Probabilistic Methods in Combinatorics
Lecture 1

Linyuan Lu

University of South Carolina

Mathematical Sciences Center at Tsinghua University

November 16, 2011 – December 30, 2011

Course information

Textbook: The probabilistic method (3rd edition) by Noga Alon and Joel H. Spencer, publisher: Wiley, 2008.

Course information

Textbook: The probabilistic method (3rd edition) by Noga Alon and Joel H. Spencer, publisher: Wiley, 2008.

Goal: We shall cover some selected topics in the textbook and some supplemental material.

Course information

Textbook: The probabilistic method (3rd edition) by Noga Alon and Joel H. Spencer, publisher: Wiley, 2008.

Goal: We shall cover some selected topics in the textbook and some supplemental material.

Lecture time: (November 16-December 30) Wednesday, Friday, 1:00pm-2:45pm

Course information

Textbook: The probabilistic method (3rd edition) by Noga Alon and Joel H. Spencer, publisher: Wiley, 2008.

Goal: We shall cover some selected topics in the textbook and some supplemental material.

Lecture time: (November 16-December 30) Wednesday, Friday, 1:00pm-2:45pm

Location: Conference room 3, floor 2.

History

Paul Erdős: 1913–1996
1525 papers
511 coauthors

History

Paul Erdős: 1913–1996
1525 papers
511 coauthors

Main contributions:

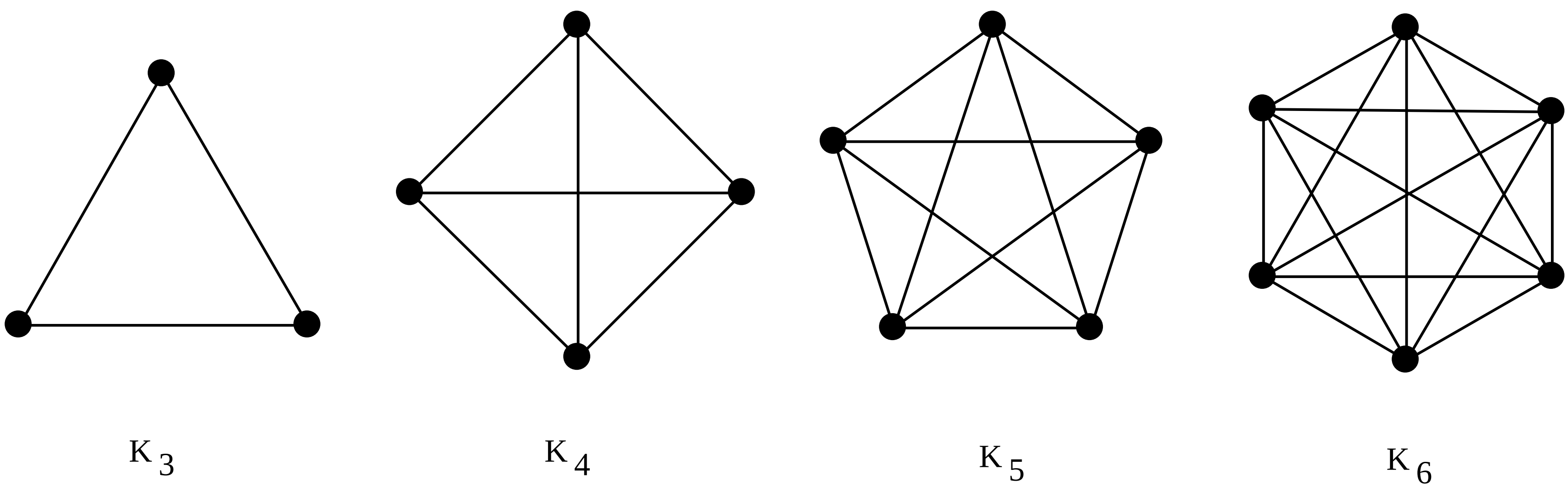
- Ramsey theory
- Probabilistic method
- Extremal combinatorics
- Additive number theory

Notation

A graph G consists of two sets V and E.

