Probabilistic Methods in Combinatorics Lecture 14

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Supercritical regimes

Now we consider $G(n,p)$ for $p = c/n$, with $c > 1$ constant.
Let $y := y(c)$ be the positive real solution of $e^{-cy} = 1 - y$.
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Claim: The probability of having any awkward component is $o(n^{-20})$.

No middle ground

Proof: We will show for any awkward t, $\Pr(|C(v)| = t) = o(n^{-22})$. Note

$$\Pr(|C(v)| = t) \leq \Pr(B(n-1, 1-(1-\frac{c}{n})^t) = t-1.$$

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If $t = o(n)$, then $1-(1-\frac{c}{n})^t \approx \frac{ct}{n}$. So the mean is about ct, which is not close to t. If $t = xn$, then $1-(1-\frac{c}{n})^t) \approx 1-e^{-cx}$. Since $1-e^{-cx} \neq x$, so the mean is not near t.

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$$\Pr\left(B(n-1, 1-(1-\frac{c}{n})^t\right) = O(e^{-Ct})$$

for some constant C. Since $t \geq K \log n$ and K large enough, this probability is $o(n^{-22})$ as required.

Escape Probability

Let $\alpha = \Pr(C(v)$ is not small $)$. Then

$$\alpha = \Pr(T_c^{po} \geq S) \approx \Pr(T_c^{po} = \infty) = y.$$

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- Each giant component has size between $(y - \delta)n$ and $(y + \delta)n$.

It remains to show the giant component is unique and of size about yn.

Sprinkling

Set $p_1 = n^{-3/2}$. Let $G_1 = G(n, p_1)$, $G = G(n, p)$, and $G^+ = G \cup G_1$. Note $G^+ \sim G(n, p^+)$ with $p^+ = p + p_1 - pp_1$.

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Since δ can be made arbitrarily small, the unique giant component has size $(1 + o(1))yn$.

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The following statements hold.

- $\Pr(\exists$ an awkward component $) = O(n^{-20})$.
- The escape probability $\alpha \approx y \approx 2\epsilon$.
- Sprinkling works with $p_1 = n^{-4/3}$.

The critical window

Now consider $G(n, p)$ with $p = \frac{1}{n} + \lambda n^{-4/3}$ for a fixed λ. This critical window has been studied by **Bollabás, Łuczak, Janson, Knuth, Pittel** and many others. It requires delicate calculations.

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For fixed $c > 0$, Let X be the number of tree components of size $k = cn^{2/3}$. Then

$$\mathrm{E}(X) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-(k-1)}.$$

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Recall

$$\ln(1+x) = x - \frac{1}{2}x^2 + O(x^3).$$

Estimation

We estimate

$$\binom{n}{k} \approx \frac{n^k}{k^k\sqrt{2\pi k}} \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right),$$

and

$$\begin{aligned}\prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right) &= e^{\sum_{i=1}^{k-1}\ln(1-i/n)} \\ &= e^{-\sum_{i=1}^{k-1}(i/n+i^2/2n^2+O(i^3/n^3))} \\ &= e^{-\frac{k^2}{2n}-\frac{k^3}{6n^2}+o(1)} \\ &= e^{-\frac{k^2}{2n}-\frac{c^3}{6}+o(1)}.\end{aligned}$$

Continue

We also estimate

$$\begin{aligned} p^{k-1} &= n^{1-k}(1+\lambda n^{-1/3})^{k-1} \\ &= n^{1-k}e^{(k-1)\ln(1+\lambda n^{-1/3})} \\ &= n^{1-k}e^{k\lambda n^{-1/3}-\frac{1}{2}c\lambda^2+o(1)}, \end{aligned}$$

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(1-p)^{k(n-k)+\binom{k}{2}-(k-1)} &= e^{(kn-k^2/2+O(k))\ln(1-p)} \\
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Putting together

We get

$$\mathrm{E}(X) \approx k^{-5/2}\sqrt{2\pi}e^{A},$$

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For any fixed a, b, λ, let X be the number of tree components of size between $an^{2/3}$ and $bn^{2/3}$. Then

$$\lim_{n\to\infty} E(X) = \int_a^b e^{A(c)}c^{-5/2}\sqrt{2\pi}dc.$$

Other components

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Let $X^{(l)}$ be the number of components on k vertices with $k-1+l$ edges. Then a similar calculation shows

$$\lim_{n\to\infty} \mathrm{E}(X^{(l)}) = \int_a^b e^{A(c)} c^{-5/2} \sqrt{2\pi} (c_l c^{3l/2}) dc.$$

