Probabilistic Methods in Combinatorics
Lecture 5

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Recoloring

Property B problem revisited:
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Theorem [Radhakrishnan-Srinivasan 2000]:

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Proof: For a fixed r-uniform hypergraph $H = (V, E)$ with $|E| = k2^{r-1}$. Let $p \in [0, 1]$ satisfying $k(1-p)^r + k^2 p < 1$.

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Here is a two-round coloring process.

- **First round:** Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected k monochromatic edges. Let U be the set of vertices in some monochromatic edges.

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- **Second round:** Consider vertices in U sequentially in the (random) order of V. A vertex $u \in U$ is **still dangerous** if there is some monochromatic edge in the first coloring and for which no vertex has yet changed color.
  - If u is not dangerous, do nothing.
  - If u is still dangerous; with probability p, flip the color of u.

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$$2 \sum_{e \in E(H)} \Pr(A_e) = k(1-p)^r.$$

Estimating $\Pr(C_e)$

For two edge e, f, we say e **blames** f if

- $e \cap f = \{v\}$ for some v.
- In the first coloring f was blue and in the final coloring e was red.
- v was the last vertex of e that changed color from blue to red.
- When v changed its color f was still entire blue.

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Call this event B_{ef}. Then

$$\sum_e \Pr(C_e) \leq \sum_{e \neq f} \Pr(B_{ef}).$$

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We have

$$\begin{aligned} \Pr(B_{ef}) &\leq 2^{1-2r} p \mathrm{E}[(1+p)^i (1-p)^j]. \\ &\leq 2^{1-2r} p. \end{aligned}$$

Estimating k

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The function $f(p) = ke^{-pr} + k^2p$ reaches its minimum at $p = \frac{\ln(r/k)}{r}$. The minimum value is less than 1 if

$$k < (1+o(1))\sqrt{\frac{2r}{\ln r}}.$$

$\square$

Continuous time

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\end{aligned}$$

The rest of proof is the same.

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If $X = \sum_{i=1}^{n} X_i$, then

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If $X_1, \ldots, X_n$ are mutually independent, then $\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$.

Chebyshev's Inequality

- $\mathrm{E}(X) = \mu$,
- $\mathrm{Var}(X) = \sigma^2$.

Theorem [Chebyshev's Inequality]: For any positive λ,

$$\Pr(|X - \mu| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}.$$

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Theorem [Chebyshev's Inequality]: For any positive λ,

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Proof:

$$\begin{aligned} \sigma^2 &= \mathrm{Var}(X) \\ &= \mathrm{E}((X-\mu)^2) \\ &\geq \lambda^2\sigma^2 \Pr(|X - \mu| \geq \lambda\sigma). \end{aligned}$$

Number theory

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Theorem [Turán (1934)]: Let $\omega(n) \to \infty$ arbitrarily slowly. Then the number of x in $[n] := \{1, 2, \ldots, n\}$ such that

$$|\nu(x) - \ln\ln x| > \omega(n)\sqrt{\ln\ln n}.$$

is $o(n)$.

Proof

Let x be randomly chosen from $[n]$. For p prime set

$$X_p = \begin{cases} 1 & \text{if } p \mid x, \\ 0 & \text{otherwise.} \end{cases}$$

Set $M = n^{1/10}$ and $X = \sum_{p \leq M} X_p$. Then

$$\nu(x) - 10 \leq X(x) \leq \nu(x).$$

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\mathrm{Cov}(X_p, X_q) &= \mathrm{E}(X_p X_q) - \mathrm{E}(X_p)\mathrm{E}(X_q) \\
&= \frac{\lfloor n/pq \rfloor}{n} - \frac{\lfloor n/p \rfloor}{n}\frac{\lfloor n/q \rfloor}{n} \\
&\le \frac{1}{pq} - \left(\frac{1}{p} - \frac{1}{n}\right)\left(\frac{1}{q} - \frac{1}{n}\right) \\
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□

Theorem [Erdős-Kac (1940):] For any fixed λ, we have

$$\lim_{n\to\infty} \frac{1}{n}\left|\{x \colon 1 \le x \le n, \nu(x) \ge \ln\ln n + \lambda\sqrt{\ln\ln n}\}\right|$$
$$= \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

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- Write $X_i \sim X_j$ if X_i and X_j are independent. Let $\Delta = \sum_{i \sim j} \Pr(A_i \wedge A_j)$. If $E(X) \to \infty$ and $\Delta = o(E(X)^2)$, then $X > 0$ almost always.

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- Let $\Delta^* = \max_i \sum_{j \sim i} \Pr(A_j | A_i)$. If $E(X) \to \infty$ and $\Delta^* = o(E(X))$, then $X > 0$ almost always.

Erdős-Rényi model $G(n, p)$

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A property of graphs is a family of graphs closed under isomorphic.

A function $r(n)$ is called a threshold function for some property P if

- If $p \ll r(n)$, then $G(n, p)$ does not satisfy P almost always.
- If $p \gg r(n)$, then $G(n, p)$ satisfy P almost always.

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If $p \ll n^{-2/3}$ then $\mathrm{E}(X) = o(1)$ and so $X = 0$ almost surely.

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$S \sim T$ if $|S \cap T| \geq 2$. Thus,

$$\Delta^* = O(n^2p^5) + O(np^3) = o(n^4p^6) = o(\mathrm{E}(X)).$$

Hence $X > 0$ almost surely. $\square$