High-ordered Random Walks and Generalized Laplacians on Hypergraphs

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Abstract

Despite of the extreme success of the spectral graph theory, there are relatively few papers applying spectral analysis to hypergraphs. Chung first introduced Laplacians for regular hypergraphs and showed some useful applications. Other researchers treated hypergraphs as weighted graphs and then studied the Laplacians of the corresponding weighted graphs. In this paper, we aim to unify these very different versions of Laplacians for hypergraphs. We introduce a set of Laplacians for hypergraphs through studying high-ordered random walks on hypergraphs. We prove the eigenvalues of these Laplacians can effectively control the mixing rate of high-ordered random walks, the generalized distances/diameters, and the edge expansions.

1 Introduction

Many complex networks have richer structures than graphs can have. Inherently they have hypergraph structures: interconnections often cross multiple nodes. Treating these networks as graphs causes a loss of some structures. Nonetheless, it is still popular to use graph tools to study these networks; one of them is the Laplacian spectrum. Let G be a graph on n vertices. The *Laplacian* $\mathcal{L}$ of G is the $(n \times n)$-matrix $I - T^{-1/2}AT^{-1/2}$, where A is the adjacency matrix and T is the diagonal matrix of degrees. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}$, indexed in non-decreasing order. It is known that $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n-1$. If G is connected, then $\lambda_1 > 0$. The first nonzero Laplacian eigenvalue λ_1 is related to many graph parameters, such as the mixing rate of random walks, the graph diameter, the neighborhood expansion, the Cheeger constant, the isoperimetric inequalities, expander graphs, quasi-random graphs, etc [1, 2, 3, 5, 6].

In this paper, we define a set of Laplacians for hypergraphs. Laplacians for regular hypergraphs was first introduced by Chung [4] in 1993 using homology approach. The first nonzero Laplacian eigenvalue can be used to derive several useful isoperimetric inequalities. It seems hard to extend Chung's definition to general hypergraphs. Other researchers treated a hypergraph as a multi-edge graph and then defined its Laplacian to be the Laplacian of the corresponding multi-edge graph. For example, Rodríguez [9] showed that the approach above had some applications to bisections, the average minimal cut, the isoperimetric number, the max-cut, the independence number, the diameter etc.

What are "right" Laplacians for hypergraphs? To answer this question, let us recall how the Laplacian was introduced in the graph theory. One of the approaches is using

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geometric/homological analogue, where the Laplacian is defined as a self-joint operator on the functions over vertices. Another approach is using random walks, where the Laplacian is the symmetrization of the transition matrix of the random walk on a graph. Chung [3] took the first approach and defined her Laplacians for regular hypergraphs. In this paper, we take the second approach and define the Laplacians through high-ordered random walks on hypergraphs.

A high-ordered walk on a hypergraph H can be roughly viewed as a sequence of overlapped oriented edges $F_1, F_2, \ldots, F_k$. For $1 \leq s \leq r-1$, we say $F_1, F_2, \ldots, F_k$ is an s-walk if $|F_i \cap F_{i+1}| = s$ for each i in $\{1, 2, 3, \ldots, k-1\}$. The choice of s enables us to define a set of Laplacian matrices $\mathcal{L}^{(s)}$ for H. For $s = 1$, our definition of Laplacian $\mathcal{L}^{(1)}$ is the same as the definition in [9]. For $s = r-1$, while we restrict to regular hypergraphs, our definition of Laplacian $\mathcal{L}^{(r-1)}$ is similar to Chung's definition [4]. We will discuss their relations in the last section.

In this paper, we show several applications of the Laplacians of hypergraphs, such as the mixing rate of high-ordered random walks, the generalized diameters, and the edge expansions. Our approach allows users to select a "right" Laplacian to fit their special need.

The rest of the paper is organized as follows. In section 2, we review and prove some useful results on the Laplacians of weighted graphs and Eulerian directed graphs. The definition of Laplacians for hypergraphs will be given in section 3. We will prove some properties of the Laplacians of hypergraphs in section 4, and consider several applications in section 5. In last section, we will comment on future directions.

2 Preliminary results

In this section, we review some results on Laplacians of weighted graphs and Eulerian directed graphs. Those results will be applied to the Laplacians of hypergraphs later on.

In this paper, we frequently switch domains from hypergraphs to weighted (undirected) graphs, and/or to directed graphs. To reduce confusion, we use the following conventions through this paper. We denote a weighted graph by G, a directed graph by D, and a hypergraph by H. The set of vertices is denoted by $V(G)$, $V(D)$, and $V(H)$, respectively. (Whenever it is clear under the context, we will write it as V for short.) The edge set is denoted by $E(G)$, $E(D)$, and $E(H)$, respectively. The degrees d_* and volumes $\mathrm{vol}(*)$ are defined separately for the weighted graph G, for the directed graph D, and for the hypergraph H. Readers are warned to interpret them carefully under the context.

For a positive integer s and a vertex set V, let $\mathrm{V}^{\underline{s}}$ be the set of all (ordered) s-tuples consisting of s distinct elements in V. Let $\binom{V}{s}$ be the set of all unordered (distinct) s-subset of V.

Let $\mathbf{1}$ be the row (or the column) vector with all entries of value 1 and I be the identity matrix. For a row (or column) vector f, the norm $\|f\|$ is always the L_2-norm of f.

2.1 Laplacians of weighted graphs

A *weighted graph* G on the vertex set V is an undirected graph associated with a weight function $w \colon V \times V \to \mathbb{R}^{\geq 0}$ satisfying $w(u,v) = w(v,u)$ for all u and v in $V(G)$. Here we always assume $w(v,v) = 0$ for every $v \in V$.

A simple graph can be viewed as a special weighted graph with weight 1 on all edges and 0 otherwise. Many concepts of simple graphs are naturally generalized to weighted graphs. If $w(u,v) > 0$, then u and v are *adjacent*, written as $x \sim y$. The graph distance $d(u,v)$ between two vertices u and v in G is the minimum integer k such that there is a path $u = v_0, v_1, \ldots, v_k = v$ in which $w(v_{i-1}, v_i) > 0$ for $1 \leq i \leq k$. If no such k exists, then we let $d(u,v) = \infty$. If the distance $d(u,v)$ is finite for every pair (u,v), then G is *connected*. For a connected weighted graph G, the *diameter* (denoted by $\text{diam}(G)$) is the smallest value of $d(u, v)$ among all pairs of vertices (u, v).

The *adjacency matrix* A of G is defined as the matrix of weights, i.e., $A(x, y) = w(x, y)$ for all x and y in V. The *degree* d_x of a vertex x is $\sum_y w(x, y)$. Let T be the diagonal matrix of degrees in G. The *Laplacian* $\mathcal{L}$ is the matrix $I - T^{-1/2}AT^{-1/2}$. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}$, indexed in the non-decreasing order. It is known [6] that $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n-1$. If G is connected, then $\lambda_1 > 0$.

From now on, we assume G is connected. The first non-trivial Laplacian eigenvalue λ_1 is the most useful one. It can be written in terms of the Rayleigh quotient as follows (see [6])

$$\lambda_1 = \inf_{f \perp T\mathbf{1}} \frac{\sum_{x \sim y}(f(x) - f(y))^2 w(x, y)}{\sum_x f(x)^2 d_x}. \tag{1}$$

Here the infimum is taken over all functions $f \colon V \to \mathbb{R}$ which is orthogonal to the degree vector $\mathbf{1}T = (d_1, d_2, \ldots, d_n)$. Similarly, the largest Laplacian eigenvalue λ_{n-1} can be defined in terms of the Rayleigh quotient as follows

$$\lambda_{n-1} = \sup_{f \perp T\mathbf{1}} \frac{\sum_{x \sim y}(f(x) - f(y))^2 w(x, y)}{\sum_x f(x)^2 d_x}. \tag{2}$$

Note that scaling the weights by a constant factor will not affect the Laplacian. A weighted graph G is *complete* if $w(u, v) = c$ for some constant c such that $c > 0$, independent of the choice of (u, v) with $u \neq v$. We say G is *bipartite* if there is a partition $V = L \cup R$ such that $w(x, y) = 0$ for all $x, y \in L$ and all $x, y \in R$.

We have the following facts (see [6]).

1. $0 \leq \lambda_i \leq 2$ for each $0 \leq i \leq n-1$.
2. The number of 0 eigenvalues equals the number of connected components in G. If G is connected, then $\lambda_1 > 0$.
3. $\lambda_{n-1} = 2$ if and only if G has a connected component which is a bipartite weighted subgraph.
4. $\lambda_{n-1} = \lambda_1$ if and only if G is a complete weighted graph.

