f(k)} = n^{-1+o(1)}. \]
Remark

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Let \( k_0 \) be the value with \( f(k_0) \geq 1 > f(k_0 + 1) \). For most of \( n \), \( f(k) \) will jump from very large to very small. With high probability, \( \omega(G) = k_0 \).
A set \( x_1, \ldots, x_k \) of positive integers is said to have distinct sums if all sums

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\sum_{i \in S} x_i, \quad S \subset \{1, \ldots, k\}
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Distinct sum

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are distinct.

- Let $f(k)$ be the smallest $k$ for which there is a set

$$\{x_1, x_2, \ldots, x_k\} \subset \{1, \ldots, n\}$$

with distinct set.

It is clear $f(n) \geq 1 + \lfloor \log_2 n \rfloor$. 
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**Theorem:**

\[ f(n) < \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1). \]
Lovász Local Lemma

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- $A_1, A_2, \ldots, A_n$: $n$ events in an arbitrary probability spaces.
- A dependency digraph $D = (V, E)$: if for each $A_i$, $A_i$ is mutually independent to all the events $\{A_j: A_i A_j \notin E\}$.

**Lovász Local Lemma, general case:** If there are real number $x_1, \ldots, x_n$ such that $0 \leq x_i < 1$ and

$$\Pr(A_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j)$$

for all $1 \leq i \leq n$. Then

$$\Pr\left(\bigwedge_{i=1}^{n} \bar{A}_i\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$
Lovász Local Lemma, symmetric case: Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event $A_i$ is mutually independent of a set of all the other event $A_j$ but at most $d$, and that $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d + 1) < 1$, then $\Pr(\bigwedge_{i=1}^{n} \bar{A}_i) > 0$. 
**Theorem:** Let $H = (V, E)$ be a hypergraph in which every edge has at least $k$ elements, and suppose that each edge of $H$ intersects at most $d$ other edges. If $e(d + 1) \leq 2^{k-1}$, then $H$ has property $B$. 
**Theorem:** Let $H = (V, E)$ be a hypergraph in which every edge has at least $k$ elements, and suppose that each edge of $H$ intersects at most $d$ other edges. If $e(d + 1) \leq 2^{k-1}$, then $H$ has property $B$.

**Proof:** Color each vertex in two colors randomly and independently. For each edge $f \in E$, let $A_f$ be the event that $f$ is monochromatic. Then

$$\Pr(A_f) = 2^{1-|f|} \leq 2^{1-k}.$$ 

$A_f$ is independent to all event but at most $d$. Apply LLL. □
Let $c: \mathbb{R} \rightarrow \{1, 2, \ldots, k\}$ be a $k$-coloring of $\mathbb{R}$. A set $T \subset \mathbb{R}$ is multicolored if $c(T) = \{1, 2, \ldots, k\}$.
$k$-coloring of $\mathbb{R}$

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**Theorem:** Let $m$ and $k$ be two positive integers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$  

Then, for any set $S$ of $m$ real numbers there is a $k$-coloring so that each translation $x + S$ (for $x \in \mathbb{R}$) is multicolored.
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Then, for any set $S$ of $m$ real numbers there is a $k$-coloring so that each translation $x + S$ (for $x \in \mathbb{R}$) is multicolored.

The condition is satisfied if $m > (3 + o(1))k \log k$. 


First we use LLL to prove “For any finite set $X \subset \mathbb{R}$, there is a $k$-coloring so that $x + S$ (for all $x \in X$) is multi-colored.”
Proof

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Let $Y = \bigcup_{x \in X} (x + S)$. Color numbers in $Y$ in $k$-colors randomly and independently. Let $A_x$ be the event that $x + S$ is not multi-colored.

$$\Pr(A_x) \leq k \left(1 - \frac{1}{k}\right)^{m-1}.$$
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$$\Pr(A_x) \leq k\left(1 - \frac{1}{k}\right)^{m-1}.$$ 

$A_x$ depends on $A_y$ if $(x + S) \cap (y + S) \neq \emptyset$. Equivalently, $y - x \in S - S$. There are at most $m(m-1)$ such events.

$$d \leq m(m - 1).$$
Applying LLL, we get

\[ \Pr(\bigwedge_{x \in X} \tilde{A}_x) > 0. \]

Then by Tikhonov’s theorem, \([k]^\mathbb{R}\) is compact. For any \(x \in \mathbb{R}\), let

\[ C_x = \{ c \in [k]^\mathbb{R} : x + S \text{ is multi-colored} \}. \]
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Now \(C_x\) is a closed set and \(\bigcap_{x \in X} C_x \neq \emptyset\) for any finite \(X\). Then \(\bigcap_{x \in \mathbb{R}} C_x \neq \emptyset\). \(\square\)
 Ramsey numbers

Theorem (Spencer, 1975)

\[ R(k, k) \geq (1 + o(1)) \frac{\sqrt{2}}{e} k 2^{k/2}. \]
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\[ R(3, k) \geq \frac{ck^2}{\log k} . \]
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\[ R(3, k) \geq \frac{c k^2}{\log k}. \]

Best bounds for \( R(r, k) \) (for fixed \( r \) and \( k \) large),

\[ c \left( \frac{k}{\log k} \right)^{(r+1)/2} < R(r, k) < (1 + o(1)) \frac{k^{r-1}}{\log^{r-2} k}. \]