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NOTE

An Upper Bound for the Turán Number $t_3(n, 4)$

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Let $t_r(n, r+1)$ denote the smallest integer m such that every r-uniform hypergraph on n vertices with $m+1$ edges must contain a complete graph on $r+1$ vertices. In this paper, we prove that

$$\lim_{n\to\infty} \frac{t_3(n,4)}{\binom{n}{3}} \leqslant \frac{3+\sqrt{17}}{12} = 0.593592....$$

1. INTRODUCTION

For an r-uniform hypergraph H (or r-graph, for short), we denote by $t_r(n, H)$ the smallest integer m such that every r-graph on n vertices with $m+1$ edges must contain H as a subgraph. When H is a complete graph on k vertices, we write $t_r(n, k) = t_r(n, H)$. In 1941, Turán [10] determined the Turán number $t_2(n, k)$ for 2-graphs and he asked the problem of determining the limit

$$\lim_{n\to\infty} \frac{t_r(n,k)}{\binom{n}{r}},$$

for $2 < r < k$. For this problem, Erdős offered \$1000 in honor of Paul Turán (see [1, 10]). Since 1941, the above problem has remained open, even for the first non-trivial case of $r = 3$ and $k = 4$. The exact value for Turán number $t_3(n, 4)$ is conjectured [10] as follows:

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Conjecture.

$$t_3(n,4)=\begin{cases} k^2(5k-3)/2 & \text{if} \quad n=3k,\\ k(5k^2+2k-1)/2 & \text{if} \quad n=3k+1,\\ k(k+1)(5k+2)/2 & \text{if} \quad n=3k+2.\end{cases}$$

For small values of n, the conjectured values for $t_3(n,4)$, $n \leqslant 13$, have been verified [8]. The above conjecture, if true, would give

$$\lim_{n\to\infty} \frac{t_3(n,4)}{\binom{n}{3}}=\frac{5}{9}.$$

For the lower bound, Kostochka [5] gave several different constructions which achieve the conjectured value for $t_3(n,4)$. For the upper bound for $t_3(n,4)/\binom{n}{3}$, de Caen [3] gave an upper bound of $0.6213\cdots$ which is the real root of $9x^3-33x^2+46x-18$. The best upper bound is due to Giraud (unpublished, see [3]) who proved

$$\lim_{n\to\infty} \frac{t_3(n,4)}{\binom{n}{3}} \leqslant \frac{\sqrt{21}-1}{6}=0.5971....$$

We will show the following:

THEOREM 1.

$$\lim_{n\to\infty} \frac{t_3(n,4)}{\binom{n}{3}} \leqslant \frac{3+\sqrt{17}}{12}=0.5936...$$

For general r-graphs, de Caen [2], Sidorenko [6], Tazawa and Shirakura [9] proved

$$\lim_{n\to\infty} \frac{t_r(n,r+1)}{\binom{n}{r}} \leqslant 1-\frac{1}{r}.$$

Giraud (unpublished) improved this upper bound to

$$\lim_{n\to\infty} \frac{t_r(n,r+1)}{\binom{n}{r}} \leqslant 1-\frac{2}{r(1+\sqrt{r/(r+4)})} \qquad \text{for odd } r.$$

By similar methods as in the proof of Theorem 1, we have the following improvement:

THEOREM 2. *For any odd* $r \geqslant 3$, *we have*

$$\lim_{n\to\infty} \frac{t_r(n,r+1)}{\binom{n}{r}} \leqslant 1-\frac{5r+12-\sqrt{9r^2+24r}}{2r(r+3)}.$$

We remark that the best lower bound was due to A. Sidorenko [7],

$$\lim_{n\to\infty} \frac{t_r(n, r+1)}{\binom{n}{r}} \geqslant 1-(0.5+o_r(1))\,\frac{\log r}{r},$$

where $o_r(1)$ denotes a quantity that goes to 0 as r approaches infinity. For smaller even $r=2s$, several constructions (due to Giraud, D. de Caen, D. L. Kreher, and J. Wiseman, see [4]) gave better lower bounds. They showed

$$\lim_{n\to\infty} \frac{t_{2s}(n, 2s+1)}{\binom{n}{2s}} \geqslant 1-\frac{1}{4}-4^{-s}, \qquad \text{for all} \quad s\geqslant 1.$$

2. PRELIMINARIES

In this section, we consider a 3-graph $G=(V, E)$ which contains no complete 3-graph on 4 vertices as a subgraph. We say that G is Δ_4-free. Let e be the number of edges and $\bar{e}$ be the number of non-edges in G. We first introduce some definitions and notations.