It turns out that λ_1 and λ_{n-1} are related to many graph parameters, such as the mixing rate of random walks, the diameter, the edge expansions, and the isoperimetric inequalities.

A random walk on a weighted graph G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that the conditional probability $\mathbf{Pr}(v_{i+1} = v \mid v_i = u) = w(u, v)/d_u$ for $0 \leq i \leq k-1$. A *vertex probability distribution* is a map $f \colon V \to \mathbb{R}$ such that $f(v) \geq 0$ for each v in G and $\sum_{v \in V} f(v) = 1$. It is convenient to write a vertex probability distribution into a row vector. A random walk maps a vertex probability distribution to a vertex probability distribution through multiplying from right a transition matrix P, where $P(u, v) = w(u, v)/d_u$ for each pair of vertices u and v. We can write $P = T^{-1}A = T^{-1/2}(I - \mathcal{L})T^{1/2}$. The second largest eigenvalue $\bar{\lambda}(P)$, denoted by $\bar{\lambda}$ for short, is $\max\{|1 - \lambda_1|, |1 - \lambda_{n-1}|\}$. Let $\pi(u) = d_u/\text{vol}(G)$ for each vertex u in G. Observe π is the stationary distribution of the random walk, i.e., $\pi P = \pi$. A random walk is *mixing* if $\lim_{i \to \infty} f_0 P^i = \pi$ for any initial vertex probability distribution f_0. It is known that a random walk is always mixing if G is connected and not a bipartite graph. To overcome the difficulty resulted from being a bipartite graph (where $\lambda_{n-1} = 2$), for $0 \leq \alpha \leq 1$, we consider an α-lazy random walk, whose transition matrix P_α is given by $P_\alpha(u, u) = \alpha$ for each u and $P_\alpha(u, v) = (1 - \alpha)w(u, v)/d_u$ for each pair of vertices u and v with $u \neq v$. Note that the transition matrix is

$$P_\alpha = \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2}(I - (1 - \alpha)\mathcal{L})T^{1/2}.$$

Let $L_\alpha = T^{1/2} P_\alpha T^{-1/2} = I - (1-\alpha)\mathcal{L}$ and $\bar{\lambda}_\alpha = \max\{|1-(1-\alpha)\lambda_1|, |1-(1-\alpha)\lambda_{n-1}|\}$. Since L_α is a symmetric matrix, we have

$$\bar{\lambda}_\alpha = \max_{u \perp T^{1/2}\mathbf{1}} \frac{\|L_\alpha u\|}{\|u\|}.$$

It turns out that the mixing rate of an α-lazy random walk is determined by $\bar{\lambda}_\alpha$.

Theorem 1 *For $0 \leq \alpha \leq 1$, the vertex probability distribution f_k of the α-lazy random walk at time k converges to the stationary distribution π in probability. In particular, we have*

$$\|(f_k - \pi)T^{-1/2}\| \leq \bar{\lambda}^k \|(f_0 - \pi)T^{-1/2}\|.$$

Here f_0 is the initial vertex probability distribution.

Proof: Notice that $f_k = f_0 P_\alpha^k$ and $(f_0 - \pi)T^{-1/2} \perp \mathbf{1}T^{1/2}$. We have

$$\begin{aligned} \|(f_k - \pi)T^{-1/2}\| &= \|(f_0 P_\alpha^k - \pi P_\alpha^k)T^{-1/2}\| \\ &= \|(f_0 - \pi)P_\alpha^k T^{-1/2}\| \\ &= \|(f_0 - \pi)T^{-1/2} L_\alpha^k\| \\ &\leq \bar{\lambda}_\alpha^k \|(f_0 - \pi)T^{-1/2}\|. \end{aligned}$$

□

For each subset X of $V(G)$, the *volume* $\mathrm{vol}(X)$ is $\sum_{x \in X} d_x$. If $X = V(G)$, then we write $\mathrm{vol}(G)$ instead of $\mathrm{vol}(V(G))$. We have

$$\mathrm{vol}(G) = \sum_{i=1}^n d_i = 2 \sum_{u \sim v} w(u,v).$$

If $\bar{X}$ is the complement set of X, then have $\mathrm{vol}(\bar{X}) = \mathrm{vol}(G) - \mathrm{vol}(X)$. For any two subsets X and Y of $V(G)$, the *distance* $d(X,Y)$ between X and Y is $\min\{d(x,y) : x \in X, y \in Y\}$.

Theorem 2 (See [3, 6]) *In a weighted graph G, for $X, Y \subseteq V(G)$ with distance at least 2, we have*

$$d(X,Y) \leq \left\lceil \frac{\log \sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}}}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil.$$

A special case of Theorem 2 is that both X and Y are single vertices, which gives an upper bound on the diameter of G.

Corollary 1 (See [6]) *If G is not a complete weighted graph, then we have*

$$diam(G) \leq \left\lceil \frac{\log(\mathrm{vol}(G)/\delta)}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil,$$

where δ is the minimum degree of G.

For $X, Y \subseteq V(G)$, let $E(X,Y)$ be the set of edges between X and Y. Namely, we have

$$E(X,Y) = \{(u,v) : u \in X, v \in Y \text{ and } uv \in E(G)\}.$$

We have the following theorem.

Theorem 3 (See [3, 6]) *If X and Y are two subsets of $V(G)$, then we have*

$$\left| |E(X,Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \bar{\lambda} \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(G)}.$$

2.2 Laplacians of Eulerian directed graphs

The Laplacian of a general directed graph was introduced by Chung [7, 8]. The theory is considerably more complicated than the one for undirected graphs, but when we consider a special class of directed graphs — Eulerian directed graphs, it turns out to be quite neat.

Let D be a directed graph with the vertex set $V(D)$ and the edge set $E(D)$. A directed edge from x to y is denoted by an ordered pair (x, y) or $x \to y$. The *out-neighborhood* $\Gamma^+(x)$ of a vertex x in D is the set $\{y\colon (x, y) \in E(D)\}$. The *out-degree* d_x^+ is $|\Gamma^+(x)|$. Similarly, the *in-neighborhood* $\Gamma^-(x)$ is $\{y\colon (y, x) \in E(D)\}$, and the *in-degree* d_x^- is $|\Gamma^-(x)|$. A directed graph D is *Eulerian* if $d_x^+ = d_x^-$ for every vertex x. In this case, we simply write $d_x = d_x^+ = d_x^-$ for each x. For a vertex subset S, the *volume* of S, denoted by $\mathrm{vol}(S)$, is $\sum_{x\in S} d_x$. In particular, we write $\mathrm{vol}(D) = \sum_{x\in V} d_x$.

Eulerian directed graphs have many good properties. For example, a Eulerian directed graph is strongly connected if and only if it is weakly connected.

The *adjacency matrix* of D is a square matrix A satisfying $A(x, y) = 1$ if $(x, y) \in E(D)$ and 0 otherwise. Let T be the diagonal matrix with $T(x, x) = d_x$ for each $x \in V(D)$. Let $\vec{\mathcal{L}} = I - T^{-1/2}AT^{-1/2}$, i.e.,

$$\vec{\mathcal{L}}(x, y) = \begin{cases} 1 & \text{if } x = y; \\ -\frac{1}{\sqrt{d_x d_y}} & \text{if } x \to y; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Note that $\vec{\mathcal{L}}$ is not symmetric. We define the Laplacian $\mathcal{L}$ of D to be the symmetrization of $\vec{\mathcal{L}}$, that is

$$\mathcal{L} = \frac{\vec{\mathcal{L}} + \vec{\mathcal{L}}'}{2}.$$

Since $\mathcal{L}$ is symmetric, its eigenvalues are real and can be listed as $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ in the non-decreasing order. Note that λ_1 can also be written in terms of Raleigh quotient (see [7]) as follows

$$\lambda_1 = \inf_{f \perp T\mathbf{1}} \frac{\sum_{x\to y}(f(x) - f(y))^2}{2\sum_x f(x)^2 d_x}. \tag{4}$$

Chung [8] proved a general theorem on the relationship between λ_1 and the diameter. After restricting to Eulerian directed graphs, it can be stated as follows.

Theorem 4 (See [8]) *Suppose D is a connected Eulerian directed graph, then the diameter of D (denoted by $diam(D)$) satisfies*

$$diam(D) \le \left\lfloor \frac{2\log(\mathrm{vol}(G)/\delta)}{\log \frac{2}{2-\lambda_1}} \right\rfloor + 1,$$

where λ_1 is the first non-trivial eigenvalue of the Laplacian, and δ is the minimum degree $\min\{d_x \mid x \in V(D)\}$.