- V is the set of vertices (or nodes).
- E is the set of edges, where each edge is a pair of vertices.

Complete graphs K_n:



Ramsey number $R(k,k)$

Ramsey number $R(k,l)$: the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k or a blue K_l.

Ramsey number $R(k,k)$

Ramsey number $R(k,l)$: the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k or a blue K_l.

Major question: How large is $R(k,k)$?

Ramsey number $R(k, k)$

Ramsey number $R(k, l)$: the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k or a blue K_l.

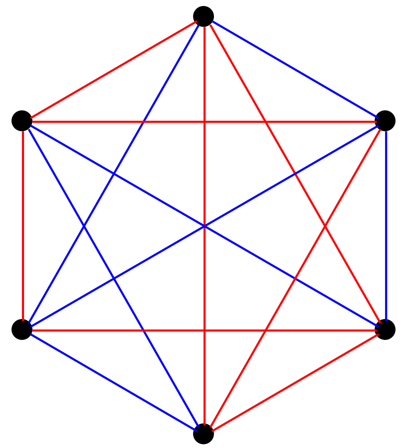
Major question: How large is $R(k, k)$?

Proposition (by Erdős): If $\binom{n}{2}2^{1-\binom{k}{2}} < 1$, then $R(k, k) > n$. Thus

$$R(k, k) > \frac{k}{e\sqrt{2}}2^{k/2}.$$

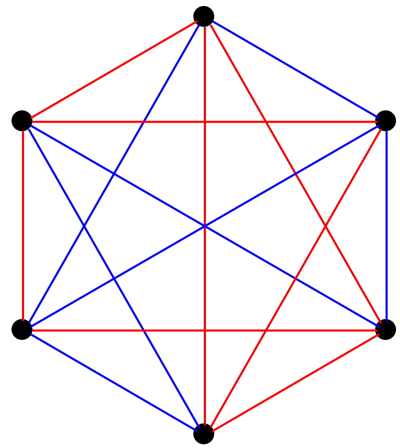
Ramsey number $R(3,3) = 6$

- If edges of K_6 are 2-colored then there exists a monochromatic triangle.

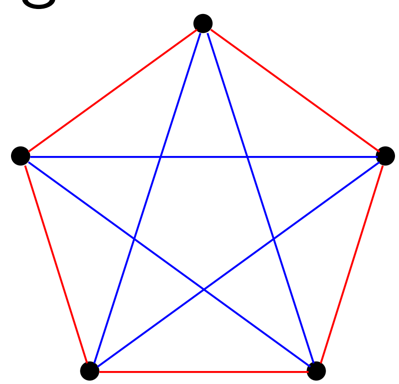


Ramsey number $R(3,3) = 6$

- If edges of K_6 are 2-colored then there exists a monochromatic triangle.



- There exists a 2-coloring of edges of K_5 with no monochromatic triangle.



Erdős' idea

To prove $R(k, k) > n$, we need construct a 2-coloring of K_n so that it contains no red K_k or blue K_n.

Erdős' idea

To prove $R(k,k) > n$, we need construct a 2-coloring of K_n so that it contains no red K_k or blue K_n.

Make the set of all 2-colorings of K_n into a probability space, then show the event " no red K_k or blue K_n" with positive probability.

Probability space

Finite probability space (Ω, P):

- $\Omega := \{s_1, s_2, \ldots, s_n\}$: a set of n elements.

Probability space

Finite probability space (Ω, P):

- $\Omega := \{s_1, s_2, \ldots, s_n\}$: a set of n elements.
- $P \colon \Omega \to [0, 1]$: a probability measure. View P as a vector $(p_1, p_2, \ldots, p_n)$, where $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$.

Probability space

Finite probability space (Ω, P):

- $\Omega := \{s_1, s_2, \ldots, s_n\}$: a set of n elements.
- $P \colon \Omega \to [0, 1]$: a probability measure. View P as a vector $(p_1, p_2, \ldots, p_n)$, where $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$.
- An event A: a subset of Ω.

