Probabilistic Methods in Combinatorics
Lecture 3

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k-sets

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- A k-set F is crossing if it contains precisely one point form each V_i.

Theorem: Suppose $h(F) = +1$ for all crossing k-sets F. Then there is an $S \subset V$ for which

$$|h(S)| \geq c_k n^k.$$

Here $c_k > 0$, independent of n.

A Lemma

Lemma: *Let P_k be the set of all homogeneous polynomials $f(p_1, \ldots, p_k)$ of degree k with all coefficients have absolute value at most one and $p_1 p_2 \cdots p_k$ having coefficient one. Then for all $f \in P_k$ there exists $p_1, \ldots, p_k \in [0, 1]$ with*

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Proof: Let $M(f) = \max_{p_1,\ldots,p_k} |f(p_1, \ldots, p_k)|$. Note P_k is compact and M is continuous. M reaches its minimum value c_k at some point f_0. We have

$$c_k = M(f_0) > 0. \qquad \square$$

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Say F has type $(a_1, \ldots, a_k)$ if $|F \cap V_i| = a_i$, $1 \le i \le k$. For these F,

$$\mathrm{E}(X_F) = h(F) p_1^{a_1} \cdots p_k^{a_k}.$$

continue

$$\mathrm{E}(X) = \sum_{\sum_{i=1}^{k} a_i = k} p_1^{a_1} \cdots p_k^{a_k} \sum_{F \text{ of type } (a_1,\ldots,a_k)} h(F).$$

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Now select $p_1, \dots, p_k \in [0, 1]$ with $|f(p_1, \dots, p_k)| \geq c_k$.
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There exists a S such that $|h(S)| \geq c_k n^k$. $\square$

Balancing vectors

Theorem: Let $v_1, \ldots, v_n$ are n unit vector in $\mathbb{R}^n$. Then there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n},$$

and also there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \geq \sqrt{n}.$$

Proof

Let $\epsilon_1, \ldots, \epsilon_n$ be selected uniformly and independently from $\{+1, -1\}$. Let $X = \|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|^2$.

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$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}(\sum_{i,j=1}^{n} \epsilon_i \epsilon_j v_i \cdot v_j) \\ &= \sum_{i,j=1}^{n} \mathrm{E}(\epsilon_i \epsilon_j) v_i \cdot v_j \\ &= \sum_{i,j=1}^{n} \delta_i^j v_i \cdot v_j \\ &= \sum_{i=1}^{n} \|v_i\|^2 = n. \end{aligned}$$

An extension

Theorem: Let $v_1, \ldots, v_n \in \mathbb{R}^n$, all $\|v_i\| \leq 1$. Let $p_1, p_2, \ldots, p_n \in [0, 1]$ be arbitrary and set $w = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n$. Then there exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ so that setting $v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$,

$$\|w - v\| \geq \frac{\sqrt{n}}{2}.$$

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Hint: Pick ϵ_i independently with

$$\Pr(\epsilon_i = 1) = p_i, \quad \Pr(\epsilon_i = 0) = 1 - p_i.$$

The proof is similar.

Unbalancing lights

Theorem: Let $a_{ij} = \pm 1$ for $1 \leq i, j \leq n$. Then there exist $x_i, y_j = \pm 1$, $1 \leq i, j \leq n$ so that

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq \left(\sqrt{\frac{2}{\pi}} + o(1) \right) n^{3/2}.$$

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Proof: Choose $y_j = 1$ or -1 randomly and independently. Let $R_i = \sum_{i=1}^{n} a_{ij} y_j$. Let x_i be the sign of R_i. Then

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} |R_i|.$$

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Each R_i has the distribution $S_n = \sum_{i=1}^{n} X_i$, where X_i's are independent uniform $\{-1, 1\}$ random variables.

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Hence,

$$\sum_{i=1}^n \mathrm{E}(|R_i|) = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{1/2}.$$

Brégman's Theorem

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- r_i: the i-th row sum.

Brégman's Theorem (1973): $\operatorname{per}(A) \leq \prod_{1 \leq i \leq n} (r_i!)^{1/r_i}$.

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Claim: $\mathrm{per}(A)) \leq G(L)$.

continue

For any fixed τ. Assume $\tau(1) = 1$. By re-ordering, assume the first row has ones in the first $r := r_1$ columns. For $1 \leq j \leq r$ let t_j be the permanent of A with the first row and j-th column removed (i.e., $\sigma(1) = j$). Let

$$t = \frac{t_1 + \cdots + t_r}{r} = \frac{\mathrm{per}(A)}{r}.$$

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By induction,

$$G(R_2 \cdots R_n | \sigma(1) = j) \ge t_j.$$

$$G(L) \ge \prod_{j=1}^{r} (rt_j)^{t_j/per(A)} = r \prod_{j=1}^{r} (t_j)^{t_j/rt}.$$

continue

Since $\left(\prod_{j=1}^{r} t_j^{t_j}\right)^{\frac{1}{r}} \geq t^t$, we have

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Now we calculate $G[L]$ conditional on a fixed σ. By reordering, assume $\sigma(i) = i$ for all i. Note

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$$G(R) = G(\prod_{i=1}^{n} R_i) = \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$