The main idea in the proof of the theorem above is using α-lazy random walks on D. A random walk on a Eulerian directed graph D is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that for $0 \le i \le k-1$, the conditional probability $\mathbf{Pr}(v_{i+1} = v \mid v_i = u)$ equals $1/d_u$ for each $v \in \Gamma^+(u)$ and 0 otherwise. For $0 \le \alpha \le 1$, the α-lazy random walk is defined similarly. The transition matrix P_α of the α-lazy random walk satisfies

$$P_\alpha = \alpha I + (1-\alpha)T^{-1}A = T^{-1/2}(I - (1-\alpha)\vec{\mathcal{L}})T^{1/2}.$$

Chung [7] considered only 1/2-lazy random walks. Here we prove some results on α-lazy random walks for $\alpha \in [0, 1)$.

Let $\pi(u) = d_u/\mathrm{vol}(D)$ for each $u \in V(D)$. Note that π is the stationary distribution, i.e. $\pi P_\alpha = \pi$. Let $L_\alpha = \alpha I + (1-\alpha)T^{-1/2}AT^{-1/2} = I - (1-\alpha)\vec{\mathcal{L}} = T^{1/2}P_\alpha T^{-1/2}$. The key observation is that there is a unit-vector ϕ_0 such that ϕ_0 is both a row eigenvector and a column eigenvector of L_α for the largest eigenvalue 1. Here let $\phi_0 = \mathbf{1}T^{1/2}/\mathrm{vol}(D) = \frac{1}{\mathrm{vol}(G)}(\sqrt{d_1}, \ldots, \sqrt{d_n})$. We have

$$\phi_0 L_\alpha = \phi_0 \text{ and } L_\alpha \phi_0' = \phi_0'.$$

Let $\phi_0^\perp$ be the orthogonal complement of ϕ_0 in R^n. It is easy to check L_α maps $\phi_0^\perp$ to $\phi_0^\perp$. Let σ_α be the spectral norm of L_α when restricting to $\phi_0^\perp$. An equivalent definition of σ_α is the second largest singular value of L_α, i.e.,

$$\sigma_\alpha = \max_{f \perp \phi_0'} \frac{\|L_\alpha f\|}{\|f\|}.$$

Lemma 1 *We have the following properties for σ_α.*

1. *For every $\beta \in \phi_0^\perp$, we have $\|L_\alpha \beta\| \le \sigma_\alpha \|\beta\|$.*
2. $(1-\lambda_1)^2 \le \sigma_0^2 \le 1$.
3. $\sigma_\alpha^2 \le \alpha^2 + 2\alpha(1-\alpha)\lambda_1 + (1-\alpha)^2\sigma_0^2$.

Proof: Item 1 is from the definition of σ_α. Since the largest eigenvalue of L_α is 1, we have $\sigma_\alpha \le 1$. In particular, $\sigma_0^2 \le 1$. Note that $L_0 = T^{-1/2}AT^{-1/2}$. Let $f = gT^{1/2}$. It follows that

$$\sigma_0^2 = \sup_{f \perp \phi_0'} \frac{\|L_0 f\|^2}{\|f\|^2} = \sup_{g \perp T\mathbf{1}} \frac{g'A'T^{-1}Ag}{g'Tg}.$$

Choose $g \in (T\mathbf{1})^\perp$ such that the Raleigh quotient (4) reaches its minimum at g, i.e.,

$$\lambda_1 = \frac{\sum_{x \to y}(g(x) - g(y))^2}{2\sum_x g(x)^2 d_x}.$$

We have

$$\begin{aligned}
\frac{g'A'T^{-1}Ag}{g'Tg} &= \frac{\sum_x \frac{1}{d_x}\left(\sum_{y \in \Gamma^+(x)} g(y)\right)^2}{\sum_x d_x g(x)^2} \\
&= \frac{\sum_x d_x g(x)^2 \sum_x \frac{1}{d_x}\left(\sum_{y \in \Gamma^+(x)} g(y)\right)^2}{(\sum_x d_x g(x)^2)^2} \\
&\ge \frac{\left(\sum_x g(x) \sum_{y \in \Gamma^+(x)} g(y)\right)^2}{(\sum_x d_x g(x)^2)^2} \\
&= \left(\frac{\sum_x g(x) \sum_{y \in \Gamma^+(x)} g(y)}{\sum_x d_x g(x)^2}\right)^2 \\
&= (1-\lambda_1)^2.
\end{aligned}$$

In the last step, we use the following argument.

$$\begin{aligned}\frac{\sum_x g(x)\sum_{y\in\Gamma^+(x)} g(y)}{\sum_x d_x g(x)^2} &= \frac{\frac{1}{2}\sum_{x\to y}\left(g(x)^2+g(y)^2-(g(x)-g(y))^2\right)}{\sum_x d_x g(x)^2}\\ &= 1-\frac{\sum_{x\to y}(g(x)-g(y))^2}{2\sum_x d_x g(x)^2}\\ &= 1-\lambda_1.\end{aligned}$$

Since σ_0 is the maximum over all $g \perp T\mathbf{1}$, we get $(1-\lambda_1)^2 \leq \sigma_0^2$.

For item 3, we have

$$\begin{aligned}\sigma_\alpha^2 &= \sup_{f\perp\phi_0'} \frac{\|L_\alpha f\|^2}{\|f\|^2}\\ &= \sup_{g\perp T\mathbf{1}} \frac{g'P_\alpha' T P_\alpha g}{g'Tg}\\ &\leq \alpha^2+\alpha(1-\alpha)\sup_{g\perp T\mathbf{1}}\frac{g'(A+A')g}{g'Tg}+(1-\alpha)^2\sup_{g\perp T\mathbf{1}}\frac{g'A'T^{-1}Ag}{g'Tg}\\ &= \alpha^2+2\alpha(1-\alpha)(1-\lambda_1)+(1-\alpha)^2\sigma_0^2.\end{aligned}$$

□

Theorem 5 *For $0<\alpha<1$, the vertex probability distribution f_k of the α-lazy random walk on a Eulerian directed graph D at time k converges to the stationary distribution π in probability. In particular, we have*

$$\|(f_k-\pi)T^{-1/2}\| \leq \sigma_\alpha^k \|(f_0-\pi)T^{-1/2}\|.$$

Here f_0 is the initial vertex probability distribution.

The proof is omitted since it is very similar to the proof of Theorem 1. Notice that when $0<\alpha<1$, we have $\sigma_\alpha < 1$ by Lemma 1. We have the α-lazy random converges to the stationary distribution exponentially fast.

For two vertex subsets X and Y of $V(D)$, let $E(X,Y)$ be the number of directed edges from X to Y, i.e., $E(X,Y)=\{(u,v): u\in X \text{ and } v\in Y\}$. We have the following theorem on the edge expansions in Eulerian directed graphs.

Theorem 6 *If X and Y are two subsets of the vertex set V of a Eulerian directed graph D, then we have*

$$\left| |E(X,Y)| - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(D)}\right| \leq \sigma_0 \frac{\sqrt{\text{vol}(X)\text{vol}(Y)\text{vol}(\bar{X})\text{vol}(\bar{Y})}}{\text{vol}(D)}.$$

Proof: Let $\mathbf{1}_X$ be the indicator variable of X, i.e., $\mathbf{1}_X(u)=1$ if $u\in X$ and 0 otherwise. We define $\mathbf{1}_Y$ similarly. Assume $\mathbf{1}_X T^{1/2} = a_0\phi_0 + a_1\phi_1$ and $\mathbf{1}_Y T^{1/2} = b_0\phi_0 + b_1\phi_2$, where $\phi_1,\phi_2 \in \phi_0^\perp$ and are unit vectors. Since ϕ_0 is a unit vector, we have

$$a_0 = \langle \mathbf{1}_X T^{1/2}, \phi_0\rangle = \frac{\text{vol}(X)}{\sqrt{\text{vol}(D)}} \tag{5}$$

and

$$a_0^2 + a_1^2 = \langle \mathbf{1}_X T^{1/2}, \mathbf{1}_X T^{1/2}\rangle = \text{vol}(X). \tag{6}$$

Thus

$$a_1 = \sqrt{\text{vol}(X)\text{vol}(\bar{X})/\text{vol}(D)}. \tag{7}$$

Similarly, we get

$$b_0 = \frac{\mathrm{vol}(Y)}{\sqrt{\mathrm{vol}(D)}}; \tag{8}$$

$$b_1 = \sqrt{\mathrm{vol}(Y)\mathrm{vol}(\bar{Y})/\mathrm{vol}(D)}. \tag{9}$$

It follows that

$$\begin{aligned}
\left| |E(X,Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(D)} \right| &= \left| \mathbf{1}_X T^{1/2}(L_0 - \phi_0'\phi_0)(\mathbf{1}_Y T^{1/2})' \right| \\
&= |(a_0\phi_0 + a_1\phi_1)(L_0 - \phi_0'\phi_0)(b_0\phi_0 + b_1\phi_2)'| \\
&= |a_1 b_1 \phi_1 L_0 \phi_2'| \\
&\leq |a_1 b_1| \|\phi_1\| \|L_0\phi_2'\| \\
&\leq |a_1 b_1| \sigma_0 \\
&= \sigma_0 \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(D)}.
\end{aligned}$$

The proof of this theorem is completed. □

If we use $\bar{\lambda}$ instead of σ_0, then we get a weaker theorem on the edge expansions. The proof will be omitted since it is very similar to the proof of Theorem 6.