Probability space

Finite probability space (Ω, P):

- $\Omega := \{s_1, s_2, \ldots, s_n\}$: a set of n elements.
- $P \colon \Omega \to [0, 1]$: a probability measure. View P as a vector $(p_1, p_2, \ldots, p_n)$, where $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$.

- An event A: a subset of Ω.
- Probability of A: $\Pr(A) = \sum_{s_i \in A} p_i$.

Probability space

Finite probability space (Ω, P):

- $\Omega := \{s_1, s_2, \dots, s_n\}$: a set of n elements.
- $P : \Omega \to [0, 1]$: a probability measure. View P as a vector $(p_1, p_2, \dots, p_n)$, where $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$.

- An event A: a subset of Ω.
- Probability of A: $\Pr(A) = \sum_{s_i \in A} p_i$.
- Two events A and B are independent if

$$\Pr(AB) = \Pr(A)\Pr(B).$$

Proof of Proposition 1:

Color every edge of K_n independently either red or blue, where each color is equally likely.

Proof of Proposition 1:

Color every edge of K_n independently either red or blue, where each color is equally likely.

For any fixed set R of k vertices, let A_R be the event that all pairs with both ends in R are either all red or all blue.

Proof of Proposition 1:

Color every edge of K_n independently either red or blue, where each color is equally likely.

For any fixed set R of k vertices, let A_R be the event that all pairs with both ends in R are either all red or all blue.

$$\Pr(A_R) = 2^{1-\binom{k}{2}}.$$

Proof of Proposition 1:

Color every edge of K_n independently either red or blue, where each color is equally likely.

For any fixed set R of k vertices, let A_R be the event that all pairs with both ends in R are either all red or all blue.

$$\Pr(A_R) = 2^{1-\binom{k}{2}}.$$

$$\Pr(\vee_R A_R) \leq \sum_R \Pr(A_R) = \binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

Proof of Proposition 1:

Color every edge of K_n independently either red or blue, where each color is equally likely.

For any fixed set R of k vertices, let A_R be the event that all pairs with both ends in R are either all red or all blue.

$$\Pr(A_R) = 2^{1-\binom{k}{2}}.$$

$$\Pr(\vee_R A_R) \le \sum_R \Pr(A_R) = \binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

Hence $\Pr(\wedge_R \bar{A}_R) = 1 - \Pr(\vee_R A_R) > 0.$

Estimation of n

Since $\binom{n}{k} \leq \frac{1}{e}\left(\frac{en}{k}\right)^k$ for all $n \geq k \geq 1$, we have

$$\begin{aligned} \binom{n}{k} 2^{1-\binom{k}{2}} &\leq \frac{1}{e}\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} \\ &\leq \frac{2}{e}\left(\frac{en}{k2^{(k-1)/2}}\right)^k \\ &< 1 \end{aligned}$$

provided $n \leq \frac{k}{e\sqrt{2}} 2^{k/2}$.

Estimation of n

Since $\binom{n}{k} \leq \frac{1}{e}\left(\frac{en}{k}\right)^k$ for all $n \geq k \geq 1$, we have

$$\begin{aligned}\binom{n}{k} 2^{1-\binom{k}{2}} &\leq \frac{1}{e}\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} \\ &\leq \frac{2}{e}\left(\frac{en}{k2^{(k-1)/2}}\right)^k \\ &< 1\end{aligned}$$

provided $n \leq \frac{k}{e\sqrt{2}} 2^{k/2}$.

Hence,

$$R(k,k) > \frac{k}{e\sqrt{2}} 2^{k/2}. \qquad \square$$

How good is the bound?

Erdős [1947]:

$$R(k,k) > (1+o(1))\frac{1}{e\sqrt{2}}k2^{k/2}.$$

How good is the bound?

Erdős [1947]:

$$R(k,k) > (1+o(1))\frac{1}{e\sqrt{2}}k2^{k/2}.$$

Spencer [1990]:

$$R(k,k) > (1+o(1))\frac{1}{e}k2^{k/2}.$$

How good is the bound?

