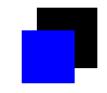


Probabilistic Methods in Combinatorics Lecture 6

Linyuan Lu University of South Carolina

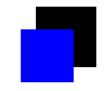


Mathematical Sciences Center at Tsinghua University November 16, 2011 – December 30, 2011



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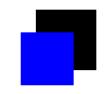




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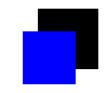




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- H is called **balanced** of for any subgraph H',

 $\rho(H') \le \rho(H).$





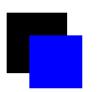
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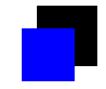
 $\rho(H') \le \rho(H).$

H is called **strictly balanced** of for any proper subgraph H',

$$\rho(H') < \rho(H).$$



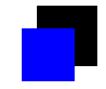




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Proof: Write $X = \sum_{S} X_{S}$. Then $E(X) = {n \choose v} p^{e}$.



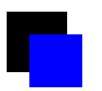


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Proof: Write $X = \sum_{S} X_{S}$. Then $E(X) = {n \choose v} p^{e}$. If $p \ll n^{-v/e}$, then E(X) = o(1); X = 0 almost surely. If $p \gg n^{-v/e}$, then $E(X) \to \infty$. We have

$$\Delta^* = O(\sum_{i=2}^{v} n^{v-i} p^{e-(ie/v)}) = o(\mathbf{E}(X)).$$



Two other results

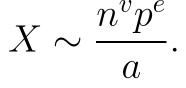
Theorem: Let H be a strictly balanced graph with v vertices and e edges and a automorphisms. Let X be the copies of H in G(n, p). Assume $p \gg n^{-v/s}$. Then almost always

$$X \sim \frac{n^v p^e}{a}.$$



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Theorem: Let H be a strictly balanced graph with v vertices and e edges and a automorphisms. Let X be the copies of H in G(n, p). Assume $p \gg n^{-v/s}$. Then almost always



Theorem: Let H be any fixed graph. For every subgraph H' of H (including H itself) let $X_{H'}$ denote the number of copies of H' in G(n, p). Assume p is such that $E(X_{H'}) \rightarrow \infty$ for every H'. Then almost surely

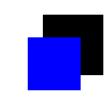
 $X_H \sim \mathrm{E}(X_H).$





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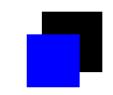
Proof: For each k-set S, let X_S be the indicator random variable that S is a clique and $X = \sum_{|S|=k} X_S$.

$$\mathcal{E}(X) = \binom{n}{k} 2^{\binom{k}{2}} = f(k).$$



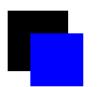


Continue

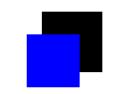


$$\begin{split} \Delta^* &= \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{\binom{i}{2} - \binom{k}{2}} \\ &\frac{\Delta^*}{E(|X|)} = \sum_{i=2}^{k-1} g(i), \end{split}$$
 where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}.$





Continue



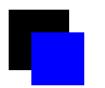
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where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$. Then

$$g(i) \le \max\{g(2), g(k-1)\} = o(n^{-1}).$$

Thus, $\Delta^* = o(\mathcal{E}(X)).$









$$\frac{f(k+1)}{f(k)} = \frac{n-k}{k+1}2^{-k}.$$

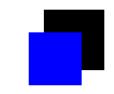
For $k \sim 2 \log_2 n$, then

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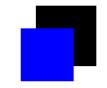
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Let k_0 be the value with $f(k_0) \ge 1 > f(k_0 + 1)$. For most of n, f(k) will jump from very large to ver small. With high probability, $\omega(G) = k_0$.





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$$\sum_{i \in S} x_i, \quad S \subset \{1, \dots, k\}$$

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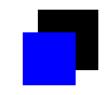
Let f(k) be the smallest k for which there is a set

$$\{x_1, x_2, \ldots, x_k\} \subset \{1, \ldots, n\}$$

with distinct set.

