

Probabilistic Methods in Combinatorics Lecture 5

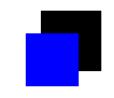
Linyuan Lu University of South Carolina



Mathematical Sciences Center at Tsinghua University November 16, 2011 – December 30, 2011



Recoloring



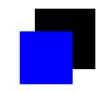
Property B problem revisited:

Let m(r) denote the minimum possible number of edges of an r-uniform hypergraph that does not have property B.





Recoloring



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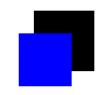
Theorem [Radhakrishnan-Srinivasan 2000]:

$$m(r) \ge \Omega\left(\left(\frac{r}{\ln r}\right)^{1/2} 2^r\right).$$





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Theorem [Radhakrishnan-Srinivasan 2000]:

$$m(r) \ge \Omega\left(\left(\frac{r}{\ln r}\right)^{1/2} 2^r\right).$$

Proof: For a fixed *r*-uniform hypergraph H = (V, E) with $|E| = k2^{r-1}$. Let $p \in [0, 1]$ satisfying $k(1-p)^r + k^2p < 1$.



Coloring process



Here is a two-round coloring process.

■ **First round:** Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected k monochromatic edges. Let U be the set of vertices in some monochromatic edges.

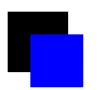


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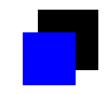
Here is a two-round coloring process.

- **First round:** Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected k monochromatic edges. Let U be the set of vertices in some monochromatic edges.
- Second round: Consider vertices in U sequentially in the (random) order of V. A vertex $u \in U$ is still dangerous if there is some monochromatic edge in the first coloring and for which no vertex has yet changed color.
 - If u is not dangerous, do nothing.
 - If u is still dangerous; with probability p, flip the color of u.



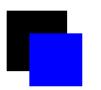






Claim: The algorithm fails with probability at most $k(1-p)^r + k^2p$.





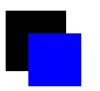


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Bad events: An edge e is red in the final coloring if

• e was red in the first coloring and remained red through the final coloring; call this event A_e .





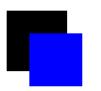


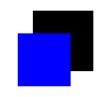
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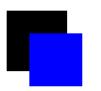
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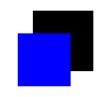
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2 $\sum_{e \in E(H)} \Pr(A_e) = k(1-p)^r.$



Estimating $Pr(C_e)$



For two edge e, f, we say e blames f if

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$$e \cap f = \{v\}$$
 for some v .

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- v was the last vertex of e that changed color from blue to red.
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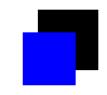
- In the first coloring f was blue and in the final coloring e was red.
- v was the last vertex of e that changed color from blue to red.
- When v changed its color f was still entire blue.

Call this event B_{ef} . Then

$$\sum_{e} \Pr(C_e) \le \sum_{e \ne f} \Pr(B_{ef}).$$







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$$\Pr(B_{ef} \mid \sigma) \le \frac{p}{2} 2^{-r+1} (1-p)^j 2^{-r+1+i} \left(\frac{1+p}{2}\right)^i$$





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We have

$$\Pr(B_{ef}) \leq 2^{1-2r} p \mathbb{E}[(1+p)^{i}(1-p)^{j}].$$

$$\leq 2^{1-2r} p.$$



Estimating k



The failure probability is at most

 $2\sum_{e \in E(H)} (\Pr(A_e) + \Pr(C_e)) \le k(1-p)^r + k^2p < ke^{-pr} + k^2p.$



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- $2\sum_{e \in E(H)} (\Pr(A_e) + \Pr(C_e)) \le k(1-p)^r + k^2p < ke^{-pr} + k^2p.$
- The function $f(p) = ke^{-pr} + k^2p$ reaches its minimum at $p = \frac{\ln(r/k)}{r}$. The minimum value is less than 1 if

$$k < (1 + o(1))\sqrt{\frac{2r}{\ln r}}.$$





$$\Pr(B_{ef}) \leq \sum_{l=0}^{r-1} \binom{r-1}{l} 2^{1-2r} \int_0^1 x^l p^{l+1} (1-xp)^{r-1} dx$$



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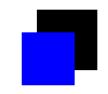
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Spencer modified the Radhakrishnan-Srinivasan's proof slightly. To assign a random ordering of the vertex in V, it is sufficient to assign each vertex v a birth time $x_v \in [0, 1]$. The birth time x_v is assigned uniformly and independently.