Erdős [1947]:

$$R(k,k) > (1+o(1))\frac{1}{e\sqrt{2}}k2^{k/2}.$$

Spencer [1990]:

$$R(k,k) > (1+o(1))\frac{1}{e}k2^{k/2}.$$

Spencer [1975] (using Lovasz Local Lemma)

$$R(k,k) > (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}.$$

Upper bound of $R(k, k)$

A trivial bound:

$$R(k, k) \leq \binom{2k - 2}{k - 1}.$$

Upper bound of $R(k,k)$

A trivial bound:

$$R(k,k) \leq \binom{2k-2}{k-1}.$$

Thomason [1988]:

$$R(k,k) \leq k^{-1/2+c/\sqrt{\log k}} \binom{2k-2}{k-1}.$$

Upper bound of $R(k,k)$

A trivial bound:

$$R(k,k) \leq \binom{2k-2}{k-1}.$$

Thomason [1988]:

$$R(k,k) \leq k^{-1/2+c/\sqrt{\log k}} \binom{2k-2}{k-1}.$$

Colon [2009]:

$$R(k,k) \leq k^{-C\frac{\log k}{\log\log k}} \binom{2k-2}{k-1}.$$

Diagonal Ramsey Problem

Erdős problems:

- \$100 for proving the limit $\lim_{k\to\infty} R(k,k)^{1/k}$ exists.

Diagonal Ramsey Problem

Erdős problems:

- \$100 for proving the limit $\lim_{k\to\infty} R(k,k)^{1/k}$ exists.
- \$250 for determining the value of $\lim_{k\to\infty} R(k,k)^{1/k}$ if it exists.

Diagonal Ramsey Problem

Erdős problems:

- \$100 for proving the limit $\lim_{k\to\infty} R(k,k)^{1/k}$ exists.
- \$250 for determining the value of $\lim_{k\to\infty} R(k,k)^{1/k}$ if it exists.

If $\lim_{k\to\infty} R(k,k)^{1/k}$ exists, then it is between $\sqrt{2}$ to 4.

Tournament

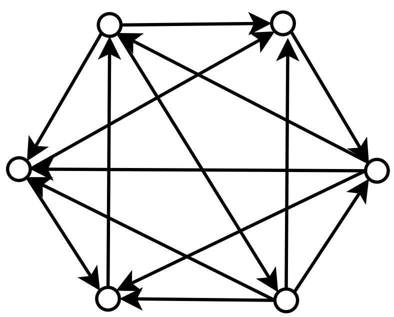
- V: a set of n players.

Tournament

- V: a set of n players.
- (x, y) means player x beats y.

Tournament

- V: a set of n players.
- (x, y) means player x beats y.
- **Tournament on V:** an orientation $T = (V, E)$ of complete graphs on V. For each pair of plays x and y, either (x, y) or (y, x) is in E.



We say T has **property** S_k if for every set of k players there is one beats all.

A question

Question (by Schütte): Is there a tournament satisfying the property S_k?

A question

Question (by Schütte): Is there a tournament satisfying the property S_k?

Theorem (Erdős [1963]) If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k.

A question

Question (by Schütte): Is there a tournament satisfying the property S_k?

Theorem (Erdős [1963]) If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k.

Proof: Consider a random tournament on V. For each pair x and y, the choice of (x, y) and (y, x) is equally likely.

A question

Question (by Schütte): Is there a tournament satisfying the property S_k?

Theorem (Erdős [1963]) If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k.

Proof: Consider a random tournament on V. For each pair x and y, the choice of (x, y) and (y, x) is equally likely.

- K: a fixed subset of size k of V.
- A_K: the event that there is no vertex that beats all the members of K.

A question

Question (by Schütte): Is there a tournament satisfying the property S_k?

Theorem (Erdős [1963]) If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k.

Proof: Consider a random tournament on V. For each pair x and y, the choice of (x, y) and (y, x) is equally likely.

