Probabilistic Methods in Combinatorics
Lecture 12

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Poisson Paradigm

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- Let X_i be the random indicator of the event B_i and $X = \sum_{i\in I} X_i$. If $\Pr(B_i)$'s are small and "mostly independent", then one may expect X follows "Poisson-like distribution". In particular,

$$\Pr(X = 0) \approx e^{-\mathrm{E}(X)}.$$

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- $M = \prod_{i \in I} \Pr(\bar{B}_i)$.

Janson inequality

The Janson inequality: Assume all $\Pr(B_i) \leq \epsilon$. Then

$$M \leq \Pr(\wedge_{i \in I} \bar{B}_i) \leq M e^{\frac{\Delta}{2(1-\epsilon)}},$$

and, further,

$$\Pr(\wedge_{i \in I} \bar{B}_i) \leq e^{-\mu + \frac{\Delta}{2}}.$$

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The Extended Janson inequality: If further $\Delta \geq \mu$, then

$$\Pr(\wedge_{i\in I}\bar{B}_i) \leq e^{\frac{-\mu^2}{2\Delta}}.$$

Proof

Proof given by Boppana and Spencer: We will use the following correlation inequality.

- For all $J \subset I$, $i \notin J$,

$$\Pr(B_i \mid \wedge_{j\in J}\bar{B}_j) \leq \Pr(B_i).$$

- For $J \subset I$, $i, k \notin J$,

$$\Pr(B_i \mid B_k \wedge \wedge_{j\in J}\bar{B}_j) \leq \Pr(B_i \mid B_k).$$

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$$\Pr(B_i \mid B_k \wedge \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).$$

Order the index set $I = \{1, 2, \ldots, m\}$.

$$\Pr(\wedge_{i \in I} \bar{B}_i) = \prod_{i=1}^{m} \Pr\left(\bar{B}_i \mid \wedge_{1 \leq j < i} \bar{B}_j\right) \geq \prod_{i=1}^{m} \Pr(\bar{B}_i).$$

Continue

For a given i renumber, for convenience, so that $i \sim j$ for $1 \leq j \leq d$ and not for $d+1 \leq j < i$. Let $A = B_i$, $B = \bar{B}_1 \wedge \cdots \wedge \bar{B}_d$, and $C = \bar{B}_{d+1} \wedge \cdots \wedge \bar{B}_{i-1}$,

$$\begin{aligned} \Pr(B_i \mid \wedge_{1 \leq j < i} \bar{B}_j) &= \Pr(A \mid B \wedge C) \\ &\leq \Pr(A \wedge B \mid C) \\ &= \Pr(A \mid C) \Pr(B \mid A \wedge C). \end{aligned}$$

Note $\Pr(A \mid C) = \Pr(A)$ and

$$\Pr(B \mid A \wedge C) \geq 1 - \sum_{j=1}^{d} \Pr(B_j \mid B_i \wedge C) \geq 1 - \sum_{j=1}^{d} \Pr(B_j \mid B_i).$$

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$$\Pr(B_i \mid \wedge_{1 \leq j < i} \bar{B}_j) \geq \Pr(B_i) - \sum_{j=1}^{d} \Pr(B_j \wedge B_i).$$

Continue

$$
\begin{aligned}
\Pr(\bar{B}_i \mid \wedge_{1\le j<i}\bar{B}_j) &\le \Pr(\bar{B}_i) + \sum_{j=1}^{d} \Pr(B_j \wedge B_i) \\
&\le \Pr(\bar{B}_i)\left(1 + \frac{1}{1-\epsilon}\sum_{j=1}^{d} \Pr(B_j \wedge B_i)\right) \\
&\le \Pr(\bar{B}_i) e^{\frac{1}{1-\epsilon}\sum_{j=1}^{d}\Pr(B_j \wedge B_i)}.
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Plug it into $\Pr(\wedge_{i\in I}\bar{B}_i) = \prod_{i=1}^{m} \Pr\left(\bar{B}_i \mid \wedge_{1\le j<i}\bar{B}_j\right)$; we get the first inequality. The second inequality use the following estimation.

