

Probabilistic Methods in Combinatorics Lecture 2

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A (k, l)-system

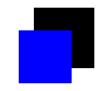


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for $1 \le i \le h$, $|A_i| = k$, $|B_i| = l$, $A_i \cap B_i = \emptyset$. for any $1 \le i \ne j \le h$, $|A_i \cap B_j| \ne \emptyset$.



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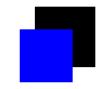
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Theorem [Bollobás 1965]: If $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^h$ is a (k, l)-system, then $h \leq \binom{k+l}{k}$.



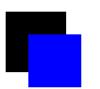


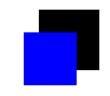




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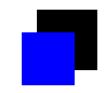
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Observe that all X_i 's are disjoint events. We have

$$1 \ge \Pr(\bigvee_{i=1}^{h} X_i) = \sum_{i=1}^{h} \Pr(X_i) = \frac{h}{\binom{k+l}{k}}.$$

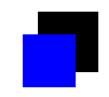




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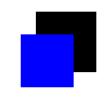


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Theorem [Erdős 1965]: Every set *B* of *n* nonzero integers contains a sum-free subset *A* of size $|A| > \frac{1}{3}n$.







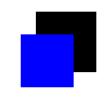
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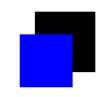
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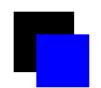
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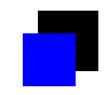
Randomly pick an integer x in [1, p - 1]. Define

$$A = \{b_i \colon xb_i (\text{ mod } p) \in C\}.$$





continue

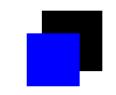


Claim: A is a sum-free set. Let X_i be the indicator random variable that $b_i \in A$.

$$\Pr(X_i) = \frac{|C|}{p-1} = \frac{k+1}{3k-1} > \frac{1}{3}.$$







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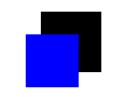
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There is a subset $A \subset B$ with greater than n/3 elements. \Box



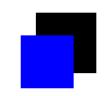
Disjoint pairs



 $\mathcal{F} \subset 2^{[n]}.$ $\mathbf{d}(\mathcal{F}) := |\{(F, F') \colon F, F' \in \mathcal{F}, F \cap F' = \emptyset\}|.$



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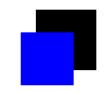


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Daykin and Erdős conjectured if $|\mathcal{F}| = 2^{(1/2+\delta)n}$ then $d(\mathcal{F}) = o(|\mathcal{F}|^2)$.



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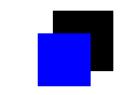
Theorem [Alon-Frankl, 1985]: If $|\mathcal{F}| = 2^{(1/2+\delta)n}$, then

 $d(\mathcal{F}) < |\mathcal{F}|^{2-\delta^2/2}.$





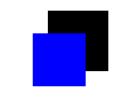




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$$\Pr(|\cup_{i=1}^{t} A_{i}| \leq \frac{n}{2})$$

$$\leq \sum_{|S|=\frac{n}{2}} \Pr(\wedge_{i=1}^{t} (A_{i} \subset S))$$

$$\leq 2^{n} \left(\frac{2^{n/2}}{2^{(1/2+\delta)n}}\right)^{t}$$

$$= 2^{n(1-\delta t)}.$$





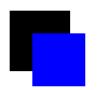




Let $v(B) = |\{A \in \mathcal{F} \colon B \cap A = \emptyset\}|$. Then

$$\sum_{B} v(B) = 2d(\mathcal{F}) \ge 2m^{2-\delta^2/2}.$$







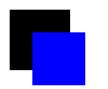


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Let Y be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all A_i $1 \le i \le t$.









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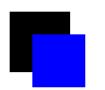
Let Y be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all A_i $1 \le i \le t$. Then

$$E(|Y|) = \sum_{B \in \mathcal{F}} \left(\frac{v(B)}{m}\right)^t$$

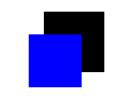
$$\geq \frac{1}{m^{t-1}} \left(\frac{\sum_B v(B)}{m}\right)^t$$

$$\geq 2m^{1-t\delta^2/2}.$$





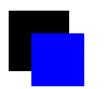




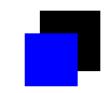
Since $Y \leq m$, we get

$$\Pr(Y \ge m^{1-t\delta^2/2}) \ge m^{-t\delta^2/2}.$$







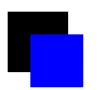


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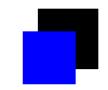
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Choose $t = \lfloor 1 + \frac{1}{\delta} \rfloor$. We have $m^{-t\delta^2/2} > 2^{n(1-\delta t)}$.

Thus, with positive probability, $|\cup_{i=1}^{t} A_i| > \frac{n}{2}$ and $\cup_{i=1}^{t} A_i$ is disjoint to more than $2^{n/2}$ members of \mathcal{F} . Contradiction. \Box



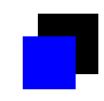
Erdős-Ko-Rado Theorem



Let $\mathcal{F} \subset {\binom{[n]}{k}}$. A family \mathcal{F} of k-sets is called **intersecting** if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.