- K: a fixed subset of size k of V.
- A_K: the event that there is no vertex that beats all the members of K.

$$\Pr(A_K) = (1-2^{-k})^{n-k}.$$

Proof continues

$$\Pr\left(\vee_{K\in\binom{V}{k}} A_K\right) \leq \sum_{K\in\binom{V}{k}} \Pr(A_K)$$
$$= \binom{n}{k}(1-2^{-k})^{n-k} < 1.$$

Therefore, with positive probability, no event A_K occurs; that is, there is a tournament on n vertices that has the property S_k.

Estimation of n

Let $f(k)$ denote the minimum possible number of vertices of a tournament that has the property S_k.
On one hand, since $\binom{n}{k} < (en/k)^k$ and $(1-2^{-k})^{n-k} < 2^{(n-k)/2^k}$, we have

$$f(k) \leq (1+o(1)) \ln 2 \cdot k^2 \cdot 2^k.$$

On the other hand, **Szekeres** proved

$$f(k) \geq c_1 k 2^k.$$

Random variable

- (Ω, P): a probability space.

Random variable

- (Ω, P): a probability space.
- $X : \Omega \to \mathbb{R}$: a random variable.

Random variable

- (Ω, P): a probability space.
- $X : \Omega \to \mathbb{R}$: a random variable.
- The **expectation** of X, denoted by $\mathrm{E}(X)$, is defined as

$$\mathrm{E}(X) = \sum_{v \in \Omega} X(v) p_v.$$

Random variable

- (Ω, P): a probability space.
- $X : \Omega \to \mathbb{R}$: a random variable.
- The **expectation** of X, denoted by $\mathrm{E}(X)$, is defined as

$$\mathrm{E}(X) = \sum_{v \in \Omega} X(v) p_v.$$

Linearity of expectation:

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y).$$

Dominating set

A **dominating set** of a graph $G = (V, E)$ is a set $U \subseteq V$ such that vertex $v \in V - U$ has at least one neighbor in U.

Dominating set

A **dominating set** of a graph $G = (V, E)$ is a set $U \subseteq V$ such that vertex $v \in V - U$ has at least one neighbor in U.

Theorem: Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of at most $\frac{1+\ln(\delta+1)}{\delta+1}n$.

Proof

- $p \in [0, 1]$: a probability chosen later.

Proof

- $p \in [0, 1]$: a probability chosen later.
- X: a random set, whose vertex is picked randomly and independently with probability p.

Proof

- $p \in [0,1]$: a probability chosen later.
- X: a random set, whose vertex is picked randomly and independently with probability p.
- $Y := Y_X$: the set of vertices in $V - X$ that do not have any neighbor in X.

Proof

- $p \in [0, 1]$: a probability chosen later.
- X: a random set, whose vertex is picked randomly and independently with probability p.
- $Y := Y_X$: the set of vertices in $V - X$ that do not have any neighbor in X.

$$E(|X|) = \sum_v \Pr(v \in X) = np.$$

Proof

- $p \in [0, 1]$: a probability chosen later.
- X: a random set, whose vertex is picked randomly and independently with probability p.
- $Y := Y_X$: the set of vertices in $V - X$ that do not have any neighbor in X.

$$E(|X|) = \sum_v \Pr(v \in X) = np.$$

$$\begin{aligned} E(|Y|) &= \sum_v \Pr(v \in Y) \\ &\leq n(1-p)^{\delta+1}. \end{aligned}$$

continue

Let $U = X \cup Y_X$. The set U is clearly a dominating set.

continue

Let $U = X \cup Y_X$. The set U is clearly a dominating set. We have

$$\begin{aligned} \mathrm{E}(|U|) &= \mathrm{E}(X) + \mathrm{E}(Y) \\ &\leq np + n(1-p)^{\delta+1} \\ &\leq n(p + e^{-p(\delta+1)}). \end{aligned}$$