$$
\begin{aligned}
\Pr(\bar{B}_i \mid \wedge_{1\le j<i}\bar{B}_j) &\le \Pr(\bar{B}_i) + \sum_{j=1}^{d} \Pr(B_j \wedge B_i) \\
&\le e^{-\Pr(B_i)+\sum_{j=1}^{d} \Pr(B_j \wedge B_i)}.
\end{aligned}
$$

Proof of second Theorem

From the Jansen inequality, we have

$$-\ln(\Pr(\wedge_{i\in I}\bar{B}_i)) \geq \sum_{i\in I}\Pr(B_i) - \frac{1}{2}\sum_{i\sim j}\Pr(B_i \wedge B_j).$$

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For any set $S \subset I$, the same inequality applied to $\{B_i\}_{i\in S}$:

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Now take S be a random subset of I given by $\Pr(i \in S) = p$, and take the expectation.

$$E\left[-\ln(\Pr(\wedge_{i\in S}\bar{B}_i))\right] \geq p\mu - p^2\frac{\delta}{2}.$$

Continue

Now choose $p = \mu/\Delta$.

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$$\Pr(\wedge_{i\in I}\bar{B}_i) \leq \Pr(\wedge_{i\in S}\bar{B}_i) \leq e^{-\frac{\mu^2}{2\Delta}}. \qquad \square$$

Brun's sieve

- X_i: the indicator random variable for B_i, for $i \in I$.
- $X := \sum_{i=1}^{m} X_i$.
- $m = m(n)$, $B_i = B_i(n)$, and $X = X(n)$.
- Let
$$S^{(r)} = \sum \Pr(B_{i_1} \wedge \cdots \wedge B_{i_r}),$$
where the sum is over all sets
$\{i_1, \ldots, i_r\} \subset \{1, 2 \ldots, m\}$.
- Let
$$X^{(r)} = X(X-1)\cdots(X-r+1).$$

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By inclusion-exclusion principle,

$$\Pr(X=0) = \Pr(\bar{B}_1 \wedge \cdots \wedge \bar{B}_m) = \sum_{r \geq 0} (-1)^r S^{(r)}.$$

Brun's sieve

Theorem: Suppose there is a constant μ so that for every fixed r,

$$\mathrm{E}\binom{X}{r} = S^{(r)} \to \frac{\mu^r}{r!}.$$

Then

$$\Pr(X = 0) \to e^{-\mu},$$

and for every t

$$\Pr(X = t) \to \frac{\mu^t}{t!}e^{-\mu}.$$

Proof

Proof: We only prove the case $t = 0$. Fix $\epsilon > 0$. Choose s so that

$$\left|\sum_{r=0}^{2s}(-1)^r\frac{\mu^r}{r!} - e^{-\mu}\right| \leq \frac{\epsilon}{2}.$$

Select n_0 so that for $n \geq n_0$,

$$|S^{(r)} - \frac{\mu^r}{r!}| \leq \frac{\epsilon}{2s(2s+1)}$$

for $0 \leq r \leq 2s$.

Continue

For such n,

$$\begin{aligned}
\Pr[X=0] &\leq \sum_{r=0}^{2s}(-1)^r S^{(r)} \\
&\leq \sum_{r=0}^{2s}(-1)^r \frac{\mu^r}{r!} + \frac{\epsilon}{2} \\
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Similarly, taking the sum to $2s+1$, we can find n_0 so that for $n \geq n_0$,

$$\Pr[X=0] \geq e^{-\mu} - \epsilon.$$

As ϵ was arbitrary $\Pr(X=0) \to e^{-\mu}$. $\square$

An application

Let $G = G(n,p)$, and EPIT represent the statement that every vertex lies in a triangle.

