# Probabilistic Methods in Combinatorics
# Lecture 2

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# A $(k, l)$-system

A family of pairs of sets $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ is called a $(k, l)$-system if

- for $1 \leq i \leq h$, $|A_i| = k$, $|B_i| = l$, $A_i \cap B_i = \emptyset$.
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**Question:** What is the maximum size that a $(k,l)$-system can have?

**Theorem [Bollobás 1965]:** If $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ is a $(k,l)$-system, then $h \le \binom{k+l}{k}$.

# Proof

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Observe that all $X_i$'s are disjoint events. We have

$$1 \geq \Pr(\vee_{i=1}^{h} X_i) = \sum_{i=1}^{h} \Pr(X_i) = \frac{h}{\binom{k+l}{k}}.$$

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**Proof:** Let $B=\{b_1,b_2,\ldots,b_n\}$. Choose a prime $p>2\max\{|b_i|\}_{i=1}^{n}$.

Let $C=\{k+1,k+2,\ldots,2k+1\}$. Then $C$ is a sum-free set of $\mathbb{Z}_p$.

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Let $C=\{k+1,k+2,\ldots,2k+1\}$. Then $C$ is a sum-free set of $\mathbb{Z}_p$.

Randomly pick an integer $x$ in $[1,p-1]$. Define

$$A=\{b_i\colon xb_i(\mod p)\in C\}.$$

# continue

**Claim:** $A$ is a sum-free set.
Let $X_i$ be the indicator random variable that $b_i \in A$.

$$\Pr(X_i) = \frac{|C|}{p-1} = \frac{k+1}{3k-1} > \frac{1}{3}.$$

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$$\mathrm{E}(|A|) = \sum_{i=1}^{n} \Pr(X_i) > \frac{n}{3}.$$

There is a subset $A \subset B$ with greater than $n/3$ elements. $\square$

# Disjoint pairs

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**Theorem [Alon-Frankl, 1985]:** If $|\mathcal{F}| = 2^{(1/2+\delta)n}$, then

$$d(\mathcal{F}) < |\mathcal{F}|^{2-\delta^2/2}.$$

# Proof

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$$
\begin{aligned}
& \Pr(|\cup_{i=1}^{t} A_i| \leq \frac{n}{2}) \\
\leq \quad & \sum_{|S|=\frac{n}{2}} \Pr(\wedge_{i=1}^{t}(A_i \subset S)) \\
\leq \quad & 2^n \left( \frac{2^{n/2}}{2^{(1/2+\delta)n}} \right)^t \\
= \quad & 2^{n(1-\delta t)}.
\end{aligned}
$$

# continue

Let $v(B) = |\{A \in \mathcal{F} \colon B \cap A = \emptyset\}|$. Then

$$\sum_B v(B) = 2d(\mathcal{F}) \geq 2m^{2-\delta^2/2}.$$

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Let $Y$ be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all $A_i$ $1 \leq i \leq t$.

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Let $Y$ be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all $A_i$ $1 \leq i \leq t$. Then

$$\begin{aligned} E(|Y|) &= \sum_{B \in \mathcal{F}} \left( \frac{v(B)}{m} \right)^t \\ &\geq \frac{1}{m^{t-1}} \left( \frac{\sum_B v(B)}{m} \right)^t \\ &\geq 2m^{1-t\delta^2/2}. \end{aligned}$$

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Since $Y \leq m$, we get

$$\Pr(Y \geq m^{1-t\delta^2/2}) \geq m^{-t\delta^2/2}.$$

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Since $Y \leq m$, we get

$$\Pr(Y \geq m^{1-t\delta^2/2}) \geq m^{-t\delta^2/2}.$$

Choose $t = \lfloor 1 + \frac{1}{\delta} \rfloor$. We have $m^{-t\delta^2/2} > 2^{n(1-\delta t)}$.
Thus, with positive probability, $|\cup_{i=1}^{t} A_i| > \frac{n}{2}$ and $\cup_{i=1}^{t} A_i$ is disjoint to more than $2^{n/2}$ members of $\mathcal{F}$. Contradiction. $\square$

# Erdős-Ko-Rado Theorem

Let $\mathcal{F} \subset \binom{[n]}{k}$. A family $\mathcal{F}$ of $k$-sets is called **intersecting** if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

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**Erdős-Ko-Rado Theorem:** If $n \geq 2k$ and $\mathcal{F}$ is an intersecting family of $k$-sets in $[n]$, then

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# Erdős-Ko-Rado Theorem

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**Erdős-Ko-Rado Theorem:** If $n \geq 2k$ and $\mathcal{F}$ is an intersecting family of $k$-sets in $[n]$, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

This is tight since we can take $\mathcal{F} = \{F \in \binom{[n]}{k} : 1 \in F\}$.

