

# Probabilistic Methods in Combinatorics Lecture 14

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Now we consider G(n, p) for p = c/n, with c > 1 constant. Let y := y(c) be the positive real solution of  $e^{-cy} = 1 - y$ . Choose a large constant K > 0 and a small constant  $\delta > 0$ . Let C(v) be the component of G(n, p) containing v.



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**Claim:** The probability of having any awkward component is  $o(n^{-20})$ .



#### No middle ground

**Proof:** We will show for any awkward t,  $Pr(|C(v)| = t) = o(n^{-22})$ . Note

$$\Pr(|C(v)| = t) \le \Pr(B(n-1, 1 - (1 - \frac{c}{n})^t) = t - 1.$$



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$$\Pr\left(B(n-1, 1 - (1 - \frac{c}{n})^t\right) = O(e^{-Ct})$$

for some constant C. Since  $t \ge K \log n$  and K large enough, this probability is  $o(n^{-22})$  as required.



Let  $\alpha = \Pr(C(v) \text{ is not small })$ . Then

$$\alpha = \Pr(T_c^{po} \ge S) \approx \Pr(T_c^{po} = \infty) = y.$$

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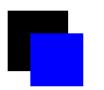
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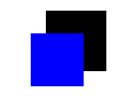
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It remains to show the giant component is unique and of size about yn.

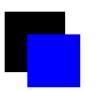






Set  $p_1 = n^{-3/2}$ . Let  $G_1 = G(n, p_1)$ , G = G(n, p), and  $G^+ = G \cup G_1$ . Note  $G^+ \sim G(n, p^+)$  with  $p^+ = p + p_1 - pp_1$ .





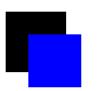


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Suppose that G has two giant components  $V_1$  and  $V_2$ . Then the probability that  $V_1$  and  $V_2$  is not connected after sprinkling is at most

$$(1-p_1)^{|V_1||V_2|} = o(1).$$







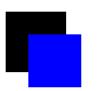
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Now  $G^+$  almost surely have a component of size at least  $2(y - \delta)n > (y + \delta)n$ . It is an awkward component for  $G^+$ . Contradiction!







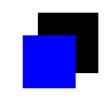
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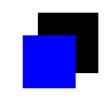
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Since  $\delta$  can be made arbitrarily small, the unique giant component has size (1 + o(1))yn.



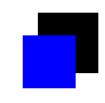
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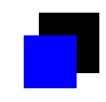




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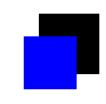




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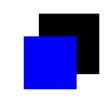




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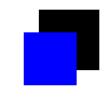
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The following statements hold.

- $Pr(\exists an awkward component) = O(n^{-20}).$
- The escape probability  $\alpha \approx y \approx 2\epsilon$ .
- Sprinkling works with  $p_1 = n^{-4/3}$ .



### The critical window



Now consider G(n, p) with  $p = \frac{1}{n} + \lambda n^{-4/3}$  for a fixed  $\lambda$ . This critical window has been studied by **Bollabás**, Łuczak, Janson, Knuth, Pittel and many others. It requires delicate calculations.



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For fixed c>0, Let X be the number of tree components of size  $k=cn^{2/3}.$  Then

$$E(X) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k) + \binom{k}{2} - (k-1)}.$$



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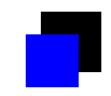
Recall

$$\ln(1+x) = x - \frac{1}{2}x^2 + O(x^3).$$





#### **Estimation**



#### We estimate

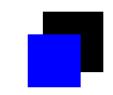
$$\binom{n}{k} \approx \frac{n^k}{k^k \sqrt{2\pi k}} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right),$$

#### $\quad \text{and} \quad$

$$\prod_{i=1}^{k-1} \left( 1 - \frac{i}{n} \right) = e^{\sum_{i=1}^{k-1} \ln(1 - i/n)}$$
$$= e^{-\sum_{i=1}^{k-1} (i/n + i^2/2n^2 + O(i^3/n^3))}$$
$$= e^{-\frac{k^2}{2n} - \frac{k^3}{6n^2} + o(1)}$$
$$= e^{-\frac{k^2}{2n} - \frac{c^3}{6} + o(1)}.$$



