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- Let \( X_i \) be the random indicator of the event \( B_i \) and \( X = \sum_{i \in I} X_i \). If \( \Pr(B_i) \)'s are small and “mostly independent”, then one may expect \( X \) follows “Poisson-like distribution”. In particular,

\[
\Pr(X = 0) \approx e^{-\mathbb{E}(X)}.
\]
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- \( M = \prod_{i \in I} \Pr(\overline{B_i}) \).
The Janson inequality: Assume all $\Pr(B_i) \leq \epsilon$. Then

$$M \leq \Pr(\bigwedge_{i \in I} \bar{B}_i) \leq M e^{\frac{\Delta}{2(1-\epsilon)}},$$

and, further,

$$\Pr(\bigwedge_{i \in I} \bar{B}_i) \leq e^{-\mu + \frac{\Delta}{2}}.$$
**Janson inequality**

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**The Extended Janson inequality**: If further $\Delta \geq \mu$, then

$$\Pr(\bigwedge_{i \in I} \bar{B}_i) \leq e^{\frac{-\mu^2}{2\Delta}}.$$
Proof given by Boppana and Spencer: We will use the following correlation inequality.

- For all $J \subset I$, $i \notin J$,

  $$\Pr(B_i \mid \bigwedge_{j \in J} \bar{B}_j) \leq \Pr(B_i).$$

- For $J \subset I$, $i, k \notin J$,

  $$\Pr(B_i \mid B_k \wedge \bigwedge_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).$$
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  \Pr(B_i \mid B_k \land \bigwedge_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).
  \]

Order the index set $I = \{1, 2, \ldots, m\}$.

\[
\Pr(\bigwedge_{i \in I} \bar{B}_i) = \prod_{i=1}^{m} \Pr(\bar{B}_i \mid \bigwedge_{1 \leq j < i} \bar{B}_j) \geq \prod_{i=1}^{m} \Pr(\bar{B}_i).
\]
For a given $i$ renumber, for convenience, so that $i \sim j$ for $1 \leq j \leq d$ and not for $d + 1 \leq j < i$. Let $A = B_i$, $B = \bar{B}_1 \land \cdots \land \bar{B}_d$, and $C = \bar{B}_{d+1} \land \cdots \land \bar{B}_{i-1}$,

$$
\Pr(B_i \mid \land_{1 \leq j < i} \bar{B}_j) = \Pr(A \mid B \land C) \\
\leq \Pr(A \land B \mid C) \\
= \Pr(A \mid C)\Pr(B \mid A \land C).
$$

Note $\Pr(A \mid C) = \Pr(A)$ and

$$
\Pr(B \mid A \land C) \geq 1 - \sum_{j=1}^{d} \Pr(B_j \mid B_i \land C) \geq 1 - \sum_{j=1}^{d} \Pr(B_j \mid B_i).
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\]

\[
\Pr(B_i \mid \land_{1 \leq j < i} \overline{B}_j) \geq \Pr(B_i) - \sum_{j=1}^{d} \Pr(B_j \land B_i).
\]
\[
\Pr(\overline{B}_i \mid \wedge_{1 \leq j < i} \overline{B}_j) \leq \Pr(\overline{B}_i) + \sum_{j=1}^{d} \Pr(B_j \wedge B_i)
\]

\[
\leq \Pr(\overline{B}_i) \left( 1 + \frac{1}{1 - \epsilon} \sum_{j=1}^{d} \Pr(B_j \wedge B_i) \right)
\]

\[
\leq \Pr(\overline{B}_i) e^{\frac{1}{1 - \epsilon} \sum_{j=1}^{d} \Pr(B_j \wedge B_i)}.
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\]
Plug it into \(\Pr(\land_{i \in I} \bar{B}_i) = \prod_{i=1}^{m} \Pr(\bar{B}_i \mid \land_{1 \leq j < i} \bar{B}_j)\); we get the first inequality. The second inequality use the following estimation.
\[
\Pr(\bar{B}_i \mid \land_{1 \leq j < i} \bar{B}_j) \leq \Pr(\bar{B}_i) + \sum_{j=1}^{d} \Pr(B_j \land B_i)
\]
\[
\leq e^{-\Pr(B_i)} + \sum_{j=1}^{d} \Pr(B_j \land B_i).
\]
Proof of second Theorem