# Katona's book proof

**Katona (1974) proof:** Consider a random permutation $\sigma \in S_n$ chosen randomly. List the elements of $[n]$ in the order of $\sigma$ on a cycle $C_\sigma$.

- For $A \in \mathcal{F}$, $X_A$ be the indicator variable that $A$ forms a consecutive block on $C_\sigma$.

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- For $A \in \mathcal{F}$, $X_A$ be the indicator variable that $A$ forms a consecutive block on $C_\sigma$.
- $X := \sum_{A\in\mathcal{F}} X_A$: the number of consecutive blocks in $\mathcal{F}$.

$$\mathrm{E}(X) = \sum_{A\in\mathcal{F}} \mathrm{E}(X_A) = \frac{n|\mathcal{F}|}{\binom{n}{k}}.$$

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- $X := \sum_{A \in \mathcal{F}} X_A$: the number of consecutive blocks in $\mathcal{F}$.

$$\mathrm{E}(X) = \sum_{A \in \mathcal{F}} \mathrm{E}(X_A) = \frac{n|\mathcal{F}|}{\binom{n}{k}}.$$

Since $\mathcal{F}$ is intersecting, $X \leq k$. We have $\frac{n|\mathcal{F}|}{\binom{n}{k}} \leq k$. □

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| liberated | divorced |
| preach | to give a mathematical lecture |
| torture | to give an oral exam to a student |

# Linearity of expectation

Let $X_1, X_2, \dots, X_n$ be random variables and $X = \sum_{i=1}^{n} c_i X_i$. Then

$$\mathrm{E}(X) = \sum_{i=1}^{n} c_i \mathrm{E}(X_i).$$

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$$\mathrm{E}(X) = \sum_{i=1}^{n} c_i \mathrm{E}(X_i).$$

**Philosophy:** There is a point in the probability space for which $X \geq \mathrm{E}(X)$ and a point for $X \leq \mathrm{E}(X)$.

# Hamilton paths

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**Proof:** Let $X$ be the number of Hamiltonian paths in a random tournament. Write $X = \sum_{\sigma \in S_n} X_\sigma$. Here $X_\sigma$ is the indicator random variable for $\sigma$ giving a Hamilton path.

$$\mathrm{E}(X_\sigma) = 2^{-(n-1)}.$$

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$$\mathrm{E}(X_\sigma) = 2^{-(n-1)}.$$

We have

$$\mathrm{E}(X) = \sum_{\sigma \in S_n} \mathrm{E}(X_\sigma) = n!2^{1-n}.$$

Done! $\square$

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Let $P(n)$ be the maximum possible number of Hamiltonian paths in a tournament on $n$ vertices.

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**Szele [1943]** proved

$$\frac{1}{2} \leq \lim_{n\to\infty} \left( \frac{P(n)}{n!} \right)^{1/n} \leq \frac{1}{2^{3/4}}.$$

He conjecture that $\lim_{n\to\infty} \left( \frac{P(n)}{n!} \right)^{1/n} = \frac{1}{2}$.

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**Theorem [Alon, 1990]:** $P(n) \leq cn^{3/2}\frac{n!}{2^{n-1}}$.

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Let $X$ be the number of crossing edges (from $L$ to $R$). Let $X_{uv}$ be the indicator variable of the edge $uv$ is crossing.

$$\mathrm{E}(X_{uv}) = \frac{1}{4}.$$

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$$\mathrm{E}(X_{uv}) = \frac{1}{4}.$$

$$\mathrm{E}(X) = \sum_{uv \in E} \mathrm{E}(X_{uv}) = \frac{m}{2}.$$

# Exercises

- If $G$ has $2n$ vertices and $m$ edges then it contains a bipartite subgraph with at least $m\frac{n}{2n-1}$ edges; if $G$ has $2n+1$ vertices and $m$ edges then it contains a bipartite subgraph with at least $m\frac{n+1}{2n+1}$ edges.

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- There is a two-coloring of $K_n$ with at most $\binom{n}{s}2^{1-\binom{s}{2}}$ monochromatic $K_s$.

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- There is a two-coloring of $K_n$ with at most $\binom{n}{s}2^{1-\binom{s}{2}}$ monochromatic $K_s$.
- There is a two-coloring of $K_{m,n}$ with at most $\binom{m}{s}\binom{n}{t}2^{1-st}$ monochromatic $K_{s,t}$.