Choose $p = \frac{\ln(\delta+1)}{\delta+1}$ to minimize the upper bound. There is a dominating set of size at most

$$\frac{1+\ln(\delta+1)}{\delta+1} n.$$

Hypergraphs

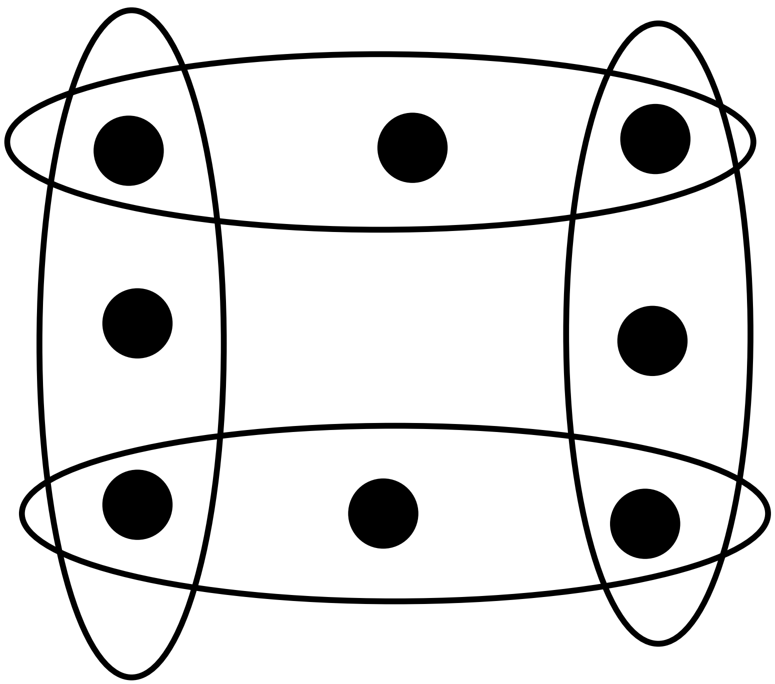
$H = (V, E)$ is an r-uniform hypergraph (r-graph, for short).

- V: the set of vertices
- E: the set of edges, each edge has cardinality r.

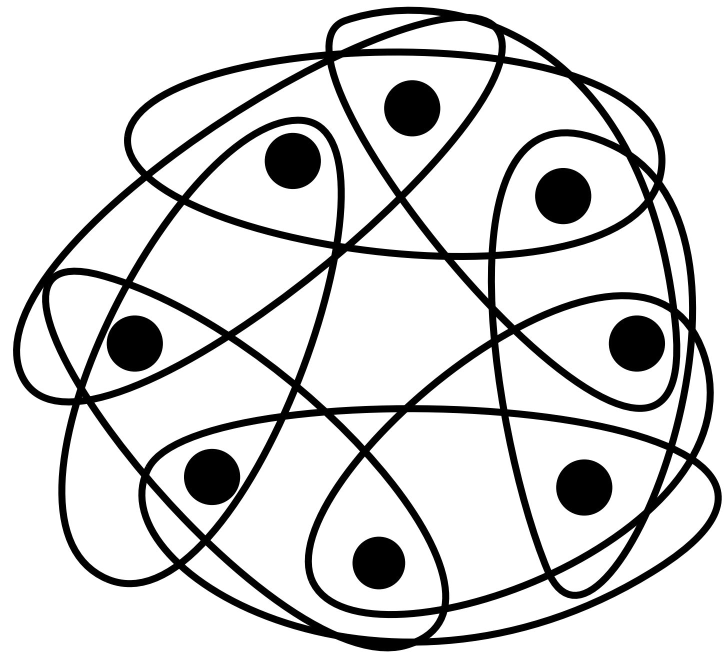
Hypergraphs

$H = (V, E)$ is an r-uniform hypergraph (r-graph, for short).

- V: the set of vertices
- E: the set of edges, each edge has cardinality r.



A 3-uniform loose cycle



A 3-uniform tight cycle

Property B problem

We say a r-uniform hypergraph H has **property B** if there is a two-coloring of V such that no edge is monochromatic.

