Probabilistic Methods in Combinatorics
Lecture 6

Linyuan Lu

University of South Carolina

Mathematical Sciences Center at Tsinghua University

November 16, 2011 – December 30, 2011

Balance graphs

- H has v vertices and e edges.

Balance graphs

- H has v vertices and e edges.
- $\rho(H) = e/v$.

Balance graphs

- H has v vertices and e edges.
- $\rho(H) = e/v$.
- H is called **balanced** of for any subgraph H',

$$\rho(H') \leq \rho(H).$$

Balance graphs

- H has v vertices and e edges.
- $\rho(H) = e/v$.
- H is called **balanced** of for any subgraph H',

$$\rho(H') \leq \rho(H).$$

- H is called **strictly balanced** of for any proper subgraph H',

$$\rho(H') < \rho(H).$$

Results

Theorem: *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G. Then $p = n^{-v/e}$ is the threshod function for A.*

Results

Theorem: *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G. Then $p = n^{-v/e}$ is the threshod function for A.*

If H is not balanced then $p = n^{-v/e}$ is the threshod function for A.

Results

Theorem: *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G. Then $p = n^{-v/e}$ is the threshod function for A.*

If H is not balanced then $p = n^{-v/e}$ is the threshod function for A.

Proof: Write $X = \sum_S X_S$. Then $\mathrm{E}(X) = \binom{n}{v} p^e$.

Results

Theorem: *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G. Then $p = n^{-v/e}$ is the threshod function for A.*

If H is not balanced then $p = n^{-v/e}$ is the threshod function for A.

Proof: Write $X = \sum_S X_S$. Then $\mathrm{E}(X) = \binom{n}{v} p^e$.

If $p \ll n^{-v/e}$, then $\mathrm{E}(X) = o(1)$; $X = 0$ almost surely.

Results

Theorem: *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G. Then $p = n^{-v/e}$ is the threshod function for A.*

If H is not balanced then $p = n^{-v/e}$ is the threshod function for A.

Proof: Write $X = \sum_S X_S$. Then $\mathrm{E}(X) = \binom{n}{v} p^e$.

If $p \ll n^{-v/e}$, then $\mathrm{E}(X) = o(1)$; $X = 0$ almost surely.

If $p \gg n^{-v/e}$, then $\mathrm{E}(X) \to \infty$. We have

$$\Delta^* = O(\sum_{i=2}^{v} n^{v-i} p^{e-(ie/v)}) = o(\mathrm{E}(X)).$$

Two other results

Theorem: *Let H be a strictly balanced graph with v vertices and e edges and a automorphisms. Let X be the copies of H in $G(n,p)$. Assume $p \gg n^{-v/s}$. Then almost always*

$$X \sim \frac{n^v p^e}{a}.$$

Two other results

Theorem: *Let H be a strictly balanced graph with v vertices and e edges and a automorphisms. Let X be the copies of H in $G(n,p)$. Assume $p \gg n^{-v/s}$. Then almost always*

$$X \sim \frac{n^v p^e}{a}.$$

Theorem: *Let H be any fixed graph. For every subgraph H' of H (including H itself) let $X_{H'}$ denote the number of copies of H' in $G(n,p)$. Assume p is such that $E(X_{H'}) \to \infty$ for every H'. Then almost surely*

$$X_H \sim \mathrm{E}(X_H).$$

Clique number of $G(n, 1/2)$

- $\omega(G)$: the clique number of G.

Clique number of $G(n, 1/2)$

- $\omega(G)$: the clique number of G.
- $f(k) = \binom{n}{k} 2^{\binom{k}{2}}$: the expected number of k-cliques.

Clique number of $G(n, 1/2)$

- $\omega(G)$: the clique number of G.
- $f(k) = \binom{n}{k} 2^{\binom{k}{2}}$: the expected number of k-cliques.

Theorem: Let $k = k(n)$ satisfying $k \sim 2\log_2 n$ and $f(k) \to \infty$. Then almost surely $\omega(G) \geq k$.

Clique number of $G(n, 1/2)$

- $\omega(G)$: the clique number of G.
- $f(k) = \binom{n}{k} 2^{\binom{k}{2}}$: the expected number of k-cliques.

Theorem: Let $k = k(n)$ satisfying $k \sim 2\log_2 n$ and $f(k) \to \infty$. Then almost surely $\omega(G) \geq k$.