Theorem 7 *Let D be a Eulerian directed graph. If X and Y are two subsets of $V(D)$, then we have*

$$\left| \frac{|E(X,Y)| + |E(Y,X)|}{2} - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(D)} \right| \leq \bar{\lambda} \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(D)}.$$

For $X, Y \subseteq V(D)$, let $d(X,Y) = \min\{d(u,v) : u \in X \text{ and } v \in Y\}$. We have the following upper bound on $d(X,Y)$.

Theorem 8 *Suppose D is a connected Eulerian directed graph. For $X, Y \subseteq V(D)$ and $0 \leq \alpha < 1$, we have*

$$d(X,Y) \leq \left\lfloor \frac{\log \sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}}}{\log \sigma_\alpha} \right\rfloor + 1.$$

In particular, for $0 \leq \alpha < 1$, the diameter of D satisfies

$$\mathrm{diam}(D) \leq \left\lceil \frac{\log(\mathrm{vol}(D)/\delta)}{\log \sigma_\alpha} \right\rceil,$$

where $\delta = \min\{d_x \colon x \in V\}$.

Remark: From lemma 1, we have

$$\sigma_\alpha^2 \leq \alpha^2 + 2\alpha(1-\alpha)\lambda_1 + (1-\alpha)^2\sigma_0^2.$$

We can choose α to minimize σ_α. If $\lambda_1 \leq 1 - \sigma_0^2$, then we choose $\alpha = 0$ and get $\sigma_\alpha = \sigma_0$; if $\lambda_1 > 1 - \sigma_0^2$, then we choose $\alpha = \frac{\lambda_1 + \sigma_0^2 - 1}{2\lambda_1 + \sigma_0^2 - 1}$ and get $\sigma_\alpha^2 \leq 1 - \frac{\lambda_1^2}{2\lambda_1 + \sigma_0^2 - 1}$. Combining two cases, we have

$$\min_{0 \leq \alpha < 1} \{\sigma_\alpha\} \leq \begin{cases} \sigma_0 & \text{if } \lambda_1 \leq 1 - \sigma_0^2; \\ \sqrt{1 - \frac{\lambda_1^2}{2\lambda_1 + \sigma_0^2 - 1}} & \text{otherwise.} \end{cases} \tag{10}$$

It is easy to check

$$\min_{0\le\alpha<1}\{\sigma_\alpha\} \le \sqrt{1-\frac{\lambda_1}{2}}.$$

Here the inequality is strict if $\sigma_0 < 1$. We have

$$d(X,Y) \le \left\lfloor \frac{\log \frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}}{\log \frac{2}{2-\lambda_1}} \right\rfloor + 1.$$

Theorem 8 is stronger than Theorem 4 in general.

Proof: Similar to the proof of Theorem 6, let $\mathbf{1}_X$ and $\mathbf{1}_Y$ be the indicator functions of X and Y, respectively. We have

$$\begin{aligned} \mathbf{1}_X T^{1/2} &= a_0\phi_0 + a_1\phi_1, \\ \mathbf{1}_Y T^{1/2} &= b_0\phi_0 + b_1\phi_2, \end{aligned}$$

where $\phi_1, \phi_2 \in \phi_0^\perp$ and are unit vectors and a_0, b_0, a_1, b_1 are given by equations (5)-(9).

Let $k = \left\lfloor \frac{\log \sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}}}{\log \sigma_\alpha} \right\rfloor + 1$. We have

$$(\mathbf{1}_X T^{1/2}) L_\alpha^k (\mathbf{1}_Y T^{1/2})' \ge a_0 b_0 + \sigma_\alpha^k a_1 b_1 > 0.$$

Thus there is a directed path starting from some vertex in X and ending at some vertex in Y, that is $d(X,Y) \le k$.

For the diameter result, we choose $X = \{x\}$ and $Y = \{y\}$. Note that $\mathrm{vol}(X) = d_x \ge \delta$, $\mathrm{vol}(Y) = d_y \ge \delta$, $\mathrm{vol}(\bar{X}) < \mathrm{vol}(G)$, and $\mathrm{vol}(\bar{y}) < \mathrm{vol}(G)$. The result follows. □

3 Definition of the s-th Laplacian

Let H be an r-uniform hypergraph with the vertex set $V(H)$ (or V for short) and the edge set $E(H)$. We assume $|V(H)| = n$ and $E(H) \subseteq \binom{V}{r}$. For a vertex subset S such that $|S| < r$, the *neighborhood* $\Gamma(S)$ is $\{T | S \cap T = \emptyset$ and $S \cup T$ is an edge in $H\}$. Let the *degree* d_S of S in H be the number of edges containing S, i.e, $d_S = |\Gamma(S)|$. For $1 \le s \le r-1$, an s-walk of length k is a sequence of vertices

$$v_1, v_2, \ldots, v_j, \ldots, v_{(r-s)(k-1)+r}$$

together with a sequence of edges $F_1, F_2, \ldots, F_k$ such that

$$F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$$

for $1 \le i \le k$. Here are some examples of s-walks as shown in Figure 1.

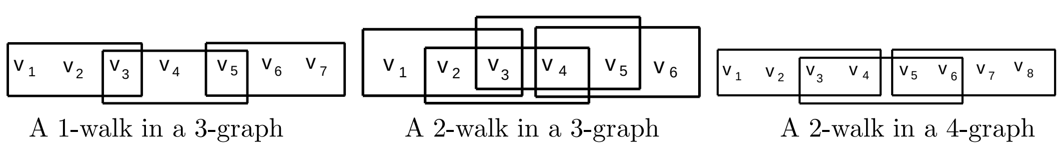

Figure 1: Three examples on an s-walk in a hypergraph

For each i in $\{0, 1, \ldots, k\}$, the *i-th stop* x_i of the s-walk is the ordered s-tuple $(v_{(r-s)i+1}, v_{(r-s)i+2}, \ldots, v_{(r-s)i+s})$. The initial stop is x_0, and the terminal stop is x_k. An

s-walk is called an s-path if every stop (as an ordered s-tuple) is different from each other. If $x_0 = x_k$, then an s-walk is *closed*. An s-cycle is a closed s-path.

For $1 \leq s \leq r-1$ and $x, y \in V^{\underline{s}}$, the s-distance $d^{(s)}(x,y)$ is the minimum integer k such that there exists an s-path of length k starting from x and ending at y. A hypergraph H is *s-connected* if $d^{(s)}(x,y)$ is finite for every pair (x,y). If H is s-connected, then the *s-diameter* of H is the maximum value of $d^{(s)}(x,y)$ for $x, y \in V^{\underline{s}}$.

A random s-walk with initial stop x_0 is an s-walk generated as follows. Let x_0 be the sequence of visited vertices at initial step. At each step, let S be the set of last s vertices in the sequence of visited vertices. A random $(r-s)$-set T is chosen from $\Gamma(S)$ uniformly; the vertex in T is added into the sequence one by one in an arbitrary order.

For $0 \leq \alpha \leq 1$, an α-lazy random s-walk is a modified random s-walk such that with probability α, one can stay at the current stop; with probability $1-\alpha$, append $r-s$ vertices to the sequence as selected in a random s-walk.

For $x \in V^{\underline{s}}$, let $[x]$ be the s-set consisting of the coordinates of x.