From the Jansen inequality, we have

\[- \ln(\Pr(\bigwedge_{i \in I} \bar{B}_i)) \geq \sum_{i \in I} \Pr(B_i) - \frac{1}{2} \sum_{i \sim j} \Pr(B_i \land B_j).\]
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For any set \( S \subset I \), the same inequality applied to \( \{B_i\}_{i \in S} \):

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Now take $S$ be a random subset of $I$ given by $\Pr(i \in S) = p$, and take the expectation.

\[E \left[- \ln(\Pr(\bigwedge_{i \in S} \bar{B}_i))\right] \geq p\mu - p^2 \frac{\delta}{2}.\]
Now choose $p = \mu / \Delta$.

\[ E \left[-\ln(\Pr(\bigwedge_{i \in S} \bar{B}_i))\right] \geq \frac{\mu^2}{2\Delta}. \]
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Then there is a specific \( S \subset I \) for which

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Then there is a specific $S \subset I$ for which

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$$\Pr(\bigcap_{i \in S} \bar{B}_i) \leq e^{-\frac{\mu^2}{2\Delta}}.$$ 

$$\Pr(\bigcap_{i \in I} \bar{B}_i) \leq \Pr(\bigcap_{i \in S} \bar{B}_i) \leq e^{-\frac{\mu^2}{2\Delta}}.$$
Brun’s sieve

- $X_i$: the indicator random variable for $B_i$, for $i \in I$.
- $X := \sum_{i=1}^{m} X_i$.
- $m = m(n)$, $B_i = B_i(n)$, and $X = X(n)$.
- Let
  \[ S^{(r)} = \sum \Pr(B_{i_1} \land \cdots \land B_{i_r}), \]
  where the sum is over all sets
  \[ \{i_1, \ldots, i_r\} \subset \{1, 2 \ldots, m\}. \]
- Let
  \[ X^{(r)} = X(X - 1) \cdots (X - r + 1). \]
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By inclusion-exclusion principle,
\[ \Pr(X = 0) = \Pr(\bar{B}_1 \land \cdots \land \bar{B}_m) = \sum_{r \geq 0} (-1)^r S^{(r)}. \]
Brun’s sieve

**Theorem:** Suppose there is a constant \( \mu \) so that for every fixed \( r \),

\[
E\left(\frac{X}{r}\right) = S(r) \rightarrow \frac{\mu^r}{r!}.
\]

Then

\[
Pr(X = 0) \rightarrow e^{-\mu},
\]

and for every \( t \)

\[
Pr(X = t) \rightarrow \frac{\mu^t}{t!}e^{-\mu}.
\]
Proof: We only prove the case $t = 0$. Fix $\epsilon > 0$. Choose $s$ so that

$$\left| \sum_{r=0}^{2s} (-1)^r \frac{\mu^r}{r!} - e^{-\mu} \right| \leq \frac{\epsilon}{2}.$$ 

Select $n_0$ so that for $n \geq n_0$,

$$|S^{(r)} - \frac{\mu^r}{r!}| \leq \frac{\epsilon}{2s(2s + 1)}$$

for $0 \leq r \leq 2s$. 
For such \( n \),

\[
\Pr[X = 0] \leq \sum_{r=0}^{2s} (-1)^r S^{(r)}
\]

\[
\leq \sum_{r=0}^{2s} (-1)^r \frac{\mu^r}{r!} + \frac{\epsilon}{2}
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\leq e^{-\mu} + \epsilon.
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\[
\leq e^{-\mu} + \epsilon.
\]

Similarly, taking the sum to \( 2s + 1 \), we can find \( n_0 \) so that for \( n \geq n_0 \),

\[
\Pr[X = 0] \geq e^{-\mu} - \epsilon.
\]

As \( \epsilon \) was arbitrary \( \Pr(X = 0) \rightarrow e^{-\mu} \).
An application

Let $G = G(n, p)$, and EPIT represent the statement that every vertex lies in a triangle.