#### Continue

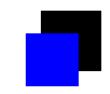


We also estimate

$$p^{k-1} = n^{1-k} (1 + \lambda n^{-1/3})^{k-1}$$
  
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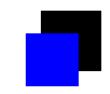
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 $\mathcal{E}(X) \approx k^{-5/2} \sqrt{2\pi} e^A,$ 

where  $A = \frac{(\lambda - c)^3 - \lambda^3}{6}$ .



#### **Putting together**

We get

$$\mathcal{E}(X) \approx k^{-5/2} \sqrt{2\pi} e^A,$$

where  $A = \frac{(\lambda - c)^3 - \lambda^3}{6}$ . For any fixed  $a, b, \lambda$ , let X be the number of tree components of size between  $an^{2/3}$  and  $bn^{2/3}$ . Then

$$\lim_{n \to \infty} \mathcal{E}(X) = \int_a^b e^{A(c)} c^{-5/2} \sqrt{2\pi} dc.$$



#### **Other components**

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Let  $X^{(l)}$  be the number of components on k vertices with k-1+l edges. Then a similar calculation shows

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Let  $X^*$  be the total number of components of size between  $an^{2/3}$  and  $bn^{2/3}$ . Let  $g(c) = \sum_{l=0}^{\infty} c_l c^{3l/2}$ . Then  $\lim_{n \to \infty} E(X^*) = \int_a^b e^{A(c)} c^{-5/2} \sqrt{2\pi} g(c) dc.$ 



#### Duality

For a fixed k, consider two random graphs G(n, p) and G(n', p'). Assume c = np > 1 and c' = n'p' < 1. We say G(n, p) and G(n', p') are **dual** to each other if  $ce^{-c} = c'e^{-c'}$ .





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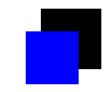
$$\lim_{n \to \infty} \Pr(C(v) = k \text{ in } G(n, p) | C(v) \text{ is small})$$

$$= \frac{1}{1-y} \frac{e^{-ck}(ck)^{k-1}}{k!} = \frac{e^{-c'k}(c'k)^{k-1}}{k!}$$
$$= \lim_{n' \to \infty} \Pr(C(v) = k \text{ in } G(n', p')).$$









Consider G(n,p) with

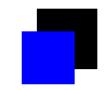
$$p = \frac{\log n}{kn} + \frac{(k-1)\log\log n}{kn} + \frac{t}{n} + o(\frac{1}{n}),$$

then there are only trees of size at most k except for the giant component. Let X be the number of trees of k vertices.









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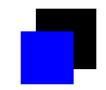
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$$E(X) = {\binom{n}{k}} k^{k-2} p^{k-1} (1-p)^{k(n-k) + {\binom{k}{2}} - k + 1}$$
$$\approx \frac{1}{k^2 p \cdot k!} (knp)^k e^{-knp} \approx \frac{e^{-kt}}{k \cdot k!}.$$









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Further, X follows the Poisson distribution.



For k = 1 and  $p = \frac{\log n}{n} + \frac{t}{n} + o(\frac{1}{n})$ , G(n, p) consists of a giant component with n - O(1) vertices and bounded number of isolated vertices.



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- The distribution of the number of isolated vertices again has a Poisson distribution with mean value  $e^{-t}$ .
- The probability that G(n,p) is connected tends to  $e^{-e^{-t}}$
- As  $t \to \infty$ , G(n, p) is almost surely connected.









Consider G(n,p) with  $p \sim \omega(n) \log n/n$  where  $\omega(n) \to \infty$ .





# Range VI

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# Range VI

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$$|X - \mathcal{E}(X)| < 2\sqrt{\omega(n)}\log n.$$

Almost surely, for all v,  $d_v$  is in the interval  $[d - 2\sqrt{\omega(n)} \log n, d + 2\sqrt{\omega(n)} \log n]$ , where d = np is the expected degree.