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Let X^* be the total number of components of size between $an^{2/3}$ and $bn^{2/3}$. Let $g(c) = \sum_{l=0}^{\infty} c_l c^{3l/2}$. Then

$$\lim_{n\to\infty} \mathrm{E}(X^*) = \int_a^b e^{A(c)} c^{-5/2} \sqrt{2\pi} g(c) dc.$$

Duality

For a fixed k, consider two random graphs $G(n, p)$ and $G(n', p')$. Assume $c = np > 1$ and $c' = n'p' < 1$. We say $G(n, p)$ and $G(n', p')$ are **dual** to each other if $ce^{-c} = c'e^{-c'}$.

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$$\begin{aligned}
&\lim_{n\to\infty} \Pr(C(v) = k \text{ in } G(n,p) | C(v) \text{ is small}) \\
&= \frac{1}{1-y} \frac{e^{-ck}(ck)^{k-1}}{k!} = \frac{e^{-c'k}(c'k)^{k-1}}{k!} \\
&= \lim_{n'\to\infty} \Pr(C(v) = k \text{ in } G(n',p')).
\end{aligned}$$

Range V

Consider $G(n,p)$ with

$$p = \frac{\log n}{kn} + \frac{(k-1)\log\log n}{kn} + \frac{t}{n} + o(\frac{1}{n}),$$

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$$\mathrm{E}(X) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1}$$

$$\approx \frac{1}{k^2 p \cdot k!} (knp)^k e^{-knp} \approx \frac{e^{-kt}}{k \cdot k!}.$$

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Further, X follows the Poisson distribution.

Threshold of connectivity

For $k = 1$ and $p = \frac{\log n}{n} + \frac{t}{n} + o(\frac{1}{n})$, $G(n, p)$ consists of a giant component with $n - O(1)$ vertices and bounded number of isolated vertices.

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- The distribution of the number of isolated vertices again has a Poisson distribution with mean value e^{-t}.
- The probability that $G(n, p)$ is connected tends to $e^{-e^{-t}}$.
- As $t \to \infty$, $G(n, p)$ is almost surely connected.

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Let $X = d_v$ be the degree of v. By Chernoff's inequality, With probability at least $1 - O(n^{-2})$, we have

$$|X - \mathrm{E}(X)| < 2\sqrt{\omega(n)} \log n.$$

Almost surely, for all v, d_v is in the interval $[d - 2\sqrt{\omega(n)} \log n, d + 2\sqrt{\omega(n)} \log n]$, where $d = np$ is the expected degree.

Subgraphs

Theorem: Let H be a strictly balanced graph with v vertices, m edges, and a automorphisms. Let $c > 0$ be arbitrary. Then with $p = cn^{-v/m}$,

$$\lim_{n\to\infty} \Pr(G(n,p) \text{ contains no } H) = e^{-c^m/a}.$$

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Proof: Let A_α, $1 \le \alpha \le \binom{n}{v} v!/a$, range over the edge sets of possible copies of H and B_α be the event $A_\alpha \subset G(n,p)$. We will apply Janson's Inequality.

$$\lim_{n\to\infty} \mu = \lim_{n\to\infty} \binom{n}{v} v! p^m / a = c^m / a.$$

$$\lim_{n\to\infty} M = e^{-c^m/a}.$$

Proof

Consider $\Delta = \sum_{\alpha \sim \beta} \Pr(B_\alpha \wedge B_\beta)$. We split the sum according to the number of vertices in $A_\alpha \cap A_\beta$. For $2 \leq j \leq v$, let f_j be the maximal number of edges in $A_\alpha \cap A_\beta$ where $\alpha \sim \beta$ and α and β intersect in j vertices. Since H is strictly balanced,

$$\frac{f_j}{j} < \frac{m}{v}.$$

There are $O(n^{2v-j})$ choices of α, β For such α, β,

$$\Pr(B_\alpha \wedge B_\beta) = p^{|A_\alpha \cup A_\beta|} \leq p^{2m-f_j}.$$

Continue

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Each term is $o(1)$ and hence $\Delta = o(1)$. By Janson's inequality,

$$\lim_{n\to\infty} \Pr(\wedge \bar{B}_\alpha) = \lim_{n\to\infty} M = e^{-c^m/a}.$$

The proof is finished. $\square$