# A new asymptotic enumeration technique: the Lovász Local Lemma

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**Abstract**

Our previous paper [14] applied a lopsided version of the Lovász Local Lemma that allows negative dependency graphs [11] to the space of random injections from an $m$-element set to an $n$-element set. (Equivalently, the same story can be told about the space of random matchings in $K_{n,m}$.) In this paper we show how the lopsided version of the Lovász Local Lemma applies to the space of random matchings in $K_{2n}$. We also prove tight upper bounds that asymptotically match the lower bound given by the Lovász Local Lemma. As a consequence, we give new proofs to a number of results on the enumeration of permutations, Latin rectangles, and regular graphs. The strength of the method is shown by a new result: enumeration of graphs by degree sequence or bipartite degree sequence and girth. As another application, we provide a new proof to the classical probabilistic result of Erdős [8] that showed the existence of graphs with arbitrary large girth and chromatic number. If the degree sequence satisfies some mild conditions, almost all graphs with this degree sequence and prescribed girth have high chromatic number.

## 1 Lovász Local Lemma with negative dependency graphs

This is a sequel to our previous paper [14] and we use the same notations. Let $A_1, A_2, \ldots, A_n$ be events in a probability space.

A *negative dependency graph* for $A_1, \ldots, A_n$ is a simple graph on $[n]$ satisfying

$$\Pr(A_i | \wedge_{j\in S} \overline{A_j}) \leq \Pr(A_i), \tag{1}$$

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for any index $i$ and any subset $S \subseteq \{j \mid ij \notin E(G)\}$, whenever the conditional probability $\Pr(A_i \mid \wedge_{j\in S}\overline{A_j})$ is well-defined, i.e. $\Pr(\wedge_{j\in S}\overline{A_j}) > 0$. We will make use of the fact that inequality (1) trivially holds when $\Pr(A_i) = 0$, otherwise the following inequality is equivalent to inequality (1):

$$\Pr(\wedge_{j\in S}\overline{A_j} \mid A_i) \leq \Pr(\wedge_{j\in S}\overline{A_j}). \tag{2}$$

For variants of the Lovász Local Lemma with increasing strength, see [10, 21, 11, 13]:

**Lemma 1** **[Lovász Local Lemma.]** *Let $A_1, \ldots, A_n$ be events with a negative dependency graph $G$. If there exist numbers $x_1, \ldots, x_n \in [0, 1)$ such that*

$$\Pr(A_i) \leq x_i \prod_{ij\in E(G)} (1 - x_j) \tag{3}$$

*for all $i$, then*

$$\Pr(\wedge_{i=1}^{n}\overline{A_i}) \geq \prod_{i=1}^{n}(1 - x_i). \tag{4}$$

The main obstacle for using Lemma 1 is the difficulty to define a useful negative dependency graph other than a dependency graph. In [14], we described a general way to create negative dependency graphs in the space of random functions $U \to V$ equipped with uniform distribution. Namely, let the events be the set of all extensions of some particular *partial* functions to functions; and create an edge for the negative dependency graph, if the partial functions have common elements in their domains or ranges, other than the agreement of the partial functions. These events also can be thought of as all extensions of (partial) matchings in the complete bipartite graph with classes $U, V$, where an edge of the negative dependency graph comes from two event-defining (partial) matchings whose union is no longer a (partial) matching after suppressing multiple edges. In [14], we used this technique to prove a new result on the Turán hypergraph problem, and we found surprising applications as proving lower bounds (matching certain asymptotic formulas) for permutation and Latin rectangle enumeration problems.

In this paper, we show an analogous construction of a negative dependency graph for events, which live in the space of random matchings of a complete graph. We require that the events are the set of all extensions of (partial) matchings in a complete graph to perfect matchings, and two event-defining partial matchings make an edge, if their union is no longer a (partial) matching after suppressing multiple edges. (Although our construction fails for extensions of partial matchings of arbitrary graphs, there might be some other graph classes providing interesting results.)

We move one step further and show some general and some specific *upper bounds* for the event estimated by the Lovász Local Lemma, and show that for large classes of problems the upper bound is asymptotically equal to the lower bound. These results apply to the permutation enumeration problems in [14], and to enumeration problems of regular graphs. Many asymptotic enumeration results that we prove are not new and typically do not give the largest known valid range of the asymptotic formula, but are nontrivial results and often more recent than the Lovász Local Lemma itself. They come out from our framework elementarily, and even easily.

The strength of the framework is shown by a new result: enumeration of graphs by degree sequence and girth, under mild condititions for the degree sequence. We also provide an analogous enumeration result for bipartite degree sequence and girth.

There is literature on some improvements on the Lovász Local Lemma using methods of statistical physics, e.g. [20], [18], that we do not touch upon this paper, as they are difficult to use and the improvement would be tiny, if present at all, in a resulting asymptotic formula.

In a forthcoming paper we will extend our negative dependency graph construction to matchings in complete $r$-uniform hypergraphs and will apply this result to hypergraph enumeration.

As another application, we revisit a classic of the probabilistic method: Erdős' proof to the existence of graphs with arbitrary large girth and chromatic number [8]. We show that if the degree sequence satisfies some mild conditions, almost all graphs with this degree sequence and prescribed girth have high chromatic number.

In a scenario of the Poisson paradigm, we estimate the probability that none of a set of rare events occur. Let $X$ be the sum of the indicator variables of these events and $\mu = \mathrm{E}(X)$. If the dependency among these events is rare, then one would expect that $X$ has a Poisson distribution with mean $\mu$. In particular, $\Pr(X = 0) \approx e^{-\mu}$. The Janson inequality and Brun's sieve method [1] are often the good choice to solve these kind of problems. Now we offer an alternative approach—using Lovász Local Lemma. Our approach can be considered as an analogue of the Janson inequality in another setting that offers plenty of applications. It is curious that the proof of Boppana and Spencer [5] for the Janson inequality (see also in [1]) uses conditional probabilities somewhat similarly to the proof of the Lovász Local Lemma. There is an inherent reason why we do not get the "second term" in asymptotic enumeration, like in (37) or (40), which extends the range of the asymptotic formula: $e^{-\mu}$ is *between* our lower and upper bounds (see Theorem 5), and therefore we cannot add a correction term to $-\mu$ in the exponent.

For further research, it would be interesting to get asymptotics for further terms from the Poisson distribution, i.e. for the probability of exactly $k$ events holding, for any fixed $k$. Lots of further applications of our framework are possible, this paper gives just a sampler of applications.

## 2 Some general results on negative and near-positive dependency graphs

These lower and upper bounds are *general* in the sense that there is no assumption on the events being defined through matchings.

All over this paper, we will be using a useful function, which cannot be expressed in a closed form. In the following lemma we summarize the properties that we will need.

**Lemma 2** (i) *For $0 \le \gamma \le 1/4$, the equation*

$$1 = ye^{-\gamma y} \tag{5}$$

*has a unique solution in $1 \le y \le 2$, and defines a function $y(\gamma)$.*

(ii) *$y(\gamma) = -\mathrm{LambertW}(-\gamma)/\gamma$, where* LambertW *is the compositional inverse of $xe^{-x}$.*

(iii) *As the Taylor series of* $\mathrm{LambertW}(\gamma)$ *is convergent for $|\gamma| < 1/e$, so does the Taylor series of $y(\gamma)$.*

(iv) *$y(\gamma)$ is strictly increasing on* $[0, 1/4]$.

(v) *For $\gamma \to 0$,*

$$y(\gamma) = 1 + \gamma + \frac{3}{2}\gamma^2 + \frac{8}{3}\gamma^3 + \frac{125}{24}\gamma^4 + \frac{54}{5}\gamma^5 + O(\gamma^6). \tag{6}$$

(vi) *For $0 \le \gamma \le 1/4$,*

$$1 + \gamma + \frac{3}{2}\gamma^2 \le y(\gamma) \le 1 + \gamma + \frac{3}{2}\gamma^2 + 66\gamma^3. \tag{7}$$

**Proof:** (ii) and (iv) can be obtained with Maple. As the RHS of (5) $< 1$ at $y = 1$ and $> 1$ at $y = 2$, there is a solution in between for (5). Using implicit differentiation, $y'(\gamma) > 0$ in $[0, 1/4]$, proving (iii) and the uniqueness claim in (i). Finally, for (v), estimates for $y'''(\gamma)$ were obtained with Maple. $\square$

In many applications we have a *sequence* of problems, where $\Pr(A_i)$ and $\sum_{ij\in E(G)} \Pr(A_j)$ are so small that one can set $x_i =: (1 + o(1))\Pr(A_i)$ to use Lemma 1.