**Theorem (a special case of Spencer’s Theorem):** Let $c > 0$ be fixed and let $p = p(n), \mu = \mu(n)$ satisfy

$$
\binom{n-1}{2} p^3 = \mu,
$$

$$
e^{-\mu} = \frac{c}{n}.
$$

Then

$$
\lim_{n \to \infty} Pr(G(n, p) \text{ satisfies } EPIT) = e^{-c}.
$$
First fix $x \in V(G)$. For each unordered $y, z \neq x$ let $B_{xyz}$ be the event that $\{x, y, z\}$ is a triangle of $G$. Let $C_x$ be the event $\wedge_{y,z} \overline{B}_{xyz}$ and $X_x$ the corresponding indicator random variable. Apply Janson’s Inequality to bound $E(X_x) = \Pr(\wedge_{y,z} \overline{B}_{xyz})$. 
Proof

First fix \( x \in V(G) \). For each unordered \( y, z \neq x \) let \( B_{xyz} \) be the event that \( \{x, y, z\} \) is a triangle of \( G \). Let \( C_x \) be the event \( \land_{y,z} \overline{B}_{xyz} \) and \( X_x \) the corresponding indicator random variable. Apply Janson’s Inequality to bound
\[
\mathbb{E}(X_x) = \text{Pr}(\land_{y,z} \overline{B}_{xyz}).
\]

\[
\Delta = \sum_{y,z,z'} \text{Pr}(B_{xyz} \land B_{xyz'}) = O(n^3 p^5) = o(1)
\]

since \( p = n^{-2/3+o(1)} \).
Proof

First fix \( x \in V(G) \). For each unordered \( y, z \neq x \) let \( B_{xyz} \) be the event that \( \{x, y, z\} \) is a triangle of \( G \). Let \( C_x \) be the event \( \bigwedge y,z \overline{B}_{xyz} \) and \( X_x \) the corresponding indicator random variable. Apply Janson’s Inequality to bound

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E(X_x) = \Pr(\bigwedge y,z \overline{B}_{xyz}).
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\[
\Delta = \sum_{y,z,z'} \Pr(B_{xyz} \land B_{xyz'}) = O(n^3 p^5) = o(1)
\]

since \( p = n^{-2/3+o(1)} \). Thus

\[
E(X_x) \approx e^{-\mu} = \frac{c}{n}.
\]
Let \( X = \sum_{x} X_{x} \), which is the number of vertices \( x \) no lying a triangle.

\[
E(X) = \sum_{x} E(X_{x}) \rightarrow c.
\]

We need to show that the Poisson Paradigm applies to \( X \).
Let \( X = \sum_x X_x \), which is the number of vertices \( x \) no lying a triangle.

\[
E(X) = \sum_x E(X_x) \rightarrow c.
\]

We need to show that the Poisson Paradigm applies to \( X \). Fix \( r \) and consider

\[
E\left(\begin{pmatrix} X \\ r \end{pmatrix}\right) = S^{(r)} = \sum \Pr(C_{x_1} \land \cdots \land C_{x_r}),
\]

where the sum is over all sets \( \{x_1, \ldots, x_r\} \).
Let $X = \sum_x X_x$, which is the number of vertices $x$ no lying a triangle.

$$E(X) = \sum_x E(X_x) \rightarrow c.$$  

We need to show that the Poisson Paradigm applies to $X$. Fix $r$ and consider

$$E\left( \binom{X}{r} \right) = S^{(r)} = \sum \Pr(C_{x_1} \land \cdots \land C_{x_r}),$$

where the sum is over all sets $\{x_1, \ldots, x_r\}$. Note

$$C_{x_1} \land \cdots \land C_{x_r} = \land_{1 \leq i \leq r, y, z} B^\infty_{x_i y z}.$$
We apply Janson’s Inequality again.