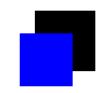
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The rest of proof is the same.



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If
$$X = \sum_{i=1}^{n} X_i$$
, then
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If X_1, \ldots, X_n are mutually independent, then $Var(X) = \sum_{i=1}^n Var(X_i).$



Chebyshev's Inequality

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$$E(X) = \mu$$
,
• $Var(X) = \sigma^2$.

Theorem [Chebyshev's Inequality]: For any positive λ ,

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Proof:

$$\sigma^{2} = \operatorname{Var}(X)$$

= $\operatorname{E}((X - \mu)^{2})$
 $\geq \lambda^{2} \sigma^{2} \operatorname{Pr}(|X - \mu| \geq \lambda \sigma).$



Number theory

 $\nu(n)$: the number of primes p dividing n.

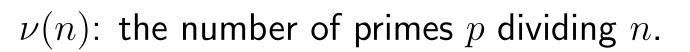


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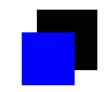
Theorem [Turán (1934)]: Let $\omega(n) \to \infty$ arbitrarily slowly. Then the number of x in $[n] := \{1, 2, ..., n\}$ such that

$$|\nu(x) - \ln \ln x| > \omega(n)\sqrt{\ln \ln n}.$$

is o(n).



Proof



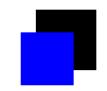
Let x be randomly chosen from [n]. For p prime set

$$X_p = \begin{cases} 1 & \text{if } p \mid x, \\ 0 & \text{otherwise.} \end{cases}$$

Set
$$M = n^{1/10}$$
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 $\nu(x) - 10 \le X(x) \le \nu(x)$



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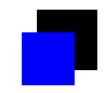
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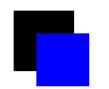


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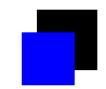
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By Chebyshev's inequality, we have

$$\Pr(|X - \ln \ln n| > \lambda \sqrt{\ln \ln n}) < \lambda^{-2} + o(1).$$



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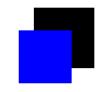
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Theorem [Erdős-Kac (1940):] For any fixed λ , we have

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ x \colon 1 \le x \le n, \nu(x) \ge \ln \ln n + \lambda \sqrt{\ln \ln n} \} \right|$$
$$= \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$





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- Write $X_i \sim X_j$ if X_i and X_j are independent. Let $\Delta = \sum_{i \sim j} \Pr(A_i \wedge A_j)$. If $E(X) \to \infty$ and $\Delta = o(E(X)^2)$, then X > 0 almost always.





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- Let $\Delta^* = \max_i \sum_{j \sim i} \Pr(A_j | A_i)$. If $E(X) \to \infty$ and $\Delta^* = o(E(X))$, then X > 0 almost always.





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A property of graphs is a family of graphs closed under isomorphic.

- A function r(n) is called a threshold function for some property ${\cal P}$ if
- If $p \ll r(n)$, then G(n, p) does not satisfy P almost always.
- If $p \gg r(n)$, then G(n,p) satisfy P almost always.



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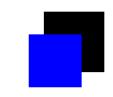
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If $p \ll n^{-2/3}$ then E(X) = o(1) and so X = 0 almost surely.



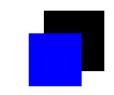




If $p \gg n^{-2/3}$, then $E(X) \to \infty$.



Continue



If $p \gg n^{-2/3}$, then $E(X) \to \infty$. $S \sim T$ if $|S \cap T| \ge 2$. Thus,

$$\Delta^* = O(n^2 p^5) + O(np^3) = o(n^4 p^6) = o(\mathbf{E}(X)).$$

Hence X > 0 almost surely.