3.1 Case $1 \leq s \leq r/2$

For $1 \leq s \leq r/2$, we define a weighted undirected graph $G^{(s)}$ over the vertex set $V^{\underline{s}}$ as follows. Let the weight $w(x,y)$ be $|\{F \in E(H) : [x] \sqcup [y] \subseteq F\}|$. Here $[x] \sqcup [y]$ is the disjoint union of $[x]$ and $[y]$. In particular, if $[x] \cap [y] \neq \emptyset$, then $w(x,y) = 0$.

For $x \in V^{\underline{s}}$, the degree of x in $G^{(s)}$, denoted by $d_x^{(s)}$, is given by

$$d_x^{(s)} = \sum_y w(x,y) = d_{[x]} \binom{r-s}{s} s!. \tag{11}$$

Here $d_{[x]}$ means the degree of the set $[x]$ in the hypergraph H. When we restrict an s-walk on H to its stops, we get a walk on $G^{(s)}$. This restriction keeps the length of the walk. Therefore, the s-distance $d^{(s)}(x,y)$ in H is simply the graph distance between x and y in $G^{(s)}$; the s-diameter of H is simply the diameter of the graph $G^{(s)}$.

A random s-walk on H is essentially a random walk on $G^{(s)}$. It can be constructed from a random walk on $G^{(s)}$ by inserting additional random $r-2s$ vertices T_i between two consecutive stops x_i and x_{i+1} at time i, where T_i is chosen uniformly from $\Gamma([x_i] \cup [x_{i+1}])$ and inserted between x_i and x_{i+1} in an arbitrary order.

Therefore, we define the *s-th Laplacian* $\mathcal{L}^{(s)}$ of H to be the Laplacian of the weighted undirected graph $G^{(s)}$.

The eigenvalues of $\mathcal{L}^{(s)}$ are listed as $\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}s!}^{(s)}$ in the non-decreasing order. Let $\lambda_{\max}^{(s)} = \lambda_{\binom{n}{s}s!}^{(s)}$ and $\bar{\lambda}^{(s)} = \max\{|1-\lambda_1^{(s)}|, |1-\lambda_{\max}^{(s)}|\}$. For some hypergraphs, the numerical values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ are shown in Table 1 at the end of this section.

3.2 The case $r/2 < s \leq r-1$

For $r/2 < s \leq r-1$, we define a directed graph $D^{(s)}$ over the vertex set $V^{\underline{s}}$ as follows. For $x, y \in V^{\underline{s}}$ such that $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, let (x,y) be a directed edge if $x_{r-s+j} = y_j$ for $1 \leq j \leq 2s-r$ and $[x] \cup [y]$ is an edge of H.

For $x \in V^{\underline{s}}$, the out-degree d_x^+ in $D^{(s)}$ and the in-degree d_x^- in $D^{(s)}$ satisfy

$$d_x^+ = d_{[x]}(r-s)! = d_x^-.$$

Thus $D^{(s)}$ is a Eulerian directed graph. We write $d_x^{(s)}$ for both d_x^+ and d_x^-. Now $D^{(s)}$ is strongly connected if and only if it is weakly connected.

Note that an s-walk on H can be naturally viewed as a walk on $D^{(s)}$ and vice versa. Thus the s-distance $d^{(s)}(x,y)$ in H is exactly the directed distance from x to y in $G^{(s)}$; the s-diameter of H is the diameter of $D^{(s)}$. A random s-walk on H is one-to-one corresponding to a random walk on $D^{(s)}$.

For $\frac{r}{2} < s \leq r-1$, we define the s-th Laplacian $\mathcal{L}^{(s)}$ as the Laplacian of the Eulerian directed graph $D^{(s)}$ (see section 2).

The eigenvalues of $\mathcal{L}^{(s)}$ are listed as $\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}s!}^{(s)}$ in the non-decreasing order. Let $\lambda_{\max}^{(s)} = \lambda_{\binom{n}{s}s!}^{(s)}$ and $\bar{\lambda}^{(s)} = \max\{|1-\lambda_1^{(s)}|, |1-\lambda_{\max}^{(s)}|\}$. For some hypergraphs, the numerical values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ are shown in Table 1 at the end of this section.

3.3 Examples

Let K_n^r be the complete r-uniform hypergraph on n vertices. Here we compute the values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ for some K_n^r (see Table 1).

H	$\lambda_1^{(4)}$	$\lambda_1^{(3)}$	$\lambda_1^{(2)}$	$\lambda_1^{(1)}$	$\lambda_{\max}^{(1)}$	$\lambda_{\max}^{(2)}$	$\lambda_{\max}^{(3)}$	$\lambda_{\max}^{(4)}$
K_6^3			3/4	6/5	6/5	3/2		
K_7^3			7/10	7/6	7/6	3/2		
K_6^4		1/3	5/6	6/5	6/5	3/2	1.76759	
K_7^4		3/8	9/10	7/6	7/6	7/5	7/4	
K_6^5	0.1464	1/2	5/6	6/5	6/5	3/2	3/2	1.809
K_7^5	0.1977	5/8	9/10	7/6	7/6	7/5	3/2	1.809

Table 1: The values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ of some complete hypergraphs K_n^r.

Remark: From the table above, we observe $\lambda_1^{(s)} = \lambda_{\max}^{(s)}$ for some complete hypergraphs. In fact, this is true for any complete hypergraph K_n^r. We point out the following fact without proofs. For an r-uniform hypergraph H and an integer s such that $1 \leq s \leq \frac{r}{2}$, $\lambda_1^{(s)}(H) = \lambda_{\max}^{(s)}(H)$ holds if and only if $s=1$ and H is a 2-design.

4 Properties of Laplacians

In this section, we prove some properties of the Laplacians for hypergraphs.

Lemma 2 *For $1 \leq s \leq r/2$, we have the following properties.*

1. *The s-th Laplacian has $\binom{n}{s}s!$ eigenvalues and all of them are in $[0,2]$.*
2. *The number of 0 eigenvalues is the number of connected components in $G^{(s)}$.*
3. *The Laplacian $L^{(s)}$ has an eigenvalue 2 if and only if $r=2s$ and $G^{(s)}$ has a bipartite component.*

Proof: Items 1 and 2 follow from the facts of the Laplacian of $G^{(s)}$. If $\mathcal{L}^{(s)}$ has an eigenvalue 2, then $G^{(s)}$ has a bipartite component T. We want to show $r=2s$. Suppose $r \geq 2s+1$. Let $\{v_0, v_2, \ldots, v_{r-1}\}$ be an edge in T. For $0 \leq i \leq 2s$ and $0 \leq j \leq s-1$, let $g(i,j) = is+j \mod (2s+1)$ and $x_i = (v_{g(i,0)}, \ldots, v_{g(i,s-1)})$. Observe $x_0, x_1, \ldots, x_{2s}$ form an odd cycle in $G^{(s)}$. Contradiction. □

The following lemma compares $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ for different s.

Lemma 3 *Suppose that H is an r-uniform hypergraph. We have*

$$\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \ldots \geq \lambda_1^{(\lfloor r/2 \rfloor)}; \tag{12}$$
$$\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \ldots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}. \tag{13}$$

Remark: We do not know whether similar inequalities hold for $s > \frac{r}{2}$.
Proof: Let T_s be the diagonal matrix of degrees in $G^{(s)}$ and $R^{(s)}(f)$ be the Rayleigh quotient of $\mathcal{L}^{(s)}$. It suffices to show $\lambda_1^{(s)} \leq \lambda_1^{(s-1)}$ for $2 \leq s \leq r/2$. Recall that $\lambda_1^{(s)}$ can be defined via the Rayleigh quotient, see equation (2). Pick a function $f : V^{\underline{(s-1)}} \to R$ such that $\langle f, T_{s-1}\mathbf{1}\rangle = 0$ and $\lambda_1^{(s-1)} = R^{(s-1)}(f)$. We define $g : \mathrm{V}^{\underline{s}} \to R$ as follows

$$g(x) = f(x'),$$

where x' is a $(s-1)$-tuple consisting of the first $(s-1)$ coordinates of x with the same order in x. Applying equation 11, we get

$$\langle g, T_s\mathbf{1}\rangle = \sum_{x\in \mathrm{V}^{\underline{s}}} d_x^{(s)} g(x) = \sum_{x\in \mathrm{V}^{\underline{s}}} g(x) d_{[x]} \binom{r-s}{s} s!.$$

We have

$$\begin{aligned}
\sum_x g(x) d_{[x]} &= \sum_x \sum_{F:[x]\subseteq F} g(x) \\
&= \sum_{x'} \sum_{F:[x']\subseteq F} (r-s+1) f(x') \\
&= \sum_{x'} d_{[x']}(r-s+1) f(x') \\
&= \frac{r-s+1}{\binom{r-s+1}{s-1}(s-1)!} \sum_{x'} f(x') d_{x'}^{(s-1)} = 0.
\end{aligned}$$