$$\sum \Pr(B_{x_iyz}) = p^3 \left( r \left( \frac{n-1}{2} \right) - O(n) \right) = r \mu + O(n^{-1+o(1)}).$$

As before $\Delta$ is $p^5$ times the number of pairs $x_iyz \sim x_jyz$; $\Delta = O(n^3 p^5) = o(1)$. 
We apply Janson’s Inequality again.

\[
\sum \Pr(B_{x_iyz}) = p^3 \left( r \left( \frac{n-1}{2} \right) - O(n) \right) = r\mu + O(n^{-1+o(1)}). 
\]

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\[
\Pr(C_{x_1} \land \cdots \land C_{x_r}) \sim e^{-r\mu} 
\]

\[
E\left( \binom{X}{r} \right) \approx \binom{n}{r} e^{-r\mu} \approx \frac{c^r}{r!}.
\]
We apply Janson’s Inequality again.

\[ \sum \Pr(B_{xyz}) = p^3 \left( r \left( \frac{n - 1}{2} \right) - O(n) \right) = r \mu + O(n^{-1+o(1)}). \]

As before \( \Delta \) is \( p^5 \) times the number of pairs \( x_iyz \sim x_jyz \);
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\[ \Pr(C_{x_1} \land \cdots \land C_{x_r}) \sim e^{-r \mu} \]
\[ E \left( \begin{pmatrix} X \\ r \end{pmatrix} \right) \approx \binom{n}{r} e^{-r \mu} \approx \frac{c^r}{r!}. \]

Applying Brun’s Sieve method, we have \( \Pr(X = 0) \rightarrow e^{-c} \).
Generalization

A sufficient condition for Janson’s Inequality:

- $I$: a dependency digraph; if for each $i \in I$ the event $B_i$ is mutually independent of $\{B_j : i \not\sim j\}$.
- $\Delta \equiv \sum_{i \sim j} \Pr(B_i \land B_j)$.
- For all $J \subset I$, $i \notin J$,
  \[
  \Pr(B_i \mid \land_{j \in J} \bar{B}_j) \leq \Pr(B_i).
  \]
- For $J \subset I$, $i, k \notin J$,
  \[
  \Pr(B_i \mid B_k \land \land_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).
  \]

Then Janson’s inequality holds.
An binary relation $\sim$ on $I$ is superdependency digraph if the following holds:

Suppose that $J_1, J_2 \subset I$ are disjoint subsets so that there is no edge between $J_1$ and $J_2$. Let $B^1$ be any Boolean combination of the events $\{B_j\}_{j \in J_1}$ and $B^2$ be any Boolean combination of the events $\{B_j\}_{j \in J_2}$. Then $B^1$ and $B^2$ are independent.
Suen’s theorem

An binary relation $\sim$ on $I$ is superdependency digraph if the following holds:
Suppose that $J_1, J_2 \subset I$ are disjoint subsets so that there is no edge between $J_1$ and $J_2$. Let $B^1$ be any Boolean combination of the events $\{B_j\}_{j \in J_1}$ and $B^2$ be any Boolean combination of the events $\{B_j\}_{j \in J_2}$. Then $B^1$ and $B^2$ are independent.

**Theorem [Suen]:** Under the above conditions,

$$\left| \Pr(\bigwedge_{i \in I} \bar{B}_i) - M \right| \leq M \left( e^{\sum_{i \sim j} y(i,j)} - 1 \right),$$

where

$$y_{i,j} = (\Pr(B_i \wedge B_j) + \Pr(B_i)\Pr(B_j)) \prod_{l \sim i \text{ or } l \sim j}(1 - \Pr(B_l))^{-1}. $$