**Theorem 1** *Let $A_1, \ldots, A_n$ be events with negative dependency graph $G$. Let us be given any $\epsilon$ with $0 < \epsilon < 1/4$. If*

$$\Pr(A_i) < \epsilon \quad \textit{and} \quad \sum_{j:ij\in E(G)} \Pr(A_j) + 2\Pr^2(A_j) < \epsilon \tag{8}$$

*for every $1 \le i \le n$, then*

(i) *for any $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, we have*

$$\Pr(\wedge_{i\in S}\overline{A_i} \mid \wedge_{j\in T}\overline{A_j}) \geq \prod_{i\in S}\Big(1 - \Pr(A_i)y(\epsilon)\Big); \tag{9}$$

(ii) *in particular, we have*

$$\Pr(\wedge_{i=1}^n \overline{A_i}) \geq \exp\left(-\sum_{i=1}^n \Pr(A_i)y(\epsilon) - \sum_{i=1}^n \Pr^2(A_i)y^2(\epsilon)\right). \tag{10}$$

**Proof:** Set $x_i = \Pr(A_i)y(\epsilon)$. It is clear that $0 \leq x_i < 1/2$. Observe that for $0 \leq x \leq 1/2$ we have $1 - x \geq e^{-x-x^2}$. To use Lemma 1, we need the condition (3). Indeed, $\Pr(A_i) = x_i/y(\epsilon) = x_i e^{-\epsilon y(\epsilon)} \leq x_i \exp\big(-\sum_{j:ij\in E(G)}(x_j + x_j^2)\big) \leq x_i \prod_{j:ij\in E(G)}(1 - x_j)$. To prove (i), we recall not the conclusion of Lovász Local Lemma, but a crucial step in the proof (see [21], [13]): for any $T \subseteq V(G)$ with $i \notin T$, we have $\Pr(A_i \mid \wedge_{j\in\mathcal{T}, j\neq i}\overline{A_j}) \leq x_i$, which in our case yields for any $i \in S$

$$\Pr(A_i \mid \wedge_{j'\in T}\overline{A_{j'}}) \leq x_i = \Pr(A_i)y(\epsilon).$$

Assume that $S = \{m_1, m_2, ..., m_s\}$. We have

$$\Pr(\overline{A_{m_1}} \wedge \overline{A_{m_2}} \wedge .... \wedge \overline{A_{m_s}} \mid \wedge_{j\in T}\overline{A_j}) =$$

$$\prod_{\ell=1}^s \left[\Pr\Big(\overline{A_{m_\ell}} \mid \overline{A_{m_1}} \wedge \overline{A_{m_2}} \wedge .... \wedge \overline{A_{m_{\ell-1}}} \wedge (\wedge_{j\in T}\overline{A_j})\Big)\right] =$$

$$\prod_{\ell=1}^s \left[1 - \Pr\Big(A_{m_\ell} \mid \overline{A_{m_1}} \wedge \overline{A_{m_2}} \wedge .... \wedge \overline{A_{m_{\ell-1}}} \wedge (\wedge_{j\in T}\overline{A_j})\Big)\right] \geq \prod_{\ell=1}^s (1 - x_{m_\ell}).$$

The conclusion of (ii) is implied by (i) with $T = \emptyset$ or by Lemma 1: $\Pr(\wedge_{i=1}^n \overline{A_i}) \geq \prod_i (1 - x_i) = \prod_i \big(1 - \Pr(A_i)y(\epsilon)\big) \geq \exp\Big(-\sum_{i=1}^n \Pr(A_i)y(\epsilon) - \sum_{i=1}^n \Pr^2(A_i)y^2(\epsilon)\Big)$.
□

Theorem 1 provided *logarithmic asymptotics* for the expected Poisson type lower bound when $\epsilon \to 0$ for a *sequence* of problems and estimations. However, we want *asymptotics*, and obtain it with slightly more assumptions:

**Corollary 1** *Set $\mu = \sum_i \Pr(A_i)$. If for a sequence of problems $\epsilon\mu \to 0$, then*

$$\Pr(\wedge_{i=1}^n \overline{A_i}) \geq (1 - o(1))e^{-\mu}. \tag{11}$$

*This holds, in particular, when $\mu$ is bounded and $\epsilon \to 0$.*

We comment here that this result does not allow a good generalization with *different* bounds on $\Pr(A_i)$ and $\sum_{j:ij\in E(G)} \Pr(A_j)$.

Next we give a crucial new definition. For the events $A_1, \ldots, A_n$ in a probability space $\Omega$, and an $\epsilon$ with $1 > \epsilon > 0$, we define an *$\epsilon$-near-positive dependency graph* to be a graph $G$ on $V(G) = [n]$ satisfying

(i) $\Pr(A_i \wedge A_j) = 0$ if $ij \in E(G)$.

(ii) For any index $i$ and any subset $i \notin T \subseteq \{j \mid ij \notin E(G)\}$,

$$\Pr(A_i \mid \wedge_{j\in T}\overline{A_j}) \geq (1-\epsilon)\Pr(A_i),$$

whenever the conditional probability is well-defined.

**Theorem 2** *Let $A_1, \ldots, A_n$ be events with an $\epsilon$-near-positive dependency graph $G$. Then we have*

$$\Pr(\wedge_{i=1}^n \overline{A_i}) \leq \prod_{i=1}^n [1-(1-\epsilon)\Pr(A_i)].$$

**Proof:** If $\Pr(\wedge_{i=1}^n \overline{A_i}) = 0$, then the conclusion holds. So we may assume without loss of generality that $\Pr(\wedge_{i=1}^n \overline{A_i}) > 0$. Now we would like to show that for any $i$ and any subset $S \subseteq V(G)$ with $i \notin S$,

$$\Pr(A_i \mid \wedge_{j\in S}\overline{A_j}) \geq (1-\epsilon)\Pr(A_i),$$

as the conditional probability above is well-defined by our assumption. Write $S = S_1 \cup S_2$, where $S_1 = S \cap N_G(i)$ and $S_2 = S \setminus S_1$. We have

$$\begin{aligned}
\Pr(A_i \mid \wedge_{j\in S}\overline{A_j}) &= \frac{\Pr(A_i \wedge (\wedge_{k\in S_1}\overline{A_k}) \mid \wedge_{j\in S_2}\overline{A_j})}{\Pr(\wedge_{k\in S_1}\overline{A_k} \mid \wedge_{j\in S_2}\overline{A_j})} \\
&= \frac{\Pr(A_i \mid \wedge_{j\in S_2}\overline{A_j})}{\Pr(\wedge_{k\in S_1}\overline{A_k} \mid \wedge_{j\in S_2}\overline{A_j})} \\
&\geq \Pr(A_i \mid \wedge_{j\in S_2}\overline{A_j}) \\
&\geq (1-\epsilon)\Pr(A_i).
\end{aligned}$$

(The first part of the definition of the $\epsilon$-near-positive dependency graph, $\Pr(A_i \wedge A_j) = 0$ for $ij$ edges, allowed the elimination of the $\wedge_{k\in S_1}\overline{A_k}$ term.) Hence, we have

$$\begin{aligned}
\Pr(\wedge_{i=1}^n \overline{A_i}) &= \prod_{i=1}^n \Pr(\overline{A_i} \mid \wedge_{k=i+1}^n \overline{A_k}) = \\
\prod_{i=1}^n [1-\Pr(A_i \mid \wedge_{k=i+1}^n \overline{A_k})] &\leq \prod_{i=1}^n (1-(1-\epsilon)\Pr(A_i)). \qquad \Box
\end{aligned}$$

# 3 Instances for negative dependency graphs: The space of random matchings of $K_N$ and $K_{N,N'}$

Let $\Omega$ denote the probability space of perfect matchings of the complete bipartite graph $K_{N,N'}$ ($N \leq N'$) or the probability space of the complete graph $K_N$ for an even integer $N$; equipped with the uniform distribution. We are going to apply the Lovász Local Lemma (Lemma 1) in $\Omega$ by identifying a class of negative dependency graphs. For any (not necessary perfect) matching $M$, let $A_M$ be the set of perfect matchings extending $M$:

$$A_M = \{F \in \Omega \mid M \subseteq F\}. \tag{12}$$

We will term an event $A_M$ in (12), with $M \neq \emptyset$, a *canonical event*. We will say that two matchings, $M_1$ and $M_2$, are in *conflict*, if $M_1 \cup M_2$ is not a matching after suppressing multiple edges. For a matching $M$, we will denote by $supp(M)$ the support set of the matching, i.e. the $2|M|$ vertices that its edges cover. We leave the following easy lemma to the reader:

**Lemma 3** (i)

$$\omega \in \overline{A_M} \quad \textit{iff} \quad \exists e \in \omega\ \exists f \in M \ \textit{with} \ \ |e \cap f| = 1. \tag{13}$$

(ii) *Matchings $M_1$ and $M_2$ are in conflict iff $A_{M_1} \wedge A_{M_2} = \emptyset$.*

(iii) *If the matchings $F$ and $M$ are not in conflict, then*

$$\overline{A_{M\setminus F}} \subseteq \overline{A_M} \quad \textit{and} \quad \overline{A_M} \wedge A_F = \overline{A_{M\setminus F}} \wedge A_F. \tag{14}$$

**Theorem 3** *Let $\mathcal{M}$ be a collection of matchings in $K_N$ or $K_{N,N'}$. The graph $G = G(\mathcal{M})$ described below is a negative dependency graph for the canonical events $\{A_M \mid M \in \mathcal{M}\}$:*

- $V(G) = \mathcal{M}$,
- $E(G) = \Big\{\{M_1, M_2\} \mid M_1 \in \mathcal{M} \textit{ and } M_2 \in \mathcal{M} \textit{ are in conflict}\Big\}$.

**Proof:** For complete bipartite graphs we proved this theorem in [14], and therefore we have to prove it now for $K_N$. We will prove the theorem by induction on $N$. The base case $N = 2$ is trivial. Throughout this paper, we always assume that the vertex set of $K_N$ is $[N] = \{1, 2, \ldots, N\}$. There is a canonical injection from $[N]$ to $[N+s]$, and consequently from $V(K_N)$ to $V(K_{N+s})$ and from $E(K_N)$ to $E(K_{N+s})$. Through this canonical injection, every matching of $K_N$ can be viewed as a matching of $K_{N+s}$. (Note that a perfect matching in $K_N$ will not be perfect in $K_{N+s}$ for $s > 0$.) To emphasize the difference in the size of the vertex set, we use $A_M^N$ to denote the event induced by the matching $M$ among the matchings of an $N$-vertex complete graph.

