Probabilistic Methods in Combinatorics Lecture 11

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- $\rho(A, \vec{x})$: Talagrand's distance from $\vec{x} \in \Omega$ to $A \subset \Omega$:

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Theorem [Talagrand's inequality]:

$$\Pr(A)(1 - \Pr(A_t)) \leq e^{-t^2/4}.$$

The distance $\rho(A, \vec{x})$

- $U(A, \vec{x}) = \{\vec{s} \in \{0,1\}^n \colon \exists \vec{y} \in A, x_i \neq y_i \Rightarrow s_i = 1\}$.

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Lemma: $\rho(A, \vec{x}) = \min_{\vec{v} \in V(A,\vec{x})} \|\vec{v}\|$.

Proof: Let $\vec{v} \in V(A, \vec{x})$ achieve this minimum. For any $\vec{s} \in V(A, \vec{x})$, we have $\vec{s} \cdot \vec{v} \geq \vec{v} \cdot \vec{v}$. Let $\vec{\alpha} = \vec{v}/\|\vec{v}\|$. We have

$$\rho(A, \vec{x}) \geq \inf_{\vec{y} \in A} \sum_{i \colon x_i \neq y_i} \alpha_i \geq \inf_{\vec{s} \in V(A,\vec{x})} \vec{s} \cdot \vec{\alpha} \geq \|\vec{v}\|.$$

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$$\rho(A, \vec{x}) \geq \inf_{\vec{y} \in A} \sum_{i:\, x_i \neq y_i} \alpha_i \geq \inf_{\vec{s} \in V(A,\vec{x})} \vec{s} \cdot \vec{\alpha} \geq \|\vec{v}\|.$$

Conversely, take any unit vector $\vec{\alpha}$. Write $\vec{v} = \sum_i \lambda_i \vec{s}_i$ for some $\vec{s}_i \in U(A, \vec{x})$, $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. Since $\|\vec{v}\| \geq \sum_i \lambda_i (\vec{\alpha} \cdot \vec{s}_i)$, we have $\alpha \cdot \vec{s}_i \leq \|\vec{v}\|$ for some i. $\square$

A general theorem

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For fixed A, consider $X = \rho(A, \vec{x})$.

$$\begin{aligned}
\Pr(\overline{A_t}) &= \Pr(X \geq t) \\
&= \Pr(e^{X^2/4} \geq e^{t^2/4}) \\
&\leq \mathrm{E}(e^{X^2/4}) e^{-t^2/4} \\
&\leq \frac{1}{\Pr(A)} e^{-t^2/4}.
\end{aligned}$$

Hence, $\Pr(A)\Pr(\overline{A_t}) \leq e^{-t^2/4}$. □

Proof

Now prove $\int_\Omega e^{\frac{1}{4}\rho^2(A,\vec{x})} d\vec{x} \leq \frac{1}{\Pr(A)}$ by induction on n.

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When $n = 1$, $\rho(A, \vec{x}) = 1$ if $\vec{x} \notin A$; and 0 if $\vec{x} \in A$.

$$\int_\Omega e^{\frac{1}{4}\rho^2(A,\vec{x})} d\vec{x} = \Pr(A) + (1 - \Pr(A))e^{1/4} \leq \frac{1}{\Pr(A)}.$$

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Assume it holds for n. For any $z \in \Omega$, write $z = (x, \omega)$ with $x \in \prod_{i=1}^n \Omega_i$ and $\omega \in \Omega_{n+1}$. Let

$$B = \{x \in \prod_{i=1}^n \Omega_i : (x, \omega) \in A \text{ for some } \omega \in \Omega_{n+1}.\}$$

$$A_\omega = \{x \in \prod_{i=1}^n \Omega_i : (x, \omega) \in A\}, \qquad \text{for } \omega \in \Omega_{n+1}.$$

Continue

Two ways to move $z = (x, \omega) \in \Omega$ to A:

- By changing ω, it reduces the problem to moving from x to B. $\vec{s} \in U(B, x) \Rightarrow (\vec{s}, 1) \in U(A, (x, \omega))$.