Proof: For each k-set S, let X_S be the indicator random variable that S is a clique and $X = \sum_{|S|=k} X_S$.

$$\mathrm{E}(X) = \binom{n}{k} 2^{\binom{k}{2}} = f(k).$$

Continue

$$\Delta^* = \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{\binom{i}{2} - \binom{k}{2}}.$$

$$\frac{\Delta^*}{E(|X|)} = \sum_{i=2}^{k-1} g(i),$$

where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$.

Continue

$$\Delta^* = \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{\binom{i}{2} - \binom{k}{2}}.$$

$$\frac{\Delta^*}{E(|X|)} = \sum_{i=2}^{k-1} g(i),$$

where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$. Then

$$g(i) \leq \max\{g(2), g(k-1)\} = o(n^{-1}).$$

Thus, $\Delta^* = o(\mathrm{E}(X))$. □

Remark

$$\frac{f(k+1)}{f(k)} = \frac{n-k}{k+1} 2^{-k}.$$

For $k \sim 2\log_2 n$, then

$$\frac{f(k+1)}{f(k)} = n^{-1+o(1)}.$$

Remark

$$\frac{f(k+1)}{f(k)} = \frac{n-k}{k+1} 2^{-k}.$$

For $k \sim 2\log_2 n$, then

$$\frac{f(k+1)}{f(k)} = n^{-1+o(1)}.$$

Let k_0 be the value with $f(k_0) \geq 1 > f(k_0+1)$. For most of n, $f(k)$ will jump from very large to ver small. With high probabilty, $\omega(G) = k_0$.

Distinct sum

- A set $x_1, \ldots, x_k$ of positive integers is said to have distinct sums if all sums

$$\sum_{i \in S} x_i, \quad S \subset \{1, \ldots, k\}$$

are distinct.

Distinct sum

- A set $x_1, \ldots, x_k$ of positive integers is said to have distinct sums if all sums

$$\sum_{i \in S} x_i, \quad S \subset \{1, \ldots, k\}$$

  are distinct.

- Let $f(k)$ be the smallest k for which there is a set

$$\{x_1, x_2, \ldots, x_k\} \subset \{1, \ldots, n\}$$

  with distinct set.

It is clear $f(n) \geq 1 + \lfloor \log_2 n \rfloor$.

Distinct sum

Erdős offered \$300 for a proof or disproof that

$$f(n) \leq \log_2 n + O(1).$$

Distinct sum

Erdős offered $300 for a proof or disproof that

$$f(n) \le \log_2 n + O(1).$$

An easy upper bound

$$f(n) < \log_2 n + \log_2 \log_2 n + O(1).$$

Distinct sum

Erdős offered \$300 for a proof or disproof that

$$f(n) \leq \log_2 n + O(1).$$

An easy upper bound

$$f(n) < \log_2 n + \log_2 \log_2 n + O(1).$$

Theorem:

$$f(n) < \log_2 n + \frac{1}{2}\log_2 \log_2 n + O(1).$$

Lovász Local Lemma

- $A_1, A_2, \ldots, A_n$: n events in an arbitrary probability spaces.

Lovász Local Lemma

- $A_1, A_2, \ldots, A_n$: n events in an arbitrary probability spaces.
- A dependency digraph $D = (V, E)$: if for each A_i, A_i is mutually independent to all the events $\{A_j : A_iA_j \notin E\}$.

Lovász Local Lemma, general case: If there are real number $x_1, \ldots, x_n$ such that $0 \leq x_i < 1$ and $\Pr(A_i) \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ for all $1 \leq i \leq n$. Then

$$\Pr\left(\wedge_{i=1}^{n} \bar{A}_i\right) \geq \prod_{i=1}^{n}(1 - x_i) > 0.$$

Symmetric Case

Lovász Local Lemma, symmetric case: Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other event A_j but at most d, and that $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) < 1$, then $\Pr(\wedge_{i=1}^{n} \bar{A}_i) > 0$.

Property B

Theorem: Let $H = (V, E)$ be a hypergraph in which every edge has at least k elements, and suppose that each edge of H intersects at most d other edges. If $e(d+1) \leq 2^{k-1}$, then H has property B.

Property B

Theorem: Let $H = (V, E)$ be a hypergraph in which every edge has at least k elements, and suppose that each edge of H intersects at most d other edges. If $e(d+1) \leq 2^{k-1}$, then H has property B.