**Lemma 4** *For any collection $\mathcal{M}$ of matchings in $K_N$, we have*

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^N}) \leq \Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}}).$$

**Proof:** We partition the space of $\Omega_{N+2}$ into $N+1$ sets as follows: for $1 \le i \le N+1$, let $\mathcal{C}_i$ be the set of perfect matchings containing the edge $i(N+2)$. We have

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}}) = \sum_{i=1}^{N+1} \Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}} \wedge \mathcal{C}_i).$$

We observe that $\mathcal{C}_i \subseteq \overline{A_M^{N+2}}$ if and only if $M$ conflicts $i(N+2)$, a one-edge matching. Let $\mathcal{B}_i$ be the subset of $\mathcal{M}$, whose elements are not in conflict with the edge $i(N+2)$. (In particular, $\mathcal{B}_{N+1} = \mathcal{M}$.) We have

$$\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}} \wedge \mathcal{C}_i = \wedge_{M\in\mathcal{B}_i}\overline{A_M^{N+2}} \wedge \mathcal{C}_i.$$

Let $\phi_i$ be the transposition $i \leftrightarrow N+1$ acting on the set $\{1,2,...,N+2\}$. Note that $\phi_i$ stabilizes $\mathcal{B}_i$, interchanges $\mathcal{C}_i$ and $\mathcal{C}_{N+1}$, and maps $\wedge_{M\in\mathcal{B}_i}\overline{A_M^{N+2}} \wedge \mathcal{C}_i$ to $\wedge_{M\in\mathcal{B}_i}\overline{A_M^{N+2}} \wedge C_{N+1}$. We have

$$\begin{aligned}
\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}}) &= \sum_{i=1}^{N+1} \Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}} \wedge \mathcal{C}_i) && (15)\\
&= \sum_{i=1}^{N+1} \Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^{N+2}} \wedge \mathcal{C}_i)\\
&= \sum_{i=1}^{N+1} \Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^{N+2}} \wedge \mathcal{C}_{N+1})\\
&= \sum_{i=1}^{N+1} \Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^{N+2}} \mid \mathcal{C}_{N+1})\Pr(\mathcal{C}_{N+1})\\
&= \frac{1}{N+1}\sum_{i=1}^{N+1} \Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^{N}}), && (16)
\end{aligned}$$

and estimating further

$$\begin{aligned}
&\ge (N+1)\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N}})\frac{1}{N+1}\\
&= \Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N}}).
\end{aligned}$$

The proof of Lemma 4 is finished. □

We are back to the proof of Theorem 3: For any fixed matching $M \in \mathcal{M}$, and a subset $\mathcal{J} \subseteq \mathcal{M}$ satisfying that for every $M' \in \mathcal{J}$, $M'$ is not in conflict with $M$, by (2) it suffices to show that

$$\Pr(\wedge_{M'\in\mathcal{J}}\overline{A_{M'}} \mid A_M) \le \Pr(\wedge_{M'\in\mathcal{J}}\overline{A_{M'}}). \tag{17}$$

Let $\mathcal{J}' = \{M' \setminus M \mid M' \in \mathcal{J}\}$. Assume first that $\emptyset \notin \mathcal{J}'$. Since every matching $M'$ in $\mathcal{J}$ is not in conflict with $M$, the vertex set $V(M' \setminus M)$ of $M' \setminus M$ is disjoint from the vertex set $V(M)$ of $M$. Let $T = V(M)$ be the set of vertices covered by the matching $M$ and $U$ be the set of vertices covered by at least one matching $F \in \mathcal{J}'$. We have $T \cap U = \emptyset$. Let $\pi$ be a permutation of $[N]$ mapping $T$ to $\{N-|T|+1, N-|T|+2, \ldots, N\}$. We have $\pi(U) \cap \pi(T) = \emptyset$. Thus, $\pi(U) \subseteq [N-|T|]$. Let $\pi(\mathcal{J}') = \{\pi(F) \mid F \in \mathcal{J}'\}$ and $F' = \pi(F)$. Each matching in $\pi(\mathcal{J}')$ is a matching in $K_{N-|T|}$. We obtain (17) using Lemma 4 repeatedly:

$$\begin{aligned}
\Pr(\wedge_{M'\in\mathcal{J}}\overline{A_{M'}} \mid A_M) &= \frac{\Pr(\wedge_{M'\in\mathcal{J}}\overline{A_{M'}} \wedge A_M)}{\Pr(A_M)} \\
&= \frac{\Pr(\wedge_{M'\in\mathcal{J}}\overline{A_{M'\setminus M}} \wedge A_M)}{\Pr(A_M)} \text{ by Lemma 3} \\
&= \frac{\Pr(\wedge_{F\in\mathcal{J}'}\overline{A_F} \wedge A_M)}{\Pr(A_M)} \\
&= \Pr(\wedge_{F\in\mathcal{J}'}\overline{A_F} \mid A_M) \\
&= \Pr(\wedge_{F'\in\pi(\mathcal{J}')}\overline{A^N_{F'}} \mid A_{\pi(M)}) \\
&= \Pr(\wedge_{F'\in\pi(\mathcal{J}')}\overline{A^{N-|T|}_{F'}}) \\
&\le \Pr(\wedge_{F'\in\pi(\mathcal{J}')}\overline{A^N_{F'}}) \text{ by Lemma 4} \\
&= \Pr(\wedge_{F\in\mathcal{J}'}\overline{A^N_F}) \\
&= \Pr(\wedge_{M'\in\mathcal{J}}\overline{A^N_{M'\setminus M}}) \\
&\le \Pr(\wedge_{M'\in\mathcal{J}}\overline{A^N_{M'}}).
\end{aligned}$$

If $\emptyset \in \mathcal{J}'$, then the LHS of the estimate above is zero, and therefore we have nothing to do. □

The following example shows that in Theorem 3 one cannot have an arbitrary graph in the place of $K_N$ or $K_{N,N'}$. Consider $G = C_6$, this graph has two perfect matchings. Let $e$ and $f$ denote two opposite edges of $C_6$. Consider the following two partial matchings: $\{e\}$ and $\{f\}$. We have $Pr(A_{\{e\}}) = Pr(A_{\{f\}}) = 1/2$. However, we have

$$Pr(A_{\{e\}} \mid \overline{A_{\{f\}}}) = \frac{Pr(A_{\{e\}} \wedge \overline{A_{\{f\}}})}{Pr(\overline{A_{\{f\}}})} \not\le Pr(A_{\{e\}}).$$

Next, we prove a partial converse of Lemma 4.

**Lemma 5** *Consider a collection $\mathcal{M}$ of matchings in $K_N$, so that their canonical events satisfy condition (8) for an $\epsilon < 1/4$, and in addition, for any $uv \in E(K_N)$*

$$\sum_{M: uv\in M\in\mathcal{M}} \Pr(A_M) + 2\Pr^2(A_M) < \epsilon. \tag{18}$$

*Then we have*

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A^{N+2}_M}) \le y^2(\epsilon)\Pr(\wedge_{M\in\mathcal{M}}\overline{A^N_M}).$$

**Proof:** Partition $\Omega_{N+2}$, introduce $\mathcal{C}_i$ and $\mathcal{B}_i$ as in the proof of Lemma 4, and use the fact derived there between (15) and (16) that

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}}) = \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^N}). \tag{19}$$

We are going to apply Theorem 1 part (i) with $S=\mathcal{M}\setminus\mathcal{B}_i$ and $T=\mathcal{B}_i$. $T=\mathcal{B}_i$ contains those matchings form $\mathcal{M}$, whose support do not contain $i$, while $S$ contains those matchings whose support do contain $i$. We are going to show

$$\frac{\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^N})}{\Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^N})} = \Pr(\wedge_{M\in\mathcal{M}\setminus\mathcal{B}_i}\overline{A_M^N}\mid\wedge_{M\in\mathcal{B}_i}\overline{A_M^N}) \geq y(\epsilon)^{-2}. \tag{20}$$

We have from (9)

$$\Pr(\wedge_{M\in\mathcal{M}\setminus\mathcal{B}_i}\overline{A_M^N}\mid\wedge_{M\in\mathcal{B}_i}\overline{A_M^N}) \geq \prod_{M\in\mathcal{M}:i\in supp(M)}\Big(1-\Pr(A_M)y(\epsilon)\Big). \tag{21}$$

If the product in (21) is empty, then we have nothing to prove. If there are $u\neq v$ such that $iu\in M_1$ and $iv\in M_2$, then $\{M\in\mathcal{M}|i\in supp(M)\}\subseteq N_G(M_1)\cup N_G(M_2)$, and the RHS of (21) has a lower bound of $\prod_{M\in N_G(M_1)}\Big(1-\Pr(A_M)y(\epsilon)\Big)\prod_{M\in N_G(M_2)}\Big(1-\Pr(A_M)y(\epsilon)\Big)\geq e^{-2\epsilon y(\epsilon)}=y(\epsilon)^{-2}$, like in the proof of Theorem 1(ii). If there is an $ij$ edge, such that $i\in supp(M)$ for $M\in\mathcal{M}$ implies $ij\in M$, then condition (18) gives a lower bound of $y(\epsilon)^{-1}$ in a similar way for the RHS of (21). We have from (19) and the estimate above:

$$\begin{aligned}
\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^{N+2}}) &= \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr(\wedge_{M\in\mathcal{B}_i}\overline{A_M^N})\\
&\leq \frac{1}{N+1}\sum_{i=1}^{N+1}\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^N})y^2(\epsilon)\\
&= y^2(\epsilon)\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M^N}).
\end{aligned}$$

The proof of Lemma 5 is finished. □

# 4 Upper bounds in the matching models

Now we consider $\Omega$, the uniform probability space of perfect matchings in $K_N$ ($N$ even) or $K_{N,N'}$ (with $N\leq N'$). Let $\mathcal{M}$ be a collection of partial matchings. For any $F\in\mathcal{M}$, let

$$\mathcal{M}_F=\{M\setminus F\mid M\in\mathcal{M}, M\neq F, M\cap F\neq\emptyset, F \text{ is not in conflict to } M\}.$$

We say that a matching $\mathcal{M}$ is *$\delta$-sparse* if

1. No matching from $\mathcal{M}$ is a subset of another matching from $\mathcal{M}$.
2. $\mathcal{M}$ satisfies (8) and (18) with $\delta$ instead of $\epsilon$.
3. For any $F \in \mathcal{M}$,

$$\sum_{H: H \in \mathcal{M}_F} \Pr_{N-2|F|}(A_H) + \Pr_{N-2|F|}(A_H)^2 < \delta, \tag{22}$$

where $\Pr_{N-2|F|}$ indicates that vertices of $F$ has been removed from the underlying vertex set $[N]$ when creating $\Omega$.