Here the second last equality follows from equation 11 and the last one follows from the choice of f. Therefore,

$$\sum_x g(x) d_x^{(s)} = (r-s+2)(r-s+1) \sum_{x'} f(x') d_{x'}^{(s-1)}.$$

Thus $\langle g, T_s\mathbf{1}\rangle = 0$. Similarly, we have

$$\sum_x g(x)^2 d_x^{(s)} = (r-s+2)(r-s+1) \sum_{x'} f(x')^2 d_{x'}^{(s-1)}.$$

Putting them together, we obtain

$$\sum_x g(x)^2 d_x^{(s)} = (r-s+2)(r-s+1) \sum_{x'} f(x')^2 d_{x'}^{(s-1)}.$$

By the similar counting method, we have

$$\begin{aligned}
\sum_{x\sim y} (g(x)-g(y))^2 w(x,y) &= \sum_{x\sim y} \sum_{F:[x]\sqcup[y]\subseteq F} (g(x)-g(y))^2 \\
&= \sum_{x'\sim y'} \sum_{F:[x']\sqcup[y']\subseteq F} (r-s+1)(r-s+2)(f(x')-f(y'))^2 \\
&= (r-s+1)(r-s+2) \sum_{x'\sim y'} (f(x')-f(y'))^2 w(x',y').
\end{aligned}$$

Thus, $R^{(s)}(g) = R^{(s-1)}(f) = \lambda_1^{(s-1)}$ by the choice of f. As $\lambda_1^{(s)}$ is the infimum over all g, we get $\lambda_1^{(s)} \leq \lambda_1^{(s-1)}$.

The inequality (13) can be proved in a similarly way. Since $\lambda_{\max}^{(s)}$ is the supremum of the Raleigh quotient, the direction of inequalities are reversed. □

Lemma 4 *For $r/2 < s \leq r-1$, we have the following facts.*

1. *The s-th Laplacian has $\binom{n}{s}s!$ eigenvalues and all of them are in $[0,2]$.*

2. *The number of 0 eigenvalues is the number of strongly connected components in $D^{(s)}$.*

3. *If 2 is an eigenvalue of $L^{(s)}$, then one of the s-connected components of H is bipartite.*

The proof is trivial and will be omitted.

5 Applications

We show some applications of Laplacians $\mathcal{L}^{(s)}$ of hypergraphs in this section.

5.1 The random s-walks on hypergraphs

For $0 \leq \alpha < 1$ and $1 \leq s \leq r/2$, after restricting an α-lazy random s-walk on a hypergraph H to its stops (see section 3), we get an α-lazy random walk on the corresponding weighted graph $G^{(s)}$. Let $\pi(x) = d_x/\mathrm{vol}(\mathrm{V}^{\underline{s}})$ for any $x \in \mathrm{V}^{\underline{s}}$, where d_x is the degree of x in $G^{(s)}$ and $\mathrm{vol}(\mathrm{V}^{\underline{s}})$ is the volume of $G^{(s)}$. Applying theorem 1, we have the following theorem.

Theorem 9 *For $1 \leq s \leq r/2$, suppose that H is an s-connected r-uniform hypergraph H and $\lambda_1^{(s)}$ (and $\lambda_{\max}^{(s)}$) is the first non-trivial (and the last) eigenvalue of the s-th Laplacian of H. For $0 \leq \alpha < 1$, the joint distribution f_k at the k-th stop of the α-lazy random walk at time k converges to the stationary distribution π in probability. In particular, we have*

$$\|(f_k - \pi)T^{-1/2}\| \leq (\bar{\lambda}_\alpha^{(s)})^k \|(f_0 - \pi)T^{-1/2}\|,$$

where $\bar{\lambda}_\alpha^{(s)} = \max\{|1-(1-\alpha)\lambda_1^{(s)}|, |(1-\alpha)\lambda_{\max}^{(s)} - 1|$, and f_0 is the probability distribution at the initial stop.

For $0 < \alpha < 1$ and $r/2 < s \leq r-1$, when restricting an α-lazy random s-walk on a hypergraph H to its stops (see section 2), we get an α-lazy random walk on the corresponding directed graph $D^{(s)}$. Let $\pi(x) = d_x/\mathrm{vol}(\mathrm{V}^{\underline{s}})$ for any $x \in \mathrm{V}^{\underline{s}}$, where d_x is the degree of x in $D^{(s)}$ and $\mathrm{vol}(\mathrm{V}^{\underline{s}})$ is the volume of $D^{(s)}$. Applying theorem 5, we have the following theorem.

Theorem 10 *For $r/2 < s \leq r-1$, suppose that H is an s-connected r-uniform hypergraph and $\lambda_1^{(s)}$ is the first non-trivial eigenvalue of the s-th Laplacian of H. For $0 < \alpha < 1$, the joint distribution f_k at the k-th stop of the α-lazy random walk at time k converges to the stationary distribution π in probability. In particular, we have*

$$\|(f_k - \pi)T^{-1/2}\| \leq (\sigma_\alpha^{(s)})^k \|(f_0 - \pi)T^{-1/2}\|,$$

where $\sigma_\alpha^{(s)} \leq \sqrt{1 - 2\alpha(1-\alpha)\lambda_1^{(s)}}$, and f_0 is the probability distribution at the initial stop.

Remark: The reason why we require $0 < \alpha < 1$ in the case $r/2 < s \leq r-1$ is $\sigma_0(D^{(s)}) = 1$ for $r/2 < s \leq r-1$.

5.2 The s-distances and s-diameters in hypergraphs

Let H be an r-uniform hypergraph. For $1 \le s \le r-1$ and $x, y \in V^{\underline{s}}$, the s-distance $d^{(s)}(x,y)$ is the minimum integer k such that there is an s-path of length k starting at x and ending at y. For $X, Y \subseteq V^{\underline{s}}$, let $d^{(s)}(X,Y) = \min\{d^{(s)}(x,y) \mid x \in X, y \in Y\}$. If H is s-connected, then the s-diameter $\mathrm{diam}^{(s)}(H)$ satisfies

$$\mathrm{diam}^{(s)}(H) = \max_{x,y \in V^{\underline{s}}} \{d^{(s)}(x,y)\}.$$

For $1 \le s \le \frac{r}{2}$, the s-distances in H (and the s-diameter of H) are simply the graph distances in $G^{(s)}$ (and the diameter of $G^{(s)}$), respectively. Applying Theorem 2 and Corollary 1, we have the following theorems.

Theorem 11 *Suppose H is an r-uniform hypergraph. For integer s such that $1 \le s \le \frac{r}{2}$, let $\lambda_1^{(s)}$ (and $\lambda_{\max}^{(s)}$) be the first non-trivial (and the last) eigenvalue of the s-th Laplacian of H. Suppose $\lambda_{\max}^{(s)} > \lambda_1^{(s)} > 0$. For $X, Y \subseteq V^{\underline{s}}$, if $d^{(s)}(X,Y) \ge 2$, then we have*

$$d^{(s)}(X,Y) \le \left\lceil \frac{\log \sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}}}{\log \frac{\lambda_{\max}^{(s)}+\lambda_1^{(s)}}{\lambda_{\max}^{(s)}-\lambda_1^{(s)}}} \right\rceil.$$

Here vol$(*)$ *are volumes in $G^{(s)}$.*

Remark: We know $\lambda_1^{(s)} > 0$ if and only if H is s-connected. The condition $\lambda_{\max}^{(s)} > \lambda_1^{(s)}$ holds unless $s = 1$ and every pair of vertices is covered by edges evenly (i.e., H is a 2-design).

Theorem 12 *Suppose H is an r-uniform hypergraph. For integer s such that $1 \le s \le \frac{r}{2}$, let $\lambda_1^{(s)}$ (and $\lambda_{\max}^{(s)}$) be the first non-trivial (and the last) eigenvalue of the s-th Laplacian of H. If $\lambda_{\max}^{(s)} > \lambda_1^{(s)} > 0$, then the s-diameter of an r-uniform hypergraph H satisfies*

$$diam^{(s)}(H) \le \left\lceil \frac{\log \frac{\mathrm{vol}(V^{\underline{s}})}{\delta^{(s)}}}{\log \frac{\lambda_{\max}^{(s)}+\lambda_1^{(s)}}{\lambda_{\max}^{(s)}-\lambda_1^{(s)}}} \right\rceil.$$

Here $\mathrm{vol}(V^{\underline{s}}) = \sum_{x \in V^{\underline{s}}} d_x = |E(H)| \frac{r!}{(r-2s)!}$ *and $\delta^{(s)}$ is the minimum degree in $G^{(s)}$.*

When $r/2 < s \le r-1$, the s-distances in H (and the s-diameter of H) is the directed distance in $D^{(s)}$ (and the diameter of $D^{(s)}$), respectively. Applying Theorem 8 and its remark, we have the following theorems.