Let d_{ij} denote the number of edges (triples) which are incident with both vertices i and j. Similarly, we denote by $\bar{d}_{ij}$ the number of non-edges which are incident with both vertices i and j. Clearly $d_{ij}+\bar{d}_{ij}=n-2$. Here are two basic equations,

$$3e=\sum_{\{i,\, j\}} d_{ij} \tag{1}$$

$$3\bar{e}=\sum_{\{i,\, j\}} \bar{d}_{ij}, \tag{2}$$

where the summation is over all the subsets of unordered two-elements of $V(G)$. Now we define

$$\varepsilon^a_{ijk}=\begin{cases} 1 & \text{if} \quad \{a, i, j\},\ \{a, i, k\}, \text{ and } \{a, j, k\} \text{ are edges while} \\ & \qquad \{i, j, k\} \text{ is non-edge,} \\ 0 & \text{otherwise.}\end{cases}$$

We associate each unordered pair $\{i, j\}$ with the weight w_{ij} defined as

$$w_{ij}=\frac{\sum_{a,\, k=1}^{n} \varepsilon^a_{ijk}}{\bar{d}_{ij}}.$$

We remark that in the extremal constructions of 3-graphs achieving the Turán number $t_3(n, 4)$, the weights w_{ij} are approximately equal for all i, j. However, the degrees d_{ij}'s are not. This illustrates the difficulties for tightening the bounds by using Cauchy–Schwarz inequalities. The main idea of our improved bounds in this paper is by utilizing the weights w_{ij} together with Giraud's bounds.

We will prove several useful facts about the relations between d_{ij}, w_{ij}, and e.

LEMMA 1. *For a Δ_4-free 3-graph G, we have*

$$\sum_{\{i,j\}} \bar{d}_{ij}(d_{ij}+w_{ij}) \geqslant 2(n-3)e. \tag{3}$$

Proof. For $i=1, 2, 3, 4$, let δ_i denote the number of the induced subgraphs of G which are isomorphic to the unique 3-graph (denoted by Δ_i) on 4 vertices with i triples. We have two basic equations:

$$(n-3)e = \delta_1 + 2\delta_2 + 3\delta_3 \tag{4}$$

$$\sum_{\{i,j\}} d_{ij}\bar{d}_{ij} = 3\delta_1 + 4\delta_2 + 3\delta_3. \tag{5}$$

By the definition of w_{ij}, we have

$$\sum_{\{i,j\}} \bar{d}_{ij} w_{ij} = 3\delta_3. \tag{6}$$

From Eqs. (4), (5), and (6), we get

$$\sum_{\{i,j\}} \bar{d}_{ij}(d_{ij}+w_{ij}) = 2(n-3)e + \delta_1.$$

The proof of inequality (3) then follows. ∎

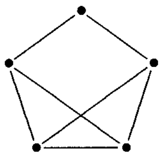
The next lemma involves further structures in a Δ_4-free 3-graph and it is particularly useful later.

LEMMA 2. *In a Δ_4-free 3-graph G, we have*

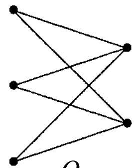
$$\sum_{\{i,j\}} d_{ij}\bar{d}_{ij}(d_{ij}+w_{ij}-1) \geqslant \sum_{\{i,j\}} \bar{d}_{ij} w_{ij}(4w_{ij}-3). \tag{7}$$

Proof. Every 3-graph H on 5 vertices is in one-to-one correspondence to a 2-graph F on 5 vertices as following. We connect two vertices in F if the other 3 vertices form an edge in the 3-graph H. Among all 3-graphs on

5 vertices, we are particularly interested in two of them, P and Q, described below:



P



Q

Let p (or q) denote the number of the induced sub-3-graphs on 5 vertices of G which are isomorphic to P (or Q). For every non-edge $\{i, j, k\}$ in G, we choose two vertices a and b from the subset consisting of vertices $x \in V(G)$ so that the induced sub-3-graph of G on vertices $\{i, j, k, x\}$ is isomorphic to Δ_3. Since G is Δ_4-free, the induced graph on vertices $\{i, j, k, a, b\}$ is isomorphic to either P or Q. By careful counting, we get

$$p+q = \sum_{\text{non-edge}\{i, j, k\}} \binom{\sum_a \varepsilon^a_{ijk}}{2}$$

$$= \frac{1}{6} \sum_{\{i, j\}} \sum_k \left(\left(\sum_a \varepsilon^a_{ijk} \right)^2 - \sum_a \varepsilon^a_{ijk} \right)$$

$$\geqslant \frac{1}{6} \sum_{\{i, j\}} (\bar{d}_{ij} w^2_{ij} - \bar{d}_{ij} w_{ij}) \tag{8}$$

and

$$\sum_{\{i, j\}} \binom{d_{ij}}{2} \bar{d}_{ij} \geqslant 7p + 9q \tag{9}$$

$$\sum_{\{i, j\}} (d_{ij} - 1)\, \bar{d}_{ij} w_{ij} \geqslant 10p + 6q. \tag{10}$$

By summing inequalities (9) twice and (10), we can use (8) to derive the inequality (7). ∎

We will further manipulate the inequalities in Lemma 1 and Lemma 2 to derive the following result.

LEMMA 5. *Suppose $e \geqslant \bar{e}$. Then for any $n \geqslant 4$, we have*

$$\sum_{\{i, j\}} \left(d_{ij} + \frac{1}{2} \right)^2 \bar{d}_{ij} \geqslant \frac{49 - 9\sqrt{17}}{32} \sum_{\{i, j\}} \bar{d}_{ij} (d_{ij} + w_{ij})^2. \tag{11}$$

Proof. We rewrite inequality (7) as

$$\tfrac{9}{4}\sum_{\{i,j\}}(d_{ij}+\tfrac{1}{2})\,\bar{d}_{ij}(d_{ij}+w_{ij})\geqslant\sum_{\{i,j\}}(d_{ij}+\tfrac{1}{2})^2\,\bar{d}_{ij}+\sum_{\{i,j\}}\bar{d}_{ij}(d_{ij}+w_{ij})^2$$
$$+\left(\tfrac{3}{8}\sum_{\{i,j\}}\bar{d}_{ij}(d_{ij}+w_{ij})-\tfrac{1}{4}\sum_{\{i,j\}}\bar{d}_{ij}\right). \quad (12)$$

By the assumptions, inequality (3) in Lemma 3 and Eq. (2), we see that the last term is always non-negative,

$$\begin{aligned}\tfrac{3}{8}\sum_{\{i,j\}}\bar{d}_{ij}(d_{ij}+w_{ij})-\tfrac{1}{4}\sum_{\{i,j\}}\bar{d}_{ij}&\geqslant\tfrac{3}{8}\,2(n-3)\,e-\tfrac{1}{4}\,3\bar{e}\\&\geqslant(\tfrac{3}{4}(n-3)-\tfrac{3}{4})\bar{e}\\&=\tfrac{3}{4}(n-4)\bar{e}\\&\geqslant 0.\end{aligned}$$

Now we use the Cauchy–Schwarz inequality

$$\left(\sum_{\{i,j\}}A_{ij}^2\right)\left(\sum_{\{i,j\}}B_{ij}^2\right)\geqslant\left(\sum_{\{i,j\}}A_{ij}B_{ij}\right)^2, \quad (13)$$

where $A_{ij}=(d_{ij}+\frac{1}{2})\sqrt{\bar{d}_{ij}}$ and $B_{ij}=\sqrt{\bar{d}_{ij}}\,(d_{ij}+w_{ij})$.

Inequality (12) can be rewritten as

$$\tfrac{9}{4}\sum_{\{i,j\}}A_{ij}B_{ij}\geqslant\sum_{\{i,j\}}A_{ij}^2+\sum_{\{i,j\}}B_{ij}^2.$$

By combining the above two inequalities, we have

$$\left(\sum_{\{i,j\}}A_{ij}^2\right)\left(\sum_{\{i,j\}}B_{ij}^2\right)\geqslant\tfrac{16}{81}\left(\sum_{\{i,j\}}A_{ij}^2+\sum_{\{i,j\}}B_{ij}^2\right)^2.$$

After solving this quadratic inequality, we get

$$\sum_{\{i,j\}}A_{ij}^2\geqslant\frac{49-9\sqrt{17}}{32}\sum_{\{i,j\}}B_{ij}^2.$$

The proof of Lemma 3 is complete. ∎

The next lemma is due to Giraud. The original version has the constant $c=0$. Here it is modified for latter usage.