For a positive integer $r$, we say that $\mathcal{M}$ is *$r$-bounded*, if for all $M \in \mathcal{M}$, $|M| \leq r$.

The main result of this section is the following theorem.

**Theorem 4** *Let $\mathcal{M}$ be a collection of matchings in $K_N$ or $K_{N,N'}$. If $\mathcal{M}$ is $\delta$-sparse and $r$-bounded, then the negative dependency graph is also an $\epsilon$-near-positive dependency graph with*

$$\epsilon = 1 - e^{-\delta y(2\delta) - \delta^2 y^2(2\delta)} y^{-2r}(2\delta) \tag{23}$$

*and therefore*

$$\Pr(\wedge_{M \in \mathcal{M}} \overline{A_M}) \leq \prod_{M \in \mathcal{M}} \left(1 - \Pr(A_M) e^{-\delta y(2\delta) - \delta^2 y^2(2\delta)} y^{-2r}(2\delta)\right). \tag{24}$$

We are going to prove Theorem 4 for $K_N$, and leave the proof for $K_{N,N'}$, which requires only negligible changes, to the Reader.

**Proof of Theorem 4:** We are going to show that the negative dependency graph $G$ defined for matchings of $K_N$ in $\mathcal{M}$ is also an $\epsilon$-near-positive dependency graph with $\epsilon$ as in (23); and then Theorem 2 together with (23) will finish the proof of (24) and Theorem 4. The first part of the definition, $\Pr(A_i \wedge A_j) = 0$ for $ij$ edges comes for free. We focus on the second part.

For any $F \in \mathcal{M}$ and a subset $S \subseteq \overline{N_G(F)}$, we need to prove

$$\Pr(A_F \mid \wedge_{M \in S} \overline{A_M}) \geq (1 - \epsilon) \Pr(A_F),$$

or equivalently,

$$\Pr(\wedge_{M \in S} \overline{A_M} \mid A_F) \geq (1 - \epsilon) \Pr(\wedge_{M \in S} \overline{A_M}).$$

Let $S_F = \{M \setminus F \mid M \in S\}$. Observe that $\emptyset \notin S_F$. Note that

$$\begin{aligned} \Pr(\wedge_{M \in S} \overline{A_M} \mid A_F) &= \frac{\Pr(\wedge_{M \in S} \overline{A_M} \wedge A_F)}{\Pr(A_F)} & (25) \\ &= \frac{\Pr(\wedge_{M \in S} \overline{A_{M \setminus F}} \wedge A_F)}{\Pr(A_F)} \\ &= \Pr(\wedge_{M \in S_F} \overline{A_M} \mid A_F). & (26) \end{aligned}$$

We have

$$\begin{aligned}
\Pr(\wedge_{M\in S_F}\overline{A_M} \mid A_F) &= \Pr(\wedge_{M\in S_F}\overline{A_M^{N-2i}}) && (27)\\
&= \Pr(\wedge_{M\in S_F}\overline{A_M^{N}}) \prod_{j=1}^{|F|} \frac{\Pr(\wedge_{M\in S_F}\overline{A_M^{N-2j}})}{\Pr(\wedge_{M\in S_F}\overline{A_M^{N-2j+2}})}\\
\text{(by Lemma 5)} \quad &\geq \Pr(\wedge_{M\in S_F}\overline{A_M^{N}}) \prod_{\ell=0}^{|F|-1} y^{-2}(2\delta)\\
&\geq \Pr(\wedge_{M\in S_F}\overline{A_M^{N}}) y^{-2r}(2\delta). && (28)
\end{aligned}$$

(Note that condition (18) is implied by assumption 3.) For any $M$, which does not conflict to $F$, we have $\overline{A_{M\setminus F}} \subset \overline{A_M}$. We have with $S_F = \{M \setminus F \mid M \in S\}$ that

$$\begin{aligned}
\frac{\Pr(\wedge_{M\in S_F}\overline{A_M^N})}{\Pr(\wedge_{M\in S}\overline{A_M^N})} &= \frac{\Pr(\wedge_{M\in S}\overline{A_{M\setminus F}^N})}{\Pr(\wedge_{M\in S}\overline{A_M^N})} && (29)\\
&= \frac{\Pr(\wedge_{M\in S}\overline{A_{M\setminus F}^N} \wedge \overline{A_M^N})}{\Pr(\wedge_{M\in S}\overline{A_M^N})}\\
&= \frac{\Pr([\wedge_{M\in S, M\cap F\neq\emptyset}\overline{A_{M\setminus F}^N}] \wedge [\wedge_{M\in S}\overline{A_M^N}])}{\Pr(\wedge_{M\in S}\overline{A_M^N})}\\
&= \Pr(\wedge_{M\in S_F\setminus S}\overline{A_M^N} \mid \wedge_{M\in S}\overline{A_M^N}). && (30)
\end{aligned}$$

Note that $S_F \setminus S = S_F$ by assumption 1. Now apply Theorem 1 part (i) to $S_F$, $S$ and $S \cup S_F$ instead of $S$, $\mathcal{T}$ and $\mathcal{M}$:

$$\begin{aligned}
&\Pr\Big(\wedge_{M\in S_F}\overline{A_M^N} \mid \wedge_{M\in S}\overline{A_M^N}\Big) \geq \prod_{M\in S_F}\big(1-\Pr(\overline{A_M^N})y(2\delta)\big)\\
\geq\ & \exp\Big(-\sum_{M\in S_F}\Pr(\overline{A_M^N})y(2\delta) - \sum_{M\in S_F}\Pr(\overline{A_M^N})^2 y^2(2\delta)\Big)\\
\geq\ & e^{-\delta y(2\delta)-\delta^2 y^2(2\delta)}. && (31)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
& \Pr(\wedge_{M\in S}\overline{A_M} \mid A_F)\\
\text{by (25-26)} \quad =\ & \Pr(\wedge_{M\in S_F}\overline{A_M} \mid A_F)\\
\text{by (27-28)} \quad \geq\ & \Pr(\wedge_{M\in S_F}\overline{A_M^N})y^{-2r}(2\delta)\\
\text{by (29-30)} \quad =\ & \Pr(\wedge_{M\in S}\overline{A_M^N})\Pr(\wedge_{M\in S'\setminus S}\overline{A_M^N} \mid \wedge_{M\in S}\overline{A_M^N})y^{-2r}(2\delta)\\
\text{by (31)} \quad \geq\ & \Pr(\wedge_{M\in S}\overline{A_M^N})e^{-\delta y(2\delta)-\delta^2 y^2(2\delta)}y^{-2r}(2\delta).
\end{aligned}$$

Thus, the negative dependency graph $G$ is also a $\epsilon$-positive dependency graph. The proof is finished by Theorem 2. $\square$

Theorem 1 provides a lower bound on $\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M})$ while Theorem 4 provides an upper bound on $\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M})$. Under proper conditions, the combination of the two theorems give asymptotics for $\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M})$, like in the following theorem.

**Theorem 5** *Let $\Omega$ be the uniform probability space of perfect matchings in $K_N$ ($N$ even) or $K_{N,N'}$ (with $N \le N'$). Let $r = r(N)$ be a positive integer and $1/16 > \epsilon = \epsilon(N) > 0$ as $N$ approaches the infinity. Let $\mathcal{M} = \mathcal{M}(N)$ be a collection of matchings in $K_N$ or $K_{N,N'}$, respectively, such that none of these matchings is a subset of another. For any $M \in \mathcal{M}$, let $A_M$ be the event consisting of perfect matchings extending $M$. Set $\mu = \mu(N) = \sum_{M\in\mathcal{M}} \Pr(A_M)$. Suppose that $\mathcal{M}$ satisfies*

1. $|M| \le r$, *for each* $M \in \mathcal{M}$.
2. $\Pr(A_M) < \epsilon$ *for each* $M \in \mathcal{M}$.
3. $\sum_{M':A_{M'}\cap A_M=\emptyset} \Pr(A_{M'}) < \epsilon$ *for each* $M \in \mathcal{M}$.
4. $\sum_{M:uv\in M\in\mathcal{M}} \Pr(A_M) < \epsilon$ *for each single edge* $uv$.
5. $\sum_{H\in\mathcal{M}_F} \Pr_{N-2r}(A_H) < \epsilon$ *for each* $F \in \mathcal{M}$.