Theorem 13 *Let H be an r-uniform hypergraph. For $r/2 < s \le r-1$ and $X, Y \subseteq V^{\underline{s}}$, if H is s-connected, then we have*

$$d^{(s)}(X,Y) \le \left\lfloor \frac{\log \frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}}{\log \frac{2}{2-\lambda_1^{(s)}}} \right\rfloor + 1.$$

Here $\lambda_1^{(s)}$ is the first non-trivial eigenvalue of the Laplacian of $D^{(s)}$, and vol$(*)$ *are volumes in $D^{(s)}$.*

Theorem 14 *For $r/2 < s \le r-1$, suppose that an r-uniform hypergraph H is s-connected. Let $\lambda_1^{(s)}$ be the smallest nonzero eigenvalue of the Laplacian of $D^{(s)}$. The s-diameter of H satisfies*

$$diam^{(s)}(H) \le \left\lceil \frac{2\log \frac{\mathrm{vol}(\mathrm{V}^{\underline{s}})}{\delta^{(s)}}}{\log \frac{2}{2-\lambda_1^{(s)}}} \right\rceil .$$

Here $\mathrm{vol}(\mathrm{V}^{\underline{s}}) = \sum_{x\in \mathrm{V}^{\underline{s}}} d_x = |E(H)|r!$ *and* $\delta^{(s)}$ *is the minimum degree in* $D^{(s)}$.

5.3 The edge expansions in hypergraphs

In this subsection, we prove some results on the edge expansions in hypergraphs.

Let H be an r-uniform hypergraph. For $S \subseteq \binom{V}{s}$, we recall that the volume of S satisfies

$$\mathrm{vol}(S) = \sum_{x\in S} d_x.$$

Here d_x is the degree of the set x in H. In particular, we have

$$\mathrm{vol}\left(\binom{V}{s}\right) = |E(H)|\binom{r}{s}.$$

The *density* $e(S)$ of S is $\frac{\mathrm{vol}(S)}{\mathrm{vol}(\binom{V}{s})}$. Let $\bar{S}$ be the complement set of S in $\binom{V}{s}$. We have

$$e(\bar{S}) = 1 - e(S).$$

For $1 \le t \le s \le r-t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let

$$E(S,T) = \{F \in E(H) : \exists x \in S, \exists y \in T, x\cap y = \emptyset, \text{ and } x \cup y \subseteq F\}.$$

Note that $|E(S,T)|$ counts the number of edges contains $x \sqcup y$ for some $x \in S$ and $y \in T$.

Particularly, we have

$$\left|E\left(\binom{V}{s},\binom{V}{t}\right)\right| = |E(H)|\frac{r!}{s!t!(r-s-t)!}.$$

Theorem 15 *For $1 \le t \le s \le \frac{r}{2}$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $e(S,T) = \frac{|E(S,T)|}{|E(\binom{V}{s},\binom{V}{t})|}$. We have*

$$|e(S,T) - e(S)e(T)| \le \bar{\lambda}^{(s)}\sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \qquad (14)$$

Proof: Let $G^{(s)}$ be the weighed undirected graph defined in section 3. Define S' and T' (sets of ordered s-tuples) as follows

$$S' = \{x \in \mathrm{V}^{\underline{s}} \mid [x] \in S\};$$

$$T' = \{(y,z) \in \mathrm{V}^{\underline{s}} \mid [y] \in T\}.$$

Let $\bar{S}'$ (or $\bar{T}'$) be the complement set of S' (or T') in $V^{\underline{s}}$, respectively. We make a convention that $\mathrm{vol}_{G^{(s)}}(*)$ denotes volumes in $G^{(s)}$ while $\mathrm{vol}(*)$ denotes volumes H. We have

$$\begin{aligned}
\mathrm{vol}_{G^{(s)}}(G^{(s)}) &= \mathrm{vol}\left(\binom{V}{s}\right)\frac{s!(r-s)!}{(r-2s)!}; &(15)\\
\mathrm{vol}_{G^{(s)}}(S') &= \mathrm{vol}(S)\frac{s!(r-s)!}{(r-2s)!}; &(16)\\
\mathrm{vol}_{G^{(s)}}(T') &= \mathrm{vol}(T)\frac{t!(r-t)!}{(r-2s)!}; &(17)\\
\mathrm{vol}_{G^{(s)}}(\bar{S}') &= \mathrm{vol}(\bar{S})\frac{s!(r-s)!}{(r-2s)!}; &(18)\\
\mathrm{vol}_{G^{(s)}}(\bar{T}') &= \mathrm{vol}(\bar{T})\frac{t!(r-t)!}{(r-2s)!}. &(19)
\end{aligned}$$

Let $E_{G^{(s)}}(S', T')$ be the number of edges between S' and T' in $G^{(s)}$. We get

$$|E_{G^{(s)}}(S', T')| = \frac{(r-s-t)!s!t!}{(r-2s)!}|E(S,T)|.$$

Applying Theorem 3 to the sets S' and T' in $G^{(s)}$, we obtain

$$\begin{aligned}
&\left||E_{G^{(s)}}(S', T')| - \frac{\mathrm{vol}_{G^{(s)}}(S')\mathrm{vol}_{G^{(s)}}(T')}{\mathrm{vol}_{G^{(s)}}(G^{(s)})}\right| \\
&\qquad \leq \bar{\lambda}_1^{(s)} \frac{\sqrt{\mathrm{vol}_{G^{(s)}}(S')\mathrm{vol}_{G^{(s)}}(T')\mathrm{vol}_{G^{(s)}}(\bar{S}')\mathrm{vol}_{G^{(s)}}(\bar{T}')}}{\mathrm{vol}_{G^{(s)}}(G^{(s)})}.
\end{aligned}$$

Combining equations (15-19) and the inequality above, we obtain inequality 14. □

Now we consider the case that $s > \frac{r}{2}$. Due to the fact that $\sigma_0^{(s)} = 1$, we have to use the weaker expansion theorem 7. Note that

$$\left|E\left(\binom{V}{s}, \binom{V}{t}\right)\right| = |E(H)|\frac{r!}{(r-s-t)!s!t!}.$$

We get the following theorem.

Theorem 16 *For $1 \leq t < \frac{r}{2} < s < s+t \leq r$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $e(S,T) = \frac{|E(S,T)|}{|E(\binom{V}{s},\binom{V}{t})|}$. If $|x \cap y| \neq \min\{t, 2s-r\}$ for any $x \in S$ and $y \in T$, then we have*

$$|\frac{1}{2}e(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)}\sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \tag{20}$$

Proof: Recall that $D^{(s)}$ is the directed graph defined in section 3. Let

$$S' = \{x \in V^{\underline{s}} \mid [x] \in S\};$$

$$T' = \{(y,z) \in V^{\underline{s}} \mid [z] \in T\}.$$

We also denote $\bar{S}'$ (or $\bar{T}'$) be the complement set of S' (or T') in $V^{\underline{s}}$, respectively. We use the convention that $\mathrm{vol}_{D^{(s)}}(*)$ denotes the volumes in $D^{(s)}$ while $\mathrm{vol}(*)$ denotes the volumes in the hypergraph H. We have

$$\begin{aligned}
\mathrm{vol}_{D^{(s)}}(D^{(s)}) &= \mathrm{vol}\left(\binom{V}{s}\right)s!(r-s)!; &(21)\\
\mathrm{vol}_{D^{(s)}}(S') &= \mathrm{vol}(S)s!(r-s)!; &(22)\\
\mathrm{vol}_{D^{(s)}}(T') &= \mathrm{vol}(T)t!(r-t)!; &(23)\\
\mathrm{vol}_{D^{(s)}}(\bar{S}') &= \mathrm{vol}(\bar{S})s!(r-s)!; &(24)\\
\mathrm{vol}_{D^{(s)}}(\bar{T}') &= \mathrm{vol}(\bar{T})s!(r-s)!. &(25)
\end{aligned}$$