## **Subgraphs**

**Theorem:** Let H be a strictly balanced graph with v vertices, m edges, and a automorphisms. Let c > 0 be arbitrary. Then with  $p = cn^{-v/m}$ ,

 $\lim_{n \to \infty} \Pr(G(n, p) \text{ contains no } H) = e^{-c^m/a}.$ 





## **Subgraphs**

**Theorem:** Let H be a strictly balanced graph with v vertices, m edges, and a automorphisms. Let c > 0 be arbitrary. Then with  $p = cn^{-v/m}$ ,

$$\lim_{n \to \infty} \Pr(G(n, p) \text{ contains no } H) = e^{-c^m/a}$$

**Proof:** Let  $A_{\alpha}$ ,  $1 \leq \alpha \leq {n \choose v} v! / a$ , range over the edge sets of possible copies of H and  $B_{\alpha}$  be the event  $A_{\alpha} \subset G(n, p)$ . We will apply Janson's Inequality.

$$\lim_{n \to \infty} \mu = \lim_{n \to \infty} \binom{n}{v} v! p^m / a = c^m / a$$
$$\lim_{n \to \infty} M = e^{-c^m / a}.$$





## Proof

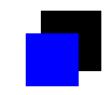
Consider  $\Delta = \sum_{\alpha \sim \beta} \Pr(B_{\alpha} \wedge B_{\beta})$ . We split the sum according to the number of vertices in  $A_{\alpha} \cap A_{\beta}$ . For  $2 \leq j \leq v$ , let  $f_j$  be the maximal number of edges in  $A_{\alpha} \cap A_{\beta}$  where  $\alpha \sim \beta$  and  $\alpha$  and  $\beta$  intersect in j vertices. Since H is strictly balanced,

$$\frac{f_j}{j} < \frac{m}{v}.$$

There are  $O(n^{2v-j})$  choices of  $\alpha$ ,  $\beta$  For such  $\alpha$ ,  $\beta$ ,

$$\Pr(B_{\alpha} \wedge B_{\beta}) = p^{|A_{\alpha} \cup A_{\beta}|} \le p^{2m - f_{j}}$$

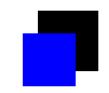


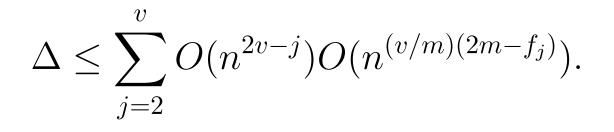


 $\Delta \le \sum_{j=2}^{\nu} O(n^{2\nu-j}) O(n^{(\nu/m)(2m-f_j)}).$ 







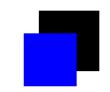


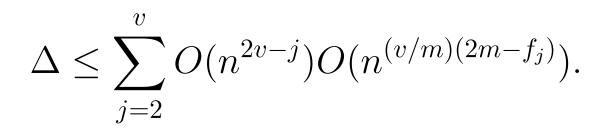
But

$$2v - j - (v/m)(2m - f_j) = \frac{vf_j}{e} - j < 0.$$









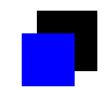
But

$$2v - j - (v/m)(2m - f_j) = \frac{vf_j}{e} - j < 0.$$

Each term is o(1) and hence  $\Delta = o(1)$ .







$$\Delta \le \sum_{j=2}^{v} O(n^{2v-j}) O(n^{(v/m)(2m-f_j)}).$$

But

$$2v - j - (v/m)(2m - f_j) = \frac{vf_j}{e} - j < 0.$$

Each term is o(1) and hence  $\Delta=o(1).$  By Janson's inequality,

$$\lim_{n \to \infty} \Pr(\wedge \bar{B}_{\alpha}) = \lim_{n \to \infty} M = e^{-c^m/a}$$

The proof is finished.