*Then we have*

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M}) = e^{-\mu+O(r\epsilon\mu)}, \tag{32}$$

*furthermore, if $r\epsilon\mu = o(1)$, then*

$$\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M}) = \Big(1 + O(r\epsilon\mu)\Big)e^{-\mu}. \tag{33}$$

**Proof:** Let $G$ be the graph defined in Theorem 3. By Theorem 3, the graph $G$ is a negative dependency graph for the family of canonical events $\{A_M\}_{M\in\mathcal{M}}$. Note that the condition (8) in Theorem 1 is satisfied with $2\epsilon$, where $\epsilon$ is from the conditions of Theorem 5, instead of $\epsilon$. Applying Theorem 3, we have

$$\begin{aligned}
\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M}) &\ge \exp\Bigg(-\sum_{M\in\mathcal{M}}\Pr(A_M)y(2\epsilon) - \sum_{M\in\mathcal{M}}\Pr^2(A_M)y^2(2\epsilon)\Bigg)\\
&> \exp\Bigg(-\sum_{M\in\mathcal{M}}\Pr(A_M)y(2\epsilon) - \sum_{M\in\mathcal{M}}\Pr(A_M)\epsilon y^2(2\epsilon)\Bigg)\\
&= \exp\Big(-\mu\big(1+3\epsilon+O(\epsilon^2)\big)\Big).
\end{aligned}$$

Now we consider the upper bound. Note that $\mathcal{M}$ is $2\epsilon$-sparse and $r$-bounded. By Theorem 4, we have

$$\begin{aligned}
\Pr(\wedge_{M\in\mathcal{M}}\overline{A_M}) &\le \prod_{M\in\mathcal{M}}\Big(1-\Pr(A_M)e^{-2\epsilon y(4\epsilon)-(2\epsilon)^2y^2(4\epsilon)}y^{-2r}(4\epsilon)\Big)\\
&\le \exp\Bigg(-\sum_{M\in\mathcal{M}}\Pr(A_M)e^{-2\epsilon y(4\epsilon)-(2\epsilon)^2y^2(4\epsilon)}y^{-2r}(4\epsilon)\Bigg)\\
&= \exp\Big(-\mu\big(1-(8r+2)\epsilon+O(r\epsilon^2)\big)\Big).
\end{aligned}$$

Combining the lower bound and the upper bound above, we obtain equation (32). $\square$

# 5 Asymptotic results in the matching models

## 5.1 Applications I: Counting $k$-cycle free permutations and Latin rectangles

It is known and easy that for any fixed $k$, the probability of a random permutation not having any $k$-cycle is asymptotically $e^{-1/k}$, see [25] or [6]. In our earlier paper, [14], we obtained an $(1-o(1))e^{-1/k}$ lower bound for this probability from Lovász Local Lemma. Now we show that the machinery that we developed in this paper actually yields the very same asymptotic formula whenever $k = o(N)$.

Let us be given two $N$-element sets with elements $\{1, 2, ..., N\}$ and $\{1', 2', ..., N'\}$. Let us identify a permutation of the first set, $\pi$, with a matching between the two sets, such that $i$ is joined to $\pi(i)'$. A $k$-cycle in the permutation can be identified with a matching between $K \subset \{1, 2, ..., N\}$ (with $|K| = k$) and $\{\ell' : \ \ell \in K\}$, which does not have a proper non-empty subset $K_1 \subset K$, such that the matching also matches $K_1$ to $\{\ell' : \ \ell \in K_1\}$. The bad events for the negative dependency graph are these $k$-element matchings; there are $\binom{N}{k}(k-1)!$ of them. We have $|\mathcal{M}| = \binom{N}{k}(k-1)!$. For each $M \in \mathcal{M}$, we have $\Pr(A_M) = \frac{1}{\binom{N}{k}k!}$. Two matchings, $M, M' \in \mathcal{M}$, $M \neq M'$, conflict each other if and only if the two cycles have non-empty intersection, i.e. have common elements.

Let $r = k$ and $\epsilon = \frac{k}{N-k+1}$. Now we will verify the conditions of Theorem 5. Items 1 and 2 are satisfied by our choice of $r$ and $\epsilon$. For item 3, we have

$$\begin{aligned}
\sum_{M' : A_{M'} \cap A_M = \emptyset} \Pr(A_{M'}) &= \left(\binom{N}{k}(k-1)! - \binom{N-k}{k}(k-1)!\right)\frac{1}{\binom{N}{k}k!} \\
&= \frac{1}{k}\left(1 - \prod_{i=1}^{k} \frac{N-k-i+1}{N-i+1}\right) \\
&= \frac{1}{k}\left(1 - \prod_{i=1}^{k} \left(1 - \frac{k}{N-i+1}\right)\right) \\
&< \frac{1}{k}\sum_{i=1}^{k} \frac{k}{N-i+1} \leq \frac{k}{N-k+1} = \epsilon. \qquad (34)
\end{aligned}$$

Now we verify item 4. For any $uv \in M \in \mathcal{M}$, a $k$-matching $M$ contains a given edge $uv$, if and only if $v = \pi(u)'$ for some $k$-cycle permutation $\pi$. The number of such $k$-cycles is $\binom{N}{k-2}(k-2)!$. We have

$$\begin{aligned}
\sum_{M : uv \in M \in \mathcal{M}} \Pr(A_M) &= \binom{N}{k-2}(k-2)!\frac{1}{\binom{N}{k}k!} \\
&= \frac{1}{(N-k+2)(N-k+1)} < \epsilon.
\end{aligned}$$

For any $F \in \mathcal{M}$, now $\mathcal{M}_F$ is empty in our special setting, hence item 5 holds trivially. All conditions of Theorem 5 are verified. Observe

$$\mu = \sum_{M \in \mathcal{M}} \Pr(A_M) = \binom{N}{k}(k-1)! \frac{1}{\binom{N}{k} k!} = \frac{1}{k}. \tag{35}$$

Therefore Theorem 5 applies, and the number of $k$-cycle-free permutations is $(1 + O(k/N))e^{-1/k}$. [25] goes further than this, and gives asymptotic formula for the number of permutations without cycles of length $r$ or less, for fixed $r$. Simple generating function arguments would not allow $k$ (or $r$) to be variables. However, our method allows the following result, which perhaps first occured in [2]:

**Theorem 6** *Let us be given a $K \subset \{1, 2, ..., N\}$ and set $r = \max K$. Assume that*

$$R = r^2 \Big( \sum_{k \in K} \frac{1}{k} \Big) \Big( \sum_{k \in K} \frac{1}{N - k + 1} \Big) \to 0.$$

*Then, the probability that a random permutation of $N$ elements do not contain any cycle, whose length belongs to $K$, is $\big(1 + O(R)\big) \exp\Big( - \sum_{k \in K} \frac{1}{k} \Big)$.*

**Proof:** The proof above goes through with minor modifications. Set $\epsilon = r \sum_{k \in K} \frac{1}{N-k+1}$, change (35) to $\mu = \sum_{k \in K} \binom{N}{k}(k-1)! \frac{1}{\binom{N}{k}k!} = \sum_{k \in K} \frac{1}{k}$, and for a matching $M$ corresponding to an $\ell$-cycle, change (34) for the estimation of $\sum_{M' : A_{M'} \cap A_M = \emptyset} \Pr(A_{M'})$ to $\sum_{k \in K} \Big( \binom{N}{k}(k-1)! - \binom{N-k}{k}(k-1)! \Big) \frac{1}{\binom{N}{k}k!}$
$= \sum_{k \in K} \frac{1}{k} \Big( 1 - \prod_{i=1}^{k} \frac{N-\ell-i+1}{N-i+1} \Big) = \sum_{k \in K} \frac{1}{k} \Big( 1 - \prod_{i=1}^{k} \Big( 1 - \frac{\ell}{N-i+1} \Big) \Big)$
$< \sum_{k \in K} \frac{1}{k} \sum_{i=1}^{k} \frac{\ell}{N-i+1} \leq \sum_{k \in K} \frac{\ell}{N-k+1} \leq \epsilon.$ $\square$

Let us turn now to the enumeration of Latin rectangles. A $k \times n$ Latin rectangle is a sequence of $k$ permutations of $\{1, 2, ..., n\}$ written in a matrix form, such that no column has any repeated entries. Let $L(k, n)$ denote the number of $k \times n$ Latin rectangles. $L(2, n)$ is just $n!$ times the number of derangements, i.e. $(n!)^2 e^{-1}$. In 1944, Riordan [19] showed that $L(3, n) \sim (n!)^3 e^{-3}$. In 1946, Erdős and Kaplansky [9] showed

$$L(k, n) \sim (n!)^k e^{-\binom{k}{2}} \tag{36}$$

for $k = o((\log n)^{3/2}$. In 1951, Yamamoto [24] extended this asymptotic formula for $k = o(n^{1/3})$. In 1978, Stein [23] refined the asymptotic formula to

$$L(k, n) \sim (n!)^k e^{-\binom{k}{2} - \frac{k^3}{6n}} \tag{37}$$

using the Chen-Stein method [7], and extended the range to $k = o(n^{1/2})$. The current best asymptotic formula is due to Godsil and McKay [12], whose further refined formula, $L(k, n) \sim (n!)^k \Big( \frac{(n)_k}{n^k} \Big)^n \Big( 1 - \frac{k}{n} \Big)^{-n/2} e^{-k/2}$ works for $k = o(n^{6/7})$.

Formula (37) has had an unexpected proof by Skau [22], who proved, for any $1 \le k \le n$, the inequality

$$(n!)^k \prod_{t=1}^{k-1} \left(1 - \frac{t}{n}\right)^n \le L(k,n) \tag{38}$$

from the van der Waerden inequality for the permanent. If $k = o((n/\log n)^{1/2})$, the lower bound in (38) is asymptotically the same as the RHS of (37). Skau's asymptotically tight upper bound [22] followed from Minc's inequality for the permanent.