Property B problem

We say a r-uniform hypergraph H has **property B** if there is a two-coloring of V such that no edge is monochromatic.

Let $m(r)$ denote the minimum possible number of edges of an r-uniform hypergraph that does not have property B.

Property B problem

We say a r-uniform hypergraph H has **property B** if there is a two-coloring of V such that no edge is monochromatic.

Let $m(r)$ denote the minimum possible number of edges of an r-uniform hypergraph that does not have property B.

Proposition [Erdős (1963)] Every r-uniform hypergraph with less than 2^{r-1} edges has property B. Therefore $m(r) \geq 2^{r-1}$.

Proof

Let H be an r-uniform hypergraph with less than 2^{r-1} edges. Color V randomly by two colors. For each edge $e \in E$, let A_e be the event that e is monochromatic.

$$\Pr(A_e) = 2^{1-r}.$$

Proof

Let H be an r-uniform hypergraph with less than 2^{r-1} edges. Color V randomly by two colors. For each edge $e \in E$, let A_e be the event that e is monochromatic.

$$\Pr(A_e) = 2^{1-r}.$$

Therefore,

$$\Pr\left(\vee_{e\in E} A_e\right) \leq \sum_{e\in E} \Pr(A_e) < 1.$$

There is a two-coloring without monochromatic edges. □

Upper bound

Theorem (Erdős [1964]): $m(r) < (1 + o(1))\frac{e \ln 2}{4} r^2 2^r$.

Upper bound

Theorem (Erdős [1964]): $m(r) < (1 + o(1))\frac{e \ln 2}{4} r^2 2^r$.

Proof: Fix V with n points. Let χ be a coloring of V with a points in one color, $b = n - a$ points in the other. Let $S \subset V$ be a uniformly selected r-set.

Upper bound

Theorem (Erdős [1964]): $m(r) < (1 + o(1))\frac{e \ln 2}{4} r^2 2^r$.

Proof: Fix V with n points. Let χ be a coloring of V with a points in one color, $b = n - a$ points in the other. Let $S \subset V$ be a uniformly selected r-set. Then

$$\Pr(S \text{ is monochromatic under } \chi) = \frac{\binom{a}{r} + \binom{b}{r}}{\binom{n}{r}}.$$

Upper bound

Theorem (Erdős [1964]): $m(r) < (1 + o(1))\frac{e \ln 2}{4} r^2 2^r$.

Proof: Fix V with n points. Let χ be a coloring of V with a points in one color, $b = n - a$ points in the other. Let $S \subset V$ be a uniformly selected r-set. Then

$$\Pr(S \text{ is monochromatic under } \chi) = \frac{\binom{a}{r} + \binom{b}{r}}{\binom{n}{r}}.$$

Assume $n = 2k$ is even. Then $\binom{a}{r} + \binom{b}{r}$ reaches the minimum when $a = b = k$. Thus

$$\Pr(S \text{ is monochromatic under } \chi) \geq \frac{2\binom{k}{r}}{\binom{n}{r}}.$$

continue

- Let $p := \frac{2\binom{k}{r}}{\binom{n}{r}}$.

continue

- Let $p := \frac{2\binom{k}{r}}{\binom{n}{r}}$.
- Pick m r-edges $S_1, \ldots, S_m$ uniformly and independently from $\binom{V}{r}$.

continue

- Let $p := \frac{2\binom{k}{r}}{\binom{n}{r}}$.
- Pick m r-edges $S_1, \ldots, S_m$ uniformly and independently from $\binom{V}{r}$.
- Let $H = (V, E)$ where $E = \{S_1, \ldots, S_m\}$.

continue

- Let $p := \frac{2\binom{k}{r}}{\binom{n}{r}}$.
- Pick m r-edges $S_1, \ldots, S_m$ uniformly and independently from $\binom{V}{r}$.
- Let $H = (V, E)$ where $E = \{S_1, \ldots, S_m\}$.