Theorem (a special case of Spencer's Theorem): Let $c > 0$ be fixed and let $p = p(n)$, $\mu = \mu(n)$ satisfy

$$\begin{aligned} \binom{n-1}{2}p^3 &= \mu, \\ e^{-\mu} &= \frac{c}{n}. \end{aligned}$$

Then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ satisfies } EPIT) = e^{-c}.$$

Proof

First fix $x \in V(G)$. For each unordered $y, z \neq x$ let B_{xyz} be the event that $\{x, y, z\}$ is a triangle of G. Let C_x be the event $\wedge_{y,z}\bar{B}_{xyz}$ and X_x the corresponding indicator random variable. Apply Janson's Inequality to bound $\mathrm{E}(X_x) = \Pr(\wedge_{y,z}\bar{B}_{xyz})$.

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$$\Delta = \sum_{y,z,z'} \Pr(B_{xyz} \wedge B_{xyz'}) = O(n^3 p^5) = o(1)$$

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since $p = n^{-2/3+o(1)}$. Thus

$$\mathrm{E}(X_x) \approx e^{-\mu} = \frac{c}{n}.$$

continue

Let $X = \sum_x X_x$, which is the number of vertices x no lying a triangle.

$$\mathrm{E}(X) = \sum_x \mathrm{E}(X_x) \to c.$$

We need to show that the Poisson Paradigm applies to X.

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Fix r and consider

$$\mathrm{E}\binom{X}{r} = S^{(r)} = \sum \Pr(C_{x_1} \wedge \cdots \wedge C_{x_r}),$$

where the sum is over all sets $\{x_1, \ldots, x_r\}$.

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where the sum is over all sets $\{x_1, \ldots, x_r\}$. Note

$$C_{x_1} \wedge \cdots \wedge C_{x_r} = \wedge_{1 \le i \le r, y, z} \overline{B_{x_i y z}}.$$

Continue

We apply Janson's Inequality again.

$$\sum \Pr(B_{x_iyz}) = p^3 \left(r \binom{n-1}{2} - O(n) \right) = r\mu + O(n^{-1+o(1)}).$$

As before Δ is p^5 times the number of pairs $x_iyz \sim x_jyz$;
$\Delta = O(n^3p^5) = o(1)$.

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$$\Pr(C_{x_1} \wedge \cdots \wedge C_{x_r}) \sim e^{-r\mu}$$

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Applying Brun's Sieve method, we have $\Pr(X = 0) \to e^{-c}$.

Generalization

A sufficient condition for Janson's Inequality:

- I: a dependency digraph; if for each $i \in I$ the event B_i is mutually independent of $\{B_j : i \not\sim j\}$.
- $\Delta := \sum_{i \sim j} \Pr(B_i \wedge B_j)$.
- For all $J \subset I$, $i \notin J$,

$$\Pr(B_i \mid \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i).$$

- For $J \subset I$, $i, k \notin J$,

$$\Pr(B_i \mid B_k \wedge \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).$$

Then Janson's inequality holds.

Suen's theorem

An binary relation $\sim$ on I is **superdenpendency digraph** if the following holds:
Suppose that $J_1, J_2 \subset I$ are disjoint subsets so that there is no edge between J_1 and J_2. Let B^1 be any Boolean combination of the events $\{B_j\}_{j \in J_1}$ and B^2 be any Boolean combination of the events $\{B_j\}_{j \in J_2}$. Then B^1 and B^2 are independent.

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Theorem [Suen]: Under the above conditions,

$$\left|\Pr(\wedge_{i \in I} \bar{B}_i) - M\right| \leq M(e^{\sum_{i \sim j} y(i,j)} - 1),$$

where
$y_{i,j} = (\Pr(B_i \wedge B_j) + \Pr(B_i)\Pr(B_j)) \prod_{l \sim i \text{ or } l \sim j} (1 - \Pr(B_l))^{-1}$.