In [14] we claimed (38) from the Lovász Local Lemma in error. However, our method still gives back Yamamoto's range for (36). Fix an arbitrary $t \times n$ Latin rectangle with rows $\pi_1, \pi_2, ..., \pi_t$. Consider a complete bipartite graph with classes $\{1, 2, ..., n\}$ an $\{1', 2', ..., n'\}$, and let $\Omega$ be the space of perfect matchings in this complete bipartite graph. Permutation $\pi_{t+1}$ of $\{1, 2, ..., n\}$ are in one-to-one correspondence with perfect matchings by $(\pi_{t+1}(j), j') : 1 \le j \le n$. Permutation $\pi_{t+1}$ fails to extend the given Latin rectangle into a $(t+1) \times n$ Latin rectangle iff there are $i, j$ such that $\pi_i(j) = \pi_{t+1}(j)$. Therefore a perfect matching provides a legal $(t+1)^{th}$ row for the Latin rectangle iff it does not contain any of the edges $(\pi_i(j), j') : 1 \le j \le n, 1 \le i \le t$. Define the event $A_{ij}$ as the canonical event in $\Omega$ corresponding to the one-edge matching $(\pi_i(j), j')$. Let $G$ be the a negative dependency graph for the family of events $A_{ij}$, according to Theorem 3. $G$ is $(t-1)$-regular. We can apply Theorem 5 with 1. $r = 1$, 2. $1/n < \epsilon$, 3. $2(t-1)/n < \epsilon$, 4. like 2., and condition 5. holds vacuously; $\mu = \frac{1}{n} \cdot (nt) = t$. Hence $\#\pi_{t+1}/n! = \exp\left(-t + O\left(\frac{t^2}{n}\right)\right)$ by (32), and $L(k,n) = \prod_{t=0}^{k-1} n! \exp\left(-t + O\left(\frac{t^2}{n}\right)\right) = (n!)^k \exp\left(-\binom{k}{2} + O\left(\frac{k^3}{n}\right)\right)$.

## 5.2 Applications II: The configuration model and the enumeration of $d$-regular graphs

For a given sequence of positive integers with an even sum, $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, the *configuration model of random multigraphs with degree sequence* $\mathbf{d}$ is defined as follows [4].

1. Let us be given a set $U$ that contains $N = \sum_{i=1}^n d_i$ distinct mini-vertices. Let $U$ be partitioned into $n$ classes such that the $i$th class consists of $d_i$ mini-vertices. This $i$th class will be associated with vertex $v_i$ after identifying its elements through a *projection*.
2. Choose a random matching $M$ of the mini-vertices in $U$ uniformly.
3. Define a random multigraph $G$ associated with $M$ as follows: For any two (not necessarily distinct) vertices $v_i$ and $v_j$, the number of edges joining $v_i$ and $v_j$ in $G$ is equal to the total number of edges in $M$ between mini-vertices associated with $v_i$ and mini-vertices associated with $v_j$.

The configuration model of random $d$-regular graphs on $n$ vertices is the instance $d_1 = d_2 = \cdots = d_n$, where $nd$ is even.

The enumeration problem of labelled $d$-regular graphs has a rich history in the literature. The first result was Bender and Canfield [3], who showed in 1978 that for any fixed $d$, with $nd$ even, the number of them is

$$\left(\sqrt{2} + o(1)\right) e^{\frac{1-d^2}{4}} \left( \frac{d^d n^d}{e^d (d!)^2} \right)^{\frac{n}{2}}.$$

The same result was discovered at the same time by Wormald. In 1980, Bollobás [4] introduced probability to this enumeration problem by defining the configuration model, and put the result in the alternative form

$$(1 + o(1)) e^{\frac{1-d^2}{4}} \frac{(dn-1)!!}{(d!)^n}. \tag{39}$$

where the term $(1 + o(1))e^{\frac{1-d^2}{4}}$ in (39) can be explained as the probability of obtaining a simple graph after the projection. The semifactorial $(dn-1)!! = \frac{(dn)!}{(dn/2)!2^{dn/2}}$ equals the number of perfect matchings on $dn$ elements, and $\frac{1}{(d!)^n}$ is just the number of ways matchings can yield the same simple graph after projection. Bollobás also extended the range of the asymptotic formula to $d < \sqrt{2 \log n}$, which was further extended to $d = o(n^{1/3})$ by McKay [15] in 1985. The strongest result is due to McKay and Wormald [16] in 1991, who refined the probability of obtaining a simple graph after the projection to

$$(1 + o(1)) e^{\frac{1-d^2}{4} - \frac{d^3}{12n} + O(\frac{d^2}{n})} \tag{40}$$

and extended the range of the asymptotic formula to $d = o(n^{1/2})$. Wormald's Theorem 2.12 in [27] (originally published in [26]) asserts that for any fixed numbers $d \geq 3$ and $g \geq 3$, the number of labelled $d$-regular graphs with girth at least $g$, is

$$(1 + o(1)) e^{-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2i}} \frac{(dn-1)!!}{(d!)^n}. \tag{41}$$

In our theorem below, we allow both $d$ and $g$ go to infinity slowly. If we set $g = 3$, we get back the asymptotic formula for the number of $d$-regular graphs up to $d = o(n^{1/3})$, giving an alternative proof to McKay's result cited above. However, our method inherently fail to extend this result as McKay and Wormald did, in fact, our method already fails to extend the lower bound. McKay, Wormald and Wysocka [17] proved the same theorem below under a slightly weaker assumption $d^{2g-3} = o(n)$:

**Theorem 7** *In the configuration model, assume $d \geq 3$ and*

$$g^6 d^{2g-3} = o(n). \tag{42}$$

*Then the probability that the random d-regular multigraph has girth at least* $g \geq 1$ *is* $(1+o(1))\exp\Big(-\sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}\Big)$, *and hence the number of d-regular graphs on n vertices with girth at least* $g \geq 3$ *is*

$$(1+o(1))e^{-\sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}}\frac{(dn-1)!!}{(d!)^n}.$$

*(The case* $g=3$ *means that the random d-regular multigraph is actually a simple graph.) Furthermore, the number of d-regular graphs not containing cycles whose length is in a set* $\mathcal{C} \subseteq \{3,4,...,g-1\}$, *is*

$$(1+o(1))e^{-\frac{d-1}{2}-\frac{(d-1)^2}{4}-\sum_{i\in\mathcal{C}}\frac{(d-1)^i}{2i}}\frac{(dn-1)!!}{(d!)^n}.$$

**Proof:** We prove the first claim. To prove the second claim, only (44) has to be adjusted, everything else remains the same. For $i=1,2,\ldots,g-1$, let $\mathcal{M}_i$ be the set of matchings of $U$ whose projection gives a cycle of length $i$; there are *exactly* $\frac{1}{2i}\binom{n}{i}i!d^i(d-1)^i$ of them. The bad events for the negative dependency graph are the union of matchings $\mathcal{M}=\cup_{i=1}^{g-1}\mathcal{M}_i$. For each $M_i \in \mathcal{M}_i$ $(i=1,2,\ldots,g-1)$, we have

$$\Pr(A_{M_i}) = \frac{1}{(nd-1)(nd-3)\cdots(nd-2i+1)}. \tag{43}$$

We have

$$\begin{aligned}\sum_{M\in\mathcal{M}}\Pr(A_M) &= \sum_{i=1}^{g-1}\frac{1}{2i}\binom{n}{i}i!d^i(d-1)^i\frac{1}{(nd-1)(nd-3)\cdots(nd-2i+1)}\\ &= \sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}\left(1+O\left(\frac{i^2}{n}\right)\right) = \left(1+O\left(\frac{g^2}{n}\right)\right)\sum_{i=1}^{g-1}\frac{(d-1)^i}{2i}.\end{aligned} \tag{44}$$

Let $r=g-1$ and $\epsilon=\frac{K'g^5(d-1)^{g-2}}{n}$ for a large constant $K'$. Now we verify the conditions of Theorem 5. Item 1 and 2 are trivial by the definition of $r$ and $\epsilon$. Item 3 can be verified as follows. For $M\in\mathcal{M}_1$, we have

$$\begin{aligned}\sum_{M':A_{M'}\cap A_M=\emptyset}\Pr(A_{M'}) &= \frac{2d-4}{nd-1}+\sum_{i=2}^{g-1}\sum_{M'\in\mathcal{M}_i:A_{M'}\cap A_M=\emptyset}\Pr(A_{M'})\\ &\leq \frac{2d-4}{nd-1}+\sum_{i=2}^{g-1}\frac{\binom{n}{i-1}(i-1)!2(d-1)^i d^{i-1}}{(nd-1)(nd-3)\cdots(nd-2i+1)}\\ &\leq \frac{2d-4}{nd-1}+\sum_{i=2}^{g-1}\frac{4(d-1)^i}{(nd-1)}\\ &< \epsilon.\end{aligned} \tag{45}$$

For $M \in \mathcal{M}_j$ $(j \neq 1)$, we have

$$\begin{aligned}
\sum_{M': A_{M'} \cap A_M = \emptyset} \Pr(A_{M'}) &= \frac{j(2d-1)}{nd-1} + \sum_{i=2}^{g-1} \sum_{M' \in \mathcal{M}_i : A_{M'} \cap A_M = \emptyset} \Pr(A_{M'}) \\
&\leq \frac{(2d-1)j}{nd-1} + j \sum_{i=2}^{g-1} \frac{\binom{n}{i-2}(i-2)!2(d-1)^{i-1}d^{i-2}}{(nd-1)(nd-3)\cdots(nd-2i+1)} \\
&< \frac{(2d-3)(g-1)}{nd-1} + (g-1) \sum_{i=2}^{g-1} \frac{4(d-1)^{i-1}}{(nd-2g+3)^2} \\
&< \epsilon. \qquad (46)
\end{aligned}$$

Now we verify item 4. For any $uv \in M \in \mathcal{M}$, we have

$$\begin{aligned}
\sum_{M: uv \in M \in \mathcal{M}} A_M &\leq \frac{1}{nd-1} + \sum_{i=2}^{g-1} \frac{\binom{n}{i-2}(i-2)!(d-1)^i d^{i-2}}{(nd-1)(nd-3)\cdots(nd-2i+1)} \\
&< \frac{1}{nd-1} + \sum_{i=2}^{g-1} \frac{4(d-1)^i}{(nd-2g+3)^2} \\
&< \epsilon/2. \qquad (47)
\end{aligned}$$