It is clear $f(n) \ge 1 + \lfloor \log_2 n \rfloor$.

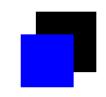




Erdős offered \$300 for a proof or disproof that

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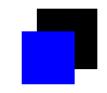
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Theorem:

$$f(n) < \log_2 n + \frac{1}{2}\log_2 \log_2 n + O(1).$$



Lovász Local Lemma

• A_1, A_2, \ldots, A_n : *n* events in an arbitrary probability spaces.



Lovász Local Lemma

- A_1, A_2, \ldots, A_n : *n* events in an arbitrary probability spaces.
- A dependency digraph D = (V, E): if for each A_i , A_i is mutually independent to all the events $\{A_j : A_i A_j \notin E\}$.

Lovász Local Lemma, general case: If there are real number x_1, \ldots, x_n such that $0 \le x_i < 1$ and $\Pr(A_i) \le x_i \prod_{(i,j) \in E} (1-x_j)$ for all $1 \le i \le n$. Then

$$\Pr\left(\wedge_{i=1}^{n}\bar{A}_{i}\right) \geq \prod_{i=1}^{n}(1-x_{i}) > 0.$$



Symmetric Case

Lovász Local Lemma, symmetric case: Let A_1, A_2, \ldots, A_n be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other event A_j but at most d, and that $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If ep(d+1) < 1, then $\Pr(\wedge_{i=1}^n \bar{A}_i) > 0$.





Property B

Theorem: Let H = (V, E) be a hypergraph in which every edge has at least k elements, and suppose that each edge of H intersects at most d other edges. If $e(d+1) \leq 2^{k-1}$, then H has property B.





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Proof: Color each vertex in two colors randomly and independently. For each edge $f \in E$, let A_f be the event that f is monochromatic. Then

$$\Pr(A_f) = 2^{1-|f|} \le 2^{1-k}.$$

 A_f is independent to all event but at most d. Aplly LLL.





$k\text{-coloring of }\mathbb{R}$

Let $c \colon \mathbb{R} \to \{1, 2, \dots, k\}$ be a k-coloring of \mathbb{R} . A set $T \subset \mathbb{R}$ is **multicolored** if $c(T) = \{1, 2, \dots, k\}$.





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Theorem: Let m and k be two positive intergers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \le 1.$$

Then, for any set S of m real numbers there is a k-coloring so that each translantion x + S (for $x \in \mathbb{R}$) is multicolored.





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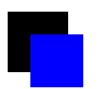
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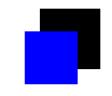
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Then, for any set S of m real numbers there is a k-coloring so that each translantion x + S (for $x \in \mathbb{R}$) is multicolored. The condition is satisfied if $m > (3 + o(1))k \log k$.



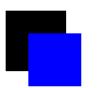


Proof



First we use LLL to prove "For any finite set $X \subset \mathbb{R}$, there is a k-coloring so that x + S (for all $x \in X$) is multi-colored."





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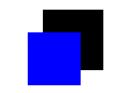
 A_x depends on A_y if $(x + S) \cap (y + S) \neq \emptyset$. Equivalently, $y - x \in S - S$. There are at most m(m - 1) such events.

 $d \le m(m-1).$









Apllying LLL, we get

$$\Pr(\wedge_{x\in X}\bar{A}_x)>0.$$

Then by Tikhonov's theorem, $[k]^{\mathbb{R}}$ is compact. For any $x\in\mathbb{R},$ let

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Now C_x is a closed set and $\bigcap_{x \in X} C_x \neq \emptyset$ for any finite X. Then $\bigcap_{x \in \mathbb{R}} C_x \neq \emptyset$.



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Theorem (Spencer, 1975)

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Best bounds for R(r, k) (for fixed r and k large),

$$c\left(\frac{k}{\log k}\right)^{(r+1)/2} < R(r,k) < (1+o(1))\frac{k^{r-1}}{\log^{r-2}k}.$$