Erdős-Ko-Rado Theorem



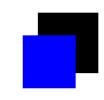
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Erdős-Ko-Rado Theorem: If $n \ge 2k$ and \mathcal{F} is an intersecting family of k-sets in [n], then

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This is tight since we can take $\mathcal{F} = \{F \in {[n] \choose k} : 1 \in F\}.$



Katona's book proof

Katona (1974) proof: Consider a random permutation $\sigma \in S_n$ chosen randomly. List the elements of [n] in the order of σ on a cycle C_{σ} .

For $A \in \mathcal{F}$, X_A be the indicator variable that A forms a consecutive block on C_{σ} .



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• $X := \sum_{A \in \mathcal{F}} X_A$: the number of consecutive blocks in \mathcal{F} .

$$E(X) = \sum_{A \in \mathcal{F}} E(X_A) = \frac{n|\mathcal{F}|}{\binom{n}{k}}$$



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$$\mathbf{E}(X) = \sum_{A \in \mathcal{F}} \mathbf{E}(X_A) = \frac{n|\mathcal{F}|}{\binom{n}{k}}.$$

Since \mathcal{F} is intersecting, $X \leq k$. We have $\frac{n|\mathcal{F}|}{\binom{n}{k}} \leq k$.



Erdős' vocabulary

Erdős's vocabulary	meaning
proof from The Book	beautiful mathematical proof



Erdős' vocabulary

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epsilon	



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preach	to give a mathematical lecture
torture	to give an oral exam to a student



Linearity of expectation

Let X_1, X_2, \ldots, X_n be random variables and $X = \sum_{i=1}^n c_i X_i$. Then

$$\mathbf{E}(X) = \sum_{i=1}^{n} c_i \mathbf{E}(X_i).$$



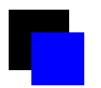
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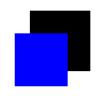
$$\mathbf{E}(X) = \sum_{i=1}^{n} c_i \mathbf{E}(X_i).$$

Philosophy: There is a point in the probability space for which $X \ge E(X)$ and a point for $X \le E(X)$.



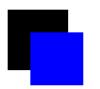


Hamilton paths



Theorem: There is a tournament T with n players and at least $n!2^{-(n-1)}$ Hamiltonian paths.





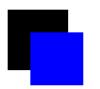
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Proof: Let X be the number of Hamiltonian paths in a random tournament. Write $X = \sum_{\sigma \in S_n} X_{\sigma}$. Here X_{σ} is the indicator random variable for σ giving a Hamilton path.

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We have

$$\mathcal{E}(X) = \sum_{\sigma \in S_n} \mathcal{E}(X_{\sigma}) = n! 2^{1-n}.$$

Done!







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Szele [1943] proved

$$\frac{1}{2} \le \lim_{n \to \infty} \left(\frac{P(n)}{n!} \right)^{1/n} \le \frac{1}{2^{3/4}}.$$

He conjecture that $\lim_{n\to\infty} \left(\frac{P(n)}{n!}\right)^{1/n} = \frac{1}{2}$.



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This conjecture was proved by Alon in 1990.



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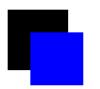
Szele [1943] proved

$$\frac{1}{2} \le \lim_{n \to \infty} \left(\frac{P(n)}{n!} \right)^{1/n} \le \frac{1}{2^{3/4}}.$$

He conjecture that $\lim_{n\to\infty} \left(\frac{P(n)}{n!}\right)^{1/n} = \frac{1}{2}$.

This conjecture was proved by Alon in 1990. **Theorem [Alon, 1990]:** $P(n) \leq cn^{3/2} \frac{n!}{2^{n-1}}$.





Theorem: Let G = (V, E) be a graph with n vertices and m edges. Then G contains a bipartite subgraph with at last m/2 edges.

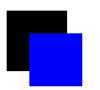




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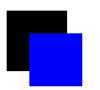
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$$\mathcal{E}(X_{uv}) = \frac{1}{4}.$$





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$$E(X_{uv}) = \frac{1}{4}.$$
$$E(X) = \sum_{uv \in E} E(X_{uv}) = \frac{m}{2}.$$



Exercises



If G has 2n vertices and m edges then it contains a bipartite subgraph with at least $m\frac{n}{2n-1}$ edges; if G has 2n+1 vertices and m edges then it contains a bipartite subgraph with at least $m\frac{n+1}{2n+1}$ edges.



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- There is a two-coloring of K_n with at most $\binom{n}{s} 2^{1-\binom{s}{2}}$ monochromatic K_s .



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 - There is a two-coloring of K_n with at most $\binom{n}{s}2^{1-\binom{s}{2}}$ monochromatic K_s .
- There is a two-coloring of $K_{m,n}$ with at most $\binom{m}{s}\binom{n}{t}2^{1-st}$ monochromatic $K_{s,t}$.