Finally, we verify item 5. For any $F \in \mathcal{M}$, we need estimate $\sum_{M \in \mathcal{M}_F} \Pr_{N-2r}(A_M)$. If the projection of $F$ is a loop, then $\mathcal{M}_F = \emptyset$ and there is nothing to do. Now we assume the projection of $F$ is a cycle $C_k$. Assume that $M' \in \mathcal{M}$ intersects $F$, $M = M' \setminus F$, and the projection of $M'$ is a cycle $C_s$ with $k, s \leq g-1$. Then the components of $C_s \cap C_k$ having at least one edge are paths $P_1, P_2, \ldots, P_t$, with $t \geq 1$. Fixing these paths, and the edges in $M' \cap F$, some additional $\ell$ vertices are joined with these $t$ paths to make $C_s$. So the number of possible $C_s$'s with these fixed paths is at most

$$\sum_{\ell \leq g-1-2t} \binom{n}{\ell} (\ell+t-1)! 2^t,$$

and the number of $M'$-s defining this particular $C_s$ with $M' \cap F$ fixed, is at most $d^\ell (d-1)^{\ell+2t}$. The $t$ paths with at least one edge can be selected in at most $2\binom{k}{2t}$ ways from $C_k$. The probability $\Pr_{N-2r}(A_M)$, where $M = M' \setminus F$, is at most $(N-3g)^{-(\ell+t)}$. We summarize that

$$\sum_{M \in \mathcal{M}_F} \Pr(A_M) \leq \sum_{t=1}^{\lfloor k/2 \rfloor} 2 \binom{k}{2t} \sum_{\ell \leq g-1-2t} \binom{n}{\ell} (\ell+t-1)! 2^t \frac{d^\ell (d-1)^{\ell+2t}}{(N-3g)^{\ell+t}}. \quad (48)$$

As $\ell + t - 1 \leq g-3$, we have $(\ell+t-1)! = \ell!(\ell+t-1)_{t-1} \leq \ell!(g-3)^{t-1}$. There is an absolute upper bound $K > \frac{(n)_\ell d^\ell}{(N-3g)^\ell}$. As $\ell + 2t \leq g-1$, the RHS of (48) can be further estimated by

$$2K(d-1)^{g-1}\sum_t \binom{k}{2t}\sum_{\ell\le g-1-2t}\left(\frac{2(g-3)}{N-3g}\right)^t \le 2Kg(d-1)^{g-1}\sum_t \binom{k}{2t}\left(\frac{2(g-3)}{N-3g}\right)^t.$$

It is easy to see that the last summation has its largest term at $t=1$, has less than $g$ terms, and is $\le 4Kg^5(d-1)^{g-1}/(N-3g) < \epsilon$.

To apply Theorem 5, we need $r\mu\epsilon = o(1)$. As $r<g$, $\mu \le (d-1)^{g-1}/2$ and $\epsilon = \frac{K'g^5(d-1)^{g-2}}{n}$, this condition boils down to $g^6(d-1)^{2g-3} = o(n)$, which was provided in (42). The neglection of error in (44) is also allowed by (42). $\square$

In the *bipartite configuration model* we have two sets, $U$ and $V$, each containing $N$ mini-vertices, a fixed partition of $U$ into $d_1,...,d_n$ element classes, and a fixed partition of $V$ into $\delta_1,...,\delta_n$ element classes. Any perfect matching between $U$ and $V$ defines a bipartite multigraph with partite sets of size $n$ after a projection contracts every class to single vertex. In the regular case, $d_1=\cdots=d_n=\delta_1=\cdots=\delta_n=d$. We prove next another theorem of McKay, Wormald and Wysocka [17]:

**Theorem 8** *In the regular case of the bipartite configuration model, assume that $g$ is even, $d\ge 3$, and*

$$g^6 d^{2g-3} = o(n). \tag{49}$$

*Then the probability that the random bipartite $d$-regular multigraph has girth at least $g\ge 2$ is $(1+o(1))\exp\Big(-\sum_{i=1}^{(g-2)/2}\frac{(d-1)^{2i}}{2i}\Big)$, and hence the number of $d$-regular biparite graphs on $n,n$ vertices with girth at least $g\ge 4$ is*

$$(1+o(1))e^{-\sum_{i=1}^{(g-2)/2}\frac{(d-1)^{2i}}{2i}}\frac{(dn)!}{(d!)^{2n}}.$$

*(The case $g=4$ means that the random $d$-regular bipartite multigraph is actually a simple bipartite graph.) Furthermore, the number of $d$-regular bipartite graphs not containing cycles whose length is in a set $\mathcal{C}\subseteq\{4,6,...,g-2\}$, is*

$$(1+o(1))e^{-\frac{(d-1)^2}{2}-\sum_{i\in\mathcal{C}}\frac{(d-1)^i}{i}}\frac{(dn)!}{(d!)^{2n}}.$$

**Proof:** We outline the proof of the first claim. For $i=1,2,\ldots,(g-2)/2$, let $\mathcal{M}_i$ be the set of matchings of $U$ and $V$, whose projection gives a cycle of length $2i$; there are *exactly* $\binom{n}{i}^2 d^{2i}(d-1)^{2i}(i-1)!^2 i$ of them. The bad events for the negative dependency graph are the union of matchings $\mathcal{M}=\cup_{i=1}^{(g-2)/2}\mathcal{M}_i$. For each $M_i\in\mathcal{M}_i$ $(i=1,2,\ldots,(g-2)/2)$, we have

$$\Pr(A_{M_i}) = \frac{(dn-2i)!}{(dn)!}. \tag{50}$$

We have

$$
\sum_{M\in\mathcal{M}} \Pr(A_M) = \sum_{i=1}^{(g-2)/2} \binom{n}{i}^2 d^{2i}(d-1)^{2i}(i-1)!^2 i \frac{(dn-2i)!}{(dn)!}
$$

$$
= \sum_{i=1}^{(g-2)/2} \frac{(d-1)^{2i}}{2i}\left(1+O\left(\frac{i^2}{n}\right)\right) = \left(1+O\left(\frac{g^2}{n}\right)\right)\sum_{i=1}^{(g-2)/2}\frac{(d-1)^{2i}}{2i}. \tag{51}
$$

All the estimates go through as in the proof of Theorem 7. To prove the second claim, only (51) has to be adjusted, everything else remains the same. □

## 5.3 Applications III: Enumeration of graphs by girth and degree sequence

McKay and Wormald [16] enumerated graphs by degree sequences. We extend this result to include the girth or the set of allowed short cycle lengths. However, our range for the degrees is not as broad as in [16]. For example, formula (40) that we could not obtain is a special case of [16].

We start with some technicalities. Let $\sigma_n^{(k)}(x_1,...,x_n)$ denote the $k^{th}$ elementary symmetric polynomial in $n$ variables. Assume that every $x_i > 0$ and set average $\bar{x} = (\sum_{i=1}^n x_i)/n$ and the second order average $\tilde{x} = (\sum_{i=1}^n x_i^2)/\bar{x}$. We claim the following:

$$
\frac{n^k}{(n)_k}\left(1-\binom{k}{2}\frac{n\tilde{x}}{\bar{x}}\right) \leq \frac{\sigma_n^{(k)}(x_1,...,x_n)}{\sigma_n^{(k)}(\bar{x},...,\bar{x})} \leq \frac{n^k}{(n)_k}. \tag{52}
$$

Formula (52) follows from the following inequalities:
(a) $\sigma_n^{(k)}(x_1,...,x_n) \leq (x_1+...+x_n)^k/k!$, and
(b) in the expansion of $(x_1+...+x_n)^k - \binom{k}{2}(\sum_{i=1}^k x_i^2)(x_1+...+x_n)^{k-2}$ all terms that contain a square or higher power of a variable, have non-positive coefficients. It follows that

$$
\sigma_n^{(k)}(x_1,...,x_n) = \frac{(n)_k(\bar{x})^k}{k!}\left(1+O\Big(\frac{k^2}{n}+k^2\frac{\tilde{x}}{\bar{x}}\Big)\right), \tag{53}
$$

whenever the quantity in the $O$-term goes to zero. Assume further that $x_1 \leq x_2 \leq ... \leq x_n$. Define a sequence by $y_i = x_{t+i}$ for $i = 1,2,...,n-t$. It is easy to see that for $t = o(n)$ and $k^2\frac{\tilde{x}}{\bar{x}} = o(1)$ we have $\bar{y} = \left(1+O\big(\frac{t}{n}\big)\right)\bar{x}$ and $\tilde{y} = \left(1+O\big(\frac{t}{n}\big)\right)\tilde{x}$. From here and (53) we conclude

$$
\sigma_{n-t}^{(k)}(y_1,...,y_{n-t}) = \frac{(n)_k(\bar{x})^k}{k!}\left(1+O\Big(\frac{k^2+kt}{n}+k^2\frac{\tilde{x}}{\bar{x}}\Big)\right). \tag{54}
$$

