



Probabilistic Methods in Combinatorics Lecture 3

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- For $S \subset V$, let $h(S) = \sum_{F \subset S} h(F)$.
- A k-set F is crossing if it contains precisely one point form each V_i .

Theorem: Suppose h(F) = +1 for all crossing k-sets F. Then there is an $S \subset V$ for which

$$|h(S)| \ge c_k n^k.$$



Here $c_k > 0$, independent of n.

A Lemma

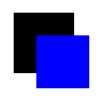


Lemma: Let P_k be the set of all homogeneous polynomials $f(p_1, \ldots, p_k)$ of degree k with all coefficients have absolute value at most one and $p_1p_2\cdots p_k$ having coefficient one. Then for all $f \in P_k$ there exists $p_1, \ldots, p_k \in [0, 1]$ with

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Here $c_k > 0$, independent of n.

Proof: Let $M(f) = \max_{p_1,...,p_k} |f(p_1,...,p_k)|$. Note P_k is compact and M is continuous. M reaches its minimum value c_k at some point f_0 . We have

$$c_k = M(f_0) > 0.$$







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Say F has type (a_1, \ldots, a_k) if $|F \cap V_i| = a_i$, $1 \le i \le k$. For these F,

$$E(X_F) = h(F)p_1^{a_1} \cdots p_k^{a_k}.$$







$$E(X) = \sum_{\substack{k \\ \sum_{i=1}^{k} a_i = k}} p_1^{a_1} \cdots p_k^{a_k} \sum_{F \text{ of type } (a_1, \dots, a_k)} h(F).$$







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Now select $p_1, \ldots, p_k \in [0, 1]$ with $|f(p_1, \ldots, p_k)| \ge c_k$. Then $\mathrm{E}(|X|) \ge |\mathrm{E}(X)| \ge c_k n^k$.







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There exists a S such that $|h(S)| \ge c_k n^k$.





Balancing vectors



Theorem: Let v_1, \ldots, v_n are n unit vector in \mathbb{R}^n . Then there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \dots + \epsilon_n v_n\| \le \sqrt{n},$$

and also there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \dots + \epsilon_n v_n\| \ge \sqrt{n}.$$





Proof



Let $\epsilon_1, \ldots, \epsilon_n$ be selected uniformly and independently from $\{+1, -1\}$. Let $X = \|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|^2$.





Proof



Let $\epsilon_1, \ldots, \epsilon_n$ be selected uniformly and independently from $\{+1, -1\}$. Let $X = \|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|^2$.

$$E(X) = E(\sum_{i,j=1} \epsilon_i \epsilon_j v_i \cdot v_j)$$

$$= \sum_{i,j=1}^{n} E(\epsilon_i \epsilon_j) v_i \cdot v_j$$

$$= \sum_{i,j=1}^{\infty} \delta_i^j v_i \cdot v_j$$

$$= \sum_{i=1}^{n} ||v_i||^2 = n.$$





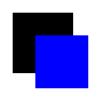
An extension



Theorem: Let $v_1, \ldots, v_n \in \mathbb{R}^n$, all $||v_i|| \leq 1$. Let $p_1, p_2, \ldots, p_n \in [0, 1]$ be arbitrary and set $w = p_1v_1 + p_2v_2 + \cdots + p_nv_n$. Then there exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ so that setting $v = \epsilon_1v_1 + \cdots + \epsilon_nv_n$,

$$||w - v|| \ge \frac{\sqrt{n}}{2}.$$





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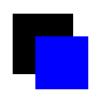
$$||w - v|| \ge \frac{\sqrt{n}}{2}.$$

Hint: Pick ϵ_i independently with

$$Pr(\epsilon_i = 1) = p_i, \quad Pr(\epsilon_i = 0) = 1 - p_i.$$

The proof is similar.





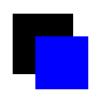
Unbalancing lights



Theorem: Let $a_{ij} = \pm 1$ for $1 \le i, j \le n$. Then there exist $x_i, y_j = \pm 1$, $1 \le i, j \le n$ so that

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j \ge \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}.$$





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Proof: Choose $y_j = 1$ or -1 randomly and independently. Let $R_i = \sum_{i=1}^n a_{ij}y_j$. Let x_i be the sign of R_i . Then

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} |R_i|.$$







Each R_i has the distribution $S_n = \sum_{i=1}^n X_i$, where X_i 's are independent uniform $\{-1,1\}$ random variables.







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Hence,

$$\sum_{i=1}^{n} E(|R_i|) = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{1/2}.$$







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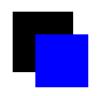




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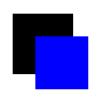
Brégman's Theorem (1973): $per(A) \le \prod_{1 \le i \le n} (r_i!)^{1/r_i}$.









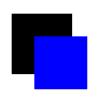




Pick $\sigma \in S$ and $\tau \in S_n$ independently and uniformly.

Let $A^{(1)} := A$; and $A^{(i)}$ is the submatrix obtained by deleting row $\tau(i-1)$ and column $\sigma(\tau(i-1))$ for 2 < i < n.

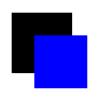






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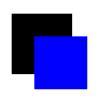






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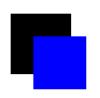






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Claim: $per(A) \le G(L)$.







For any fixed τ . Assume $\tau(1)=1$. By re-ordering, assume the first row has ones in the first $r:=r_1$ columns. For $1 \leq j \leq r$ let t_j be the permanent of A with the first row and j-th column removed (i.e., $\sigma(1)=j$). Let

$$t = \frac{t_1 + \dots + t_r}{r} = \frac{\operatorname{per}(A)}{r}.$$







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By induction,

$$G(R_2 \cdots R_n | \sigma(1) = j) \ge t_j$$
.

$$G(L) \ge \prod_{j=1}^{r} (rt_j)^{t_j/per(A)} = r \prod_{j=1}^{r} (t_j)^{t_j/rt}.$$







Since
$$\left(\prod_{j=1}^r t_j^{t_j}\right) \frac{1}{r} \geq t^t$$
, we have

$$G(L) \ge r \prod_{j=1}^{r} t_j^{t_j/rt} \ge r(t^t)^{1/t} = rt = \text{per}(A).$$







Since $\left(\prod_{j=1}^r t_j^{t_j}\right) \frac{1}{r} \geq t^t$, we have

$$G(L) \ge r \prod_{j=1}^{r} t_j^{t_j/rt} \ge r(t^t)^{1/t} = rt = \text{per}(A).$$

Now we calculate G[L] conditional on a fixed σ . By reordering, assume $\sigma(i)=i$ for all i. Note

$$G(R_i) = (r_i!)^{1/r_i}.$$







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$$G(R) = G(\prod_{i=1}^{n} R_i) = \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