LEMMA 4. *For a constant c with $d_{ij} \geqslant c$ for all pairs $\{i, j\}$, if $e \geqslant \frac{1}{3}((n/2)+(c/2)-1)\binom{n}{2}$, we have*

$$3\bar{e}\left(\frac{3e}{\binom{n}{2}}-c\right)^2 \geqslant \sum_{\{i,j\}} (d_{ij}-c)^2\, \bar{d}_{ij}. \tag{14}$$

Proof. Observe the fact that the following function $f(x)$ is convex,

$$f(x)=\begin{cases}\left(\frac{n}{2}-\frac{c}{2}-1\right)^2 (x-c) & \text{if } \; c \leqslant x \leqslant \frac{n}{2}+\frac{c}{2}-1 \\ (x-c)^2\,(n-2-x) & \text{if } \; \frac{n}{2}+\frac{c}{2}-1 \leqslant x \leqslant n-2,\end{cases}$$

and that $f(x) \geqslant (x-c)^2\,(n-2-x)$, for all $c \leqslant x \leqslant n-2$. Since the average $\sum_{\{i,j\}} d_{ij}/\binom{n}{2} \geqslant (n/2)+(c/2)-1$, the proof of this lemma follows from the convexity of $f(x)$. ∎

3. THE MAIN THEOREM

We are now ready to prove the main theorem.

Proof of Theorem 1. Let G be a $\varDelta_4$-free 3-graph with the maximum number of triples. If $e \leqslant \frac{1}{2}\binom{n}{3}$, then we are done since $\frac{1}{2} < (3+\sqrt{17})/12$. From now on we may assume that $e \geqslant \frac{1}{2}\binom{n}{3}$. Hence $e \geqslant \bar{e}$. All assumptions of Lemma 3 are satisfied. We can use inequality (11).

Let $c = -\frac{1}{2}$. Since $n \geqslant 4$, we have $e \geqslant \frac{1}{2}\binom{n}{3} \geqslant \frac{1}{3}((n/2)+(c/2)-1)\binom{n}{2}$. By inequality (14) in Lemma 4, we get

$$3\bar{e}\left(\frac{3e}{\binom{n}{2}}+\frac{1}{2}\right)^2 \geqslant \sum_{\{i,j\}} \left(d_{ij}+\frac{1}{2}\right)^2 \bar{d}_{ij}. \tag{15}$$

On the other hand, we can use the Cauchy–Schwartz inequality

$$\left(\sum_{\{i,j\}} \bar{d}_{ij}\right)\left(\sum_{\{i,j\}} \bar{d}_{ij}(d_{ij}+w_{ij})^2\right) \geqslant \left(\sum_{\{i,j\}} \bar{d}_{ij}(d_{ij}+w_{ij})\right)^2. \tag{16}$$

Combining inequalities (15) and (16) as well as inequality (3) of Lemma 1 and inequality (11) of Lemma 3, we have

$$9\bar{e}^2\left(\frac{3e}{\binom{n}{2}}+\frac{1}{2}\right)^2 \geqslant \frac{49-9\sqrt{17}}{32}\,(2(n-3)e-\bar{e})^2. \tag{17}$$

By taking square root on both sides, we get

$$3\bar{e}\left(\frac{3e}{\binom{n}{2}}+\frac{1}{2}\right)\geqslant\frac{9-\sqrt{17}}{8}\,(2(n-3)\,e-\bar{e}). \tag{18}$$

Let x denote $\lim_{n\to\infty} e/\binom{n}{3}$. We divide by $(n-2)\binom{n}{2}$ on both sides of (18) and let n approach infinity. Then

$$3(1-x)\,x\geqslant\frac{9-\sqrt{17}}{8}\,2x.$$

Hence, we have $x\leqslant(3+\sqrt{17})/12$ and therefore

$$\lim_{n\to\infty}\frac{t_3(n,4)}{\binom{n}{3}}\leqslant\frac{3+\sqrt{17}}{12}.$$

We have proved Theorem 1. ∎

4. CONCLUDING REMARKS

The result in Theorem 1 can be generlized for any odd $r\geqslant 3$ by using the same technique. Although the formulation is more complicated, the application is quite staightforward. We will omit the proof here.

THEOREM 2. *For any odd $r\geqslant 3$, we have*

$$\lim_{n\to\infty}\frac{t_r(n,r+1)}{\binom{n}{r}}\leqslant 1-\frac{5r+12-\sqrt{9r^2+24r}}{2r(r+3)}.$$

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