Let us return to the configuration model as described at the beginning of Subsection 5.2 and try to do in more generality the steps of the proof of Theorem 7. The combinatorial structures are the same, but different class sizes have to be taken into account. Assume now that $d_1 \leq d_2 \leq ... \leq d_n$ and set $D_j = d_j(d_j-1)$. If the projection provides a *graph* with degree sequence $d_1, d_2, ..., d_n$ (as opposed to a multigraph), then *exactly* $d_1! d_2! \cdots d_n!$ matchings on the set of $N = d_1 + ... + d_n$ mini-vertices yield this graph. We want to compute the probability that after the projection we obtain a graph with girth at least $g$ ($g \geq 3$). For $i = 1, 2, \ldots, g-1$, let $\mathcal{M}_i$ be the set of matchings of $U$ whose projection gives a cycle of length $i$; there are *exactly* $\frac{(i-1)!}{2}\sigma_n^{(i)}\big(D_1, ..., D_n\big)$ of them. The bad events for the negative dependency graph are the union of matchings $\mathcal{M} = \cup_{i=1}^{g-1}\mathcal{M}_i$. For each $M_i \in \mathcal{M}_i$ ($i = 1, 2, \ldots, g-1$), we have

$$\Pr(A_{M_i}) = \frac{1}{(N-1)(N-3)\cdots(N-2i+1)}, \tag{55}$$

where $N = n\bar{d}$. We have

$$\begin{aligned}\sum_{M\in\mathcal{M}} \Pr(A_M) &= \sum_{i=1}^{g-1} \frac{(i-1)!}{2} \cdot \frac{\sigma_n^{(i)}\big(D_1, ..., D_n\big)}{(N-1)(N-3)\cdots(N-2i+1)} \\ = & \sum_{i=1}^{g-1} \frac{(n)_i(\bar{D})^i}{2i(N-1)(N-3)\cdots(N-2i+1)}\left(1 + O\Big(\frac{i^2}{n} + i^2\frac{\tilde{D}}{\bar{D}}\Big)\right).\end{aligned} \tag{56}$$

The estimate in (45) changes to

$$\frac{2d_n - 4}{n\bar{d}-1} + \sum_{i=2}^{g-1} \frac{(i-1)!2(d_1-1)\sigma_{n-1}^{(i-1)}(D_2, ..., D_n)}{(n\bar{d}-1)(n\bar{d}-3)\cdots(n\bar{d}-2i+1)}. \tag{57}$$

The estimate in (46) changes to

$$\frac{(2d_n - 1)j}{n\bar{d}-1} + j\sum_{i=2}^{g-1} \frac{(i-2)!2(d_2-1)\sigma_{n-2}^{(i-2)}(D_3, ..., D_n)}{(n\bar{d}-1)(n\bar{d}-3)\cdots(n\bar{d}-2i+1)}. \tag{58}$$

The estimate in (47) changes to

$$\frac{1}{n\bar{d}-1} + \sum_{i=2}^{g-1} \frac{(d_1-1)(d_2-1)\sigma_{n-2}^{(i-2)}(D_3, ..., D_n)}{(n\bar{d}-1)(n\bar{d}-3)\cdots(n\bar{d}-2i+1)}. \tag{59}$$

The estimate in (48) changes to

$$\begin{aligned}&\sum_{t=1}^{\lfloor k/2\rfloor} 2\binom{k}{2t} \sum_{\ell \leq g-1-2t} \frac{(\ell+t-1)!2^t(d_n-1)^{2t}\sigma_{n-2t}^{(\ell)}(D_{2t+1}, ..., D_n)}{(N-3g)^{\ell+t}} \\ \leq \quad & \sum_{t=1}^{\lfloor k/2\rfloor} 2\binom{k}{2t} \sum_{\ell \leq g-1-2t} \left[\frac{4(g-3)(d_n-1)^2}{N-3g}\right]^t \left(\frac{\bar{D}}{\bar{d}}\right)^\ell\end{aligned}$$

and as the $\ell^{th}$ power is maximized at $\ell = g-1-2t$, and the sum is maximized at $t=1$, our bound is $\epsilon = Kg^5\left(\frac{\bar{D}}{\bar{d}}\right)^{g-3}\frac{(d_n-1)^2}{N-3g}$ for some constant $K$. This $\epsilon$ also provides a bound for (59), (58), and (57). The least trivial is the last one, it follows from the boundedness of $\frac{d_1\bar{D}}{\bar{d}(d_n-1)^2}$. We are in a position to claim to the generalization of Theorem 7 for other than constant degree sequences:

**Theorem 9** *Assume that $N = d_1 + ... + d_n$ is even, $\bar{d} \geq 3$, every $d_i \geq 1$. In the configuration model, assume*

$$g^2 \frac{\tilde{D}}{\bar{D}}\left(\frac{\bar{D}}{\bar{d}}\right)^{g-1} = o(1) \quad \textit{and} \quad g^6\left(\frac{\bar{D}}{\bar{d}}\right)^{2g-4} d_n^2 = o(N). \tag{60}$$

*Then the probability that the random multigraph with degrees $d_1, d_2, ..., d_n$ after the projection has girth at least $g \geq 1$ is*

$$(1+o(1))\exp\Big(-\sum_{i=1}^{g-1}\frac{(n)_i(\bar{D})^i(N-2i-1)!!}{2i(N-1)!!}\Big), \tag{61}$$

*and hence the number of graphs on $n$ vertices with degrees $d_1, d_2, ..., d_n$ and girth at least $g \geq 3$ is*

$$(1+o(1))\frac{(N-1)!!}{\prod_i d_i!}\exp\left(-\sum_{i=1}^{g-1}\frac{(n)_i(\bar{D})^i(N-2i-1)!!}{2i(N-1)!!}\right).$$

*(The case $g=3$ means that the random multigraph is actually a simple graph, and hence $d_1, d_2, ..., d_n$ is a graph degree sequence.) Furthermore, the number of graphs with degrees $d_1, d_2, ..., d_n$ not containing cycles whose length is in a set $\mathcal{C} \subseteq \{3, 4, ..., g-1\}$, is*

$$(1+o(1))\frac{(N-1)!!}{\prod_i d_i!}\exp\left(-\frac{n\bar{D}}{2(N-1)} - \frac{n(n-1)(\bar{D})^2}{4(N-1)(N-3)} - \sum_{i\in\mathcal{C}}\frac{(n)_i(\bar{D})^i(N-2i-1)!!}{2i(N-1)!!}\right).$$

**Proof:** We proved the first claim before stating the theorem. The proof of the second claim is analogous except the calculation of $\mu$. It is easy to see that $\mu = O\left(\frac{\bar{D}}{\bar{d}}\right)^{g-1}$, therefore (60) implies $r\epsilon\mu = o(1)$ as required in the condition above (33). Note that the first part of (60) allows the approximation in (56). $\Box$

It is not difficult to obtain a degree sequence version of Theorem 8. As the proof is just a combination of the proofs of Theorems 8 and 9, we leave the details to the reader.

**Theorem 10** *In the bipartite configuration model, assume that $g$ is even, the class sizes are $1 \leq d_1 \leq \cdots \leq d_n$ and $1 \leq \delta_1 \leq \cdots \leq \delta_n$, $N = \sum_i d_i = \sum_i \delta_i$, $\bar{d} = \bar{\delta} \geq 3$, $D_j = d_j(d_j-1)$ and $\Delta_j = \delta_j(\delta_j-1)$. Assume further that*

$$g^2\left(\frac{\tilde{D}}{\bar{D}} + \frac{\tilde{\Delta}}{\bar{\delta}}\right)\left(\frac{\bar{D}\cdot\bar{\Delta}}{\bar{d}\cdot\bar{\delta}}\right)^{(g-2)/2} = o(1) \quad \textit{and} \quad g^6(d_n^2+\delta_n^2)\left(\frac{\bar{D}}{\bar{d}}\right)^{g-3}\left(\frac{\bar{\Delta}}{\bar{\delta}}\right)^{g-3} = o(N). \tag{62}$$

*Then the probability that the random bipartite multigraph with the prescribed degree sequence has girth at least $g \geq 2$ is*

$$(1+o(1))\exp\Big(-\sum_{i=1}^{(g-2)/2}\frac{(n)_i^2(\bar{D})^i(\bar{\Delta})^i}{2i(N)_{2i}}\Big),$$

*and hence the number of bipartite graphs with the prescribed degree sequence and girth at least $g \geq 4$ is*

$$(1+o(1))\frac{N!}{\prod_i d_i!\delta_i!}\exp\Big(-\sum_{i=1}^{(g-2)/2}\frac{(n)_i^2(\bar{D})^i(\bar{\Delta})^i}{2i(N)_{2i}}\Big).$$

*(The case $g = 4$ means that the random bipartite multigraph with the given degree sequence is actually a simple bipartite graph, and hence given sequence* is *a bipartite graph degree sequence.) Furthermore, the number of bipartite graphs with the prescribed degree sequence that do not contain cycles whose length is in a set $\mathcal{C} \subseteq \{4, 6, ..., g-2\}$, is*

$$(1+o(1))\frac{N!}{\prod_i d_i!\delta_i!}\exp\Big(-\frac{n^2\bar{D}\bar{\Delta}}{2N(N-1)}-\sum_{i\in\mathcal{C}}\frac{(n)_i^2(\bar{D})^i(\bar{\Delta})^i}{2i(N)_{2i}}\Big).$$

# 6 Revisiting girth and chromatic number: high girth and high chromatic number graphs on a given degree sequence

An early result of Erdős [8] asserts that for every $k$ and $g$, there is a graph $G$ with $girth(G) \geq g$ and chromatic number $\chi(G) \geq k$. In Theorem 11 we refine this result of Erdős, changing the existential quantifier to universal.

We start with some technicalities. Let $N$ be an even positive integer. For a set $S \subset [N]$, we say that a perfect matching $M$ of $K_N$ *traverses* $S$, if every edge in $M$ is incident to at most one vertex in $S$, in other words no edge has two endpoints in $S$.

**Lemma 6** *For a fixed set $S$ of size $s$, the probability that $S$ is traversed, equals to*

$$\frac{2^s\binom{\frac{N}{2}}{s}}{\binom{N}{s}}.$$

**Proof:** Clearly the probability in question does not depend on the choice of $S$, just depends on the cardinality $s$. Therefore the probability does not change if we average it out for all $s$-subsets, and hence it is

$$\frac{