Proof: Color each vertex in two colors randomly and independently. For each edge $f \in E$, let A_f be the event that f is monochromatic. Then

$$\Pr(A_f) = 2^{1-|f|} \leq 2^{1-k}.$$

A_f is independent to all event but at most d. Aplly LLL. $\square$

k-coloring of $\mathbb{R}$

Let $c\colon \mathbb{R} \to \{1, 2, \ldots, k\}$ be a k-coloring of $\mathbb{R}$. A set $T \subset \mathbb{R}$ is **multicolored** if $c(T) = \{1, 2, \ldots, k\}$.

k-coloring of $\mathbb{R}$

Let $c\colon \mathbb{R} \to \{1, 2, \ldots, k\}$ be a k-coloring of $\mathbb{R}$. A set $T \subset \mathbb{R}$ is **multicolored** if $c(T) = \{1, 2, \ldots, k\}$.

Theorem: Let m and k be two positive intergers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$

Then, for any set S of m real numbers there is a k-coloring so that each translantion $x + S$ (for $x \in \mathbb{R}$) is multicolored.

k-coloring of $\mathbb{R}$

Let $c\colon \mathbb{R} \to \{1, 2, \ldots, k\}$ be a k-coloring of $\mathbb{R}$. A set $T \subset \mathbb{R}$ is **multicolored** if $c(T) = \{1, 2, \ldots, k\}$.

Theorem: Let m and k be two positive intergers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$

Then, for any set S of m real numbers there is a k-coloring so that each translantion $x + S$ (for $x \in \mathbb{R}$) is multicolored.

The condition is satisfied if $m > (3 + o(1))k \log k$.

Proof

First we use LLL to prove "For any finite set $X \subset \mathbb{R}$, there is a k-coloring so that $x + S$ (for all $x \in X$) is multi-colored."

Proof

First we use LLL to prove "For any finite set $X \subset \mathbb{R}$, there is a k-coloring so that $x + S$ (for all $x \in X$) is multi-colored."

Let $Y = \cup_{x \in X}(x + S)$. Color numbers in Y in k-colors randomly and independently. Let A_x be the event that $x + S$ is not multi-colored.

$$\Pr(A_x) \leq k(1 - \frac{1}{k})^{m-1}.$$

Proof

First we use LLL to prove "For any finite set $X \subset \mathbb{R}$, there is a k-coloring so that $x + S$ (for all $x \in X$) is multi-colored."

Let $Y = \cup_{x \in X}(x + S)$. Color numbers in Y in k-colors randomly and independently. Let A_x be the event that $x + S$ is not multi-colored.

$$\Pr(A_x) \leq k(1 - \frac{1}{k})^{m-1}.$$

A_x depends on A_y if $(x + S) \cap (y + S) \neq \emptyset$. Equivalently, $y - x \in S - S$. There are at most $m(m-1)$ such events.

$$d \leq m(m-1).$$

continue

Apllying LLL, we get

$$\Pr(\wedge_{x \in X} \bar{A}_x) > 0.$$

Then by Tikhonov's theorem, $[k]^{\mathbb{R}}$ is compact. For any $x \in \mathbb{R}$, let

$$C_x = \{c \in [k]^{\mathbb{R}} \colon x + S \text{ is multi-colored}\}.$$

continue

Apllying LLL, we get

$$\Pr(\wedge_{x \in X} \bar{A}_x) > 0.$$

Then by Tikhonov's theorem, $[k]^{\mathbb{R}}$ is compact. For any $x \in \mathbb{R}$, let

$$C_x = \{c \in [k]^{\mathbb{R}} \colon x + S \text{ is multi-colored}\}.$$

Now C_x is a closed set and $\cap_{x \in X} C_x \neq \emptyset$ for any finite X. Then $\cap_{x \in \mathbb{R}} C_x \neq \emptyset$. □

Ramsey numbers

Theorem (Spencer, 1975)

$$R(k,k) \geq (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}.$$

Ramsey numbers

Theorem (Spencer, 1975)

$$R(k,k) \geq (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}.$$

Theorem (Spencer, 1975)

$$R(3,k) \geq \frac{ck^2}{\log k}.$$

Ramsey numbers

Theorem (Spencer, 1975)

$$R(k,k) \geq (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}.$$

Theorem (Spencer, 1975)

$$R(3,k) \geq \frac{ck^2}{\log k}.$$

Best bounds for $R(r,k)$ (for fixed r and k large),

$$c\left(\frac{k}{\log k}\right)^{(r+1)/2} < R(r,k) < (1+o(1))\frac{k^{r-1}}{\log^{r-2} k}.$$