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Taking the convex hulls, if $\vec{s} \in V(B, x)$ and $\vec{t} \in V(A_\omega, x)$, then for any $\lambda \in [0, 1]$,

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$$\begin{aligned} \rho^2(A, (x, \omega)) &\leq (1-\lambda)^2 + \|(1-\lambda)\vec{s} + \lambda\vec{t}\|^2 \\ &\leq (1-\lambda)^2 + (1-\lambda)\|\vec{s}\|^2 + \lambda\|\vec{t}\|^2. \end{aligned}$$

Continue

Minimizing $\|\vec{s}\|$ and $\|\vec{t}\|$, we get

$$\rho^2(A,(x,\omega)) \leq (1-\lambda)^2 + (1-\lambda)\rho^2(A_\omega, x) + \lambda\rho^2(B,x).$$

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$$\begin{aligned}
&\int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx \\
&\leq e^{\frac{(1-\lambda)^2}{4}} \int_x \left(e^{\frac{1}{4}\rho^2(A_\omega,x)}\right)^\lambda \left(e^{\frac{1}{4}\rho^2(B,x)}\right)^{1-\lambda} dx \\
&\leq e^{\frac{(1-\lambda)^2}{4}} \left(\int_x e^{\frac{1}{4}\rho^2(A_\omega,x)} dx\right)^\lambda \left(\int_x e^{\frac{1}{4}\rho^2(B,x)} dx\right)^{1-\lambda} \\
&\leq e^{\frac{(1-\lambda)^2}{4}} \left(\frac{1}{\Pr(A_\omega)}\right)^\lambda \left(\frac{1}{\Pr(B)}\right)^{1-\lambda}
\end{aligned}$$

Continue

Let $r = \frac{\Pr(A_\omega)}{\Pr(B)} \leq 1$ and $f(\lambda, r) = e^{(1-\lambda)^2/4} r^{-\lambda}$. Then

$$\int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx \leq \frac{1}{\Pr(B)} f(\lambda, r).$$

Choose $\lambda = 1 + 2\ln r$ for $e^{-1/2} \leq r \leq 1$ and $\lambda = 0$ otherwise. One can show $f(\lambda, r) \leq 2 - r$. Thus,

$$\int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx \leq \frac{1}{\Pr(B)} \left(2 - \frac{\Pr(A_\omega)}{\Pr(B)}\right).$$

$$\int_w \int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx d\omega \leq \frac{1}{\Pr(B)} \left(2 - \frac{\Pr(A)}{\Pr(B)}\right) \leq \frac{1}{\Pr(A)}. \qquad \square$$

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Theorem [Alon, Krivelevich, Vu, (2002)]: For every positive integer $1 \le s \le n$, the probability that λ_s deviates from its median by more than t is at most $4e^{-\frac{t^2}{32s^2}}$. The same estimate holds for the probability that λ_{n+1-s} deviates from its median by more than t.

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It suffices to show $\mathcal{B}_{t'} \cap \mathcal{A} = \emptyset$. I.e., for any $B \in \mathcal{B}$ find an vector $\alpha = (\alpha_{ij})$, for any $A \in \mathcal{A}$, show

$$\sum_{(i,j):a_{ij} \neq b_{ij}} \alpha_{ij} \ge t' \left(\sum_{1 \le i \le j \le n} \alpha_{ij} \right)^{1/2}.$$

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- For $1 \leq i \leq n$, let

$$\alpha_{ii} = \sum_{p=1}^{s} (v_i^{(p)})^2.$$

For $1 \leq i < j \leq n$, let

$$\alpha_{ij} = 2\sqrt{\sum_{p=1}^{s} (v_i^{(p)})^2}\sqrt{\sum_{p=1}^{s} (v_j^{(p)})^2}.$$