Let $E_{D^{(s)}}(S',T')$ (or $E_{D^{(s)}}(T',S')$) be the number of directed edges from S' to T' (or from T' to S') in $D^{(s)}$, respectively. We get

$$|E_{D^{(s)}}(S',T')| = (r-s-t)!s!t!|E(S,T)|.$$

From the condition $|x \cap y| \neq \min\{t, 2s-r\}$ for each $x \in S$ and each $y \in T$, we observe

$$E_{D^{(s)}}(T',S') = 0.$$

Applying Theorem 7 to the sets S' and T' in $D^{(s)}$, we obtain

$$\left| \frac{|E_{D^{(s)}}(S',T')| + |E_{D^{(s)}}(T',S')|}{2} - \frac{\mathrm{vol}_{D^{(s)}}(S')\mathrm{vol}_{D^{(s)}}(T')}{\mathrm{vol}_{D^{(s)}}(D^{(s)})} \right| \leq \bar{\lambda}_1^{(s)} \frac{\sqrt{\mathrm{vol}_{D^{(s)}}(S')\mathrm{vol}_{D^{(s)}}(T')\mathrm{vol}_{D^{(s)}}(\bar{S}')\mathrm{vol}_{D^{(s)}}(\bar{T}')}}{\mathrm{vol}_{D^{(s)}}(D^{(s)})}.$$

Combining equations (21-25) and the inequality above, we get inequality 20. □

Nevertheless, we have the following strong edge expansion theorem for $\frac{r}{2} < s \leq r-1$. For $S, T \subseteq \binom{V}{s}$, let $E'(S,T)$ be the set of edges of the form $x \cup y$ for some $x \in S$ and $y \in T$. Namely,

$$E'(S,T) = \{F \in E(H) \mid \exists x \in S, \exists y \in T, F = x \cup y\}.$$

Observe that

$$\left| E'\left(\binom{V}{s}, \binom{V}{s}\right) \right| = |E(H)| \frac{r!}{(r-s)!(2s-r)!(r-s)!}.$$

Theorem 17 *For $\frac{r}{2} < s \leq r-1$ and $S, T \subseteq \binom{V}{s}$, let $e'(S,T) = \frac{|E'(S,T)|}{|E'(\binom{V}{s}, \binom{V}{s})|}$. We have*

$$|e'(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \tag{26}$$

Proof: Let

$$S' = \{x \in \mathrm{V}^{\underline{s}} \mid [x] \in S\};$$
$$T' = \{y \in \mathrm{V}^{\underline{s}} \mid [y[\in T\}.$$

Let $\bar{S}'$ (or $\bar{T}'$) be the complement set of S' (or T' respectively) in $\mathrm{V}^{\underline{s}}$. We use the convention that $\mathrm{vol}_{D^{(s)}}(*)$ denotes the volumes in $D^{(s)}$ while $\mathrm{vol}(*)$ denotes the volumes in the hypergraph H. We have

$$\mathrm{vol}_{D^{(s)}}(D^{(s)}) = \mathrm{vol}\left(\binom{V}{s}\right) s!(r-s)!; \tag{27}$$
$$\mathrm{vol}_{D^{(s)}}(S') = \mathrm{vol}(S)s!(r-s)!; \tag{28}$$
$$\mathrm{vol}_{D^{(s)}}(T') = \mathrm{vol}(T)s!(r-s)!; \tag{29}$$
$$\mathrm{vol}_{D^{(s)}}(\bar{S}') = \mathrm{vol}(\bar{S})s!(r-s)!; \tag{30}$$
$$\mathrm{vol}_{D^{(s)}}(\bar{T}') = \mathrm{vol}(\bar{T})s!(r-s)!. \tag{31}$$

Let $E_{D^{(s)}}(S',T')$ (or $E_{D^{(s)}}(T',S')$) be the number of directed edges from S' to T' (or from T' to S') in $D^{(s)}$, respectively. We get

$$|E_{D^{(s)}}(S',T')| = |E_{D^{(s)}}(T',S')| = (r-s)!(2s-r)!(r-s)!|E'(S,T)|.$$

Applying Theorem 7 to the sets S' and T' on $D^{(s)}$, we obtain

$$\left| \frac{|E_{D^{(s)}}(S',T')| + |E_{D^{(s)}}(T',S')|}{2} - \frac{\mathrm{vol}_{D^{(s)}}(S')\mathrm{vol}_{D^{(s)}}(T')}{\mathrm{vol}_{D^{(s)}}(D^{(s)})} \right| \\ \leq \bar{\lambda}_1^{(s)} \frac{\sqrt{\mathrm{vol}_{D^{(s)}}(S')\mathrm{vol}_{D^{(s)}}(T')\mathrm{vol}_{D^{(s)}}(\bar{S}')\mathrm{vol}_{D^{(s)}}(\bar{T}')}}{\mathrm{vol}_{D^{(s)}}(D^{(s)})}.$$

Combining equations (27-31) and the inequality above, we get inequality 26. □

6 Concluding Remarks

In this paper, we introduced a set of Laplacians for r-uniform hypergraphs. For $1 \leq s \leq r-1$, the s-Laplacian $\mathcal{L}^{(s)}$ is derived from the random s-walks on hypergraphs. For $1 \leq s \leq \frac{r}{2}$, the s-th Laplacian $\mathcal{L}^{(s)}$ is defined to be the Laplacian of the corresponding weighted graph $G^{(s)}$. The first Laplacian $\mathcal{L}^{(1)}$ is exactly the Laplacian introduced by Rodrìguez [9].

For $\frac{r}{2} \leq s \leq r-1$, the $\mathcal{L}^{(s)}$ is defined to be the Laplacian of the corresponding Eulerian directed graph $D^{(s)}$. At first glimpse, $\sigma_0(D^{(s)})$ might be a good parameter. However, it is not hard to show that $\sigma_0(D^{(s)}) = 1$ always holds, which makes Theorem 6 useless for hypergraphs. We can use weaker Theorem 7 for hypergraphs. Our work is based on (with some improvements) Chung's recent work [7, 8] on directed graphs.

Let us recall Chung's definition of Laplacians [4] for regular hypergraphs. An r-uniform hypergraph H is d-regular if $d_x = d$ for every $x \in \mathrm{V}^{\underline{r-1}}$. Let G be a graph on the vertex set $\mathrm{V}^{\underline{r-1}}$. For $x, y \in \mathrm{V}^{\underline{r-1}}$, let xy be an edge if $x = x_1x_2,\ldots,x_{r-1}$ and $y = y_1x_2,\ldots,x_{r-1}$ such that $\{x_1, y_1, x_2, \ldots, x_{r-1}\}$ is an edge of H. Let A be the adjacency matrix of G, T be the diagonal matrix of degrees in G, and K be the adjacency matrix of the complete graph on the edge set $\mathrm{V}^{\underline{r-1}}$. Chung [4] defined the Laplacian $\mathcal{L}$ such that

$$\mathcal{L} = T - A + \frac{d}{n}(K + (r-1)I).$$

This definition comes from the homology theory of hypergraphs. Firstly, $\mathcal{L}$ is not normalized in Chung's definition, i.e., the eigenvalues are not in the interval $[0, 2]$. Secondly, the add-on term $\frac{d}{n}(K + (r-1)I)$ is not related to the structures of H. If we ignore the add-on term and normalize the matrix, we essentially get the Laplacian of the graph G. Note G is disconnected, then $\lambda_1(G) = 0$ and it is not interesting. Thus Chung added the additional term. The graph G is actually very closed to our Eulerian directed graph $D^{(r-1)}$. Let B be the adjacency matrix of $D^{(r-1)}$. In fact we have $B = QA$, where Q is a rotation which maps $x = x_1, x_2 \ldots, x_{r-1}$ to $x' = x_2 \ldots, x_{r-1}, x_1$. Since $d_x = d_{x'}$, Q and T commute, we have

$$\begin{aligned} (T^{-1/2}BT^{-1/2})'(T^{-1/2}BT^{-1/2}) &= T^{-1/2}B'T^{-1}BT^{-1/2} \\ &= T^{-1/2}A'Q'T^{-1}QAT^{-1/2} \\ &= T^{-1/2}A'T^{-1}Q'QAT^{-1/2} \\ &= T^{-1/2}A'T^{-1}AT^{-1/2}. \end{aligned}$$

Here we use the fact $Q'Q = I$. This identity means that the singular values of $I - \mathcal{L}^{(r-1)}$ is precisely equal to 1 minus the Laplacian eigenvalues of the graph G.

Our definitions of Laplacians $\mathcal{L}^{(s)}$ are clearly related to the quasi-randomness of hypergraphs. We are very interested in this direction. Many concepts such as the s-walk, the s-path, the s-distance, and the s-diameter, have their independent interest.

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