For each coloring χ, let A_χ be the event that none of S_i are monochromatic.

$$\Pr(A_\chi) \leq (1 - p)^m.$$

continue

- Let $p := \frac{2\binom{k}{r}}{\binom{n}{r}}$.
- Pick m r-edges $S_1, \ldots, S_m$ uniformly and independently from $\binom{V}{r}$.
- Let $H = (V, E)$ where $E = \{S_1, \ldots, S_m\}$.

For each coloring χ, let A_χ be the event that none of S_i are monochromatic.

$$\Pr(A_\chi) \leq (1-p)^m.$$

$$\Pr(\vee_\chi A_\chi) \leq \sum_\chi \Pr(A_\chi) \leq 2^n (1-p)^m.$$

continue

Choose $m = \lceil \frac{n \ln 2}{p} \rceil$. Then $2^n(1-p)^m < 1$. There is a positive probability that H does not have property B.

continue

Choose $m = \lceil \frac{n \ln 2}{p} \rceil$. Then $2^n(1-p)^m < 1$. There is a positive probability that H does not have property B. Hence,

$$m(r) \leq \lceil \frac{n \ln 2}{p} \rceil.$$

continue

Choose $m = \lceil \frac{n \ln 2}{p} \rceil$. Then $2^n(1-p)^m < 1$. There is a positive probability that H does not have property B. Hence,

$$m(r) \leq \lceil \frac{n \ln 2}{p} \rceil.$$

$$\begin{aligned} p &= \frac{2\binom{k}{r}}{\binom{n}{r}} \\ &= 2^{1-r} \prod_{i=0}^{r-1} \frac{n-2i}{n-i} \\ &\approx 2^{1-r} e^{-r^2/2n}. \end{aligned}$$

Optimization

Choose $n = \frac{r^2}{2}$ to minimize n/p.

Optimization

Choose $n = \frac{r^2}{2}$ to minimize n/p. We get

$$\begin{aligned} m &= \lceil \frac{n \ln 2}{p} \rceil \\ &\approx (\ln 2) 2^{r-1} n e^{r^2/2n} \\ &\approx \frac{e \ln 2}{4} r^2 2^r. \end{aligned}$$

Optimization

Choose $n = \frac{r^2}{2}$ to minimize n/p. We get

$$\begin{aligned} m &= \lceil \frac{n \ln 2}{p} \rceil \\ &\approx (\ln 2) 2^{r-1} n e^{r^2/2n} \\ &\approx \frac{e \ln 2}{4} r^2 2^r. \end{aligned}$$

Hence $m(r) < (1 + o(1)) \frac{e \ln 2}{4} r^2 2^r$. $\square$

Property B problem

Beck [1978]:

$$m(r) \geq r^{1/3-\epsilon} 2^r.$$

Property B problem

Beck [1978]:

$$m(r) \geq r^{1/3-\epsilon} 2^r.$$

Radhakrishnan-Srinivasan [2000]: (best lower bound)

$$m(r) \geq \Omega\left(\left(\frac{r}{\ln r}\right)^{1/2} 2^r\right).$$

Property B problem

Beck [1978]:

$$m(r) \geq r^{1/3-\epsilon}2^r.$$

Radhakrishnan-Srinivasan [2000]: (best lower bound)

$$m(r) \geq \Omega\left(\left(\frac{r}{\ln r}\right)^{1/2} 2^r\right).$$

Theorem (Erdős [1964]): (best upper bound)

$$m(r) < (1+o(1))\frac{e \ln 2}{4} r^2 2^r.$$

Property B problem

Beck [1978]:

$$m(r) \geq r^{1/3-\epsilon} 2^r.$$

Radhakrishnan-Srinivasan [2000]: (best lower bound)

$$m(r) \geq \Omega\left(\left(\frac{r}{\ln r}\right)^{1/2} 2^r\right).$$

Theorem (Erdős [1964]): (best upper bound)

$$m(r) < (1 + o(1)) \frac{e \ln 2}{4} r^2 2^r.$$

$m(2) = 3$, $m(3) = 7$, $20 \leq m(4) \leq 23$.