Claim 1

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$$\sum_{1\le i\le j\le n} \alpha_{ij}^2 = \sum_{i=1}^{n}\left(\sum_{p=1}^{s}(v_i^{(p)})^2\right)$$

$$+ 4\sum_{1\le i<j\le n}\left(\sum_{p=1}^{s}(v_i^{(p)})^2\right)\left(\sum_{p=1}^{s}(v_j^{(p)})^2\right)$$

$$\le 2\left(\sum_{i=1}^{n}\sum_{p=1}^{s}(v_i^{(p)})^2\right)^2$$

$$= 2s^2.$$

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Fix $A \in \mathcal{A}$. Let $u = \sum_{p=1}^{s} c_p v^{(p)}$ be a unit vector in the span of the vectors $v(p)$ which is orthogonal to the eigenvectors of the largest $s-1$ eigenvalues of A. Then $\sum_{p=1}^{s} c_p^2 = 1$, $u'Au \leq \lambda_s(A) \leq M$, and $u'Bu \geq \lambda_s(B) \geq M + t$.

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$$
\begin{aligned}
t &\leq u'(B-A)u \\
&= \sum_{a_{ij} \neq b_{ij}} (b_{ij} - a_{ij}) \sum_{p=1}^{s} c_p v_i^{(p)} \sum_{p=1}^{s} c_p v_j^{(p)} \\
&\leq 2 \sum_{a_{ij} \neq b_{ij}} \left| \sum_{p=1}^{s} c_p v_i^{(p)} \sum_{p=1}^{s} c_p v_j^{(p)} \right| \leq 2 \sum_{a_{ij} \neq b_{ij}} \alpha_{ij}.
\end{aligned}
$$

Putting together

$$\sum_{(i,j):a_{ij}\neq b_{ij}} \alpha_{ij} \geq \frac{t}{2\sqrt{2}s} \left(\sum_{1\leq i\leq j\leq n} \alpha_{ij} \right)^{1/2}.$$

The Talagrand distance between $\mathcal{A}$ and $\mathcal{B}$ is at least $\frac{t}{2\sqrt{2}s}$.

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The Talagrand distance between $\mathcal{A}$ and $\mathcal{B}$ is at least $\frac{t}{2\sqrt{2}s}$. Applying Talagrand's inequality, we get

$$\Pr(\mathcal{A})\Pr(\mathcal{B}) \leq e^{-t^2/32s^2}.$$

Hence, $\Pr(\lambda_s \geq m + t) \leq 2e^{-t^2/32s^2}$. Similar we get $\Pr(\lambda_s \leq m - t) \leq e^{-t^2/32s^2}$. Hence

$$\Pr(|\lambda_s - m| \geq t) \leq 4e^{-t^2/32s^2}. \qquad \square$$

General applications

- $\Omega := \prod_{i=1}^{n} \Omega_i$.
- $h \colon \Omega \to \mathbb{R}$: a Lipschitz function.
- Given $f \colon N \to N$, h is f-certifiable if whenever $h(x) \geq s$ there exists $I \subset [n]$ with $|I| \leq f(s)$ so that all $y \in \Omega$ that agree with x on the coordinates I have $h(y) \geq s$.

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Example: Let $\Omega = G(n, p)$ and $h(G)$ be the number of triangles in G. Then h is f-certifiable with $f(s) = 3s$.

Theorem

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Proof: Set $A = \{x \colon h(x) \leq b - t\sqrt{f(b)}\}$. We claim for any y with $h(y) \geq b$, $y \notin A_t$.

Let I be a set of indices of size at most $f(b)$ that certifies $h(y) \geq b$. Define $\alpha_i = |I|^{-1/2}$ if $i \in I$, and 0 otherwise. For any $x \in A$, $\sum_{x_i \neq y_i} \alpha_i \geq t\sqrt{f(b)}|I|^{-1/2} \geq t$. By Talagrand's inequality,

$$\Pr(X \leq b - t\sqrt{f(b)})\Pr(X \geq b) \leq e^{-t^2/4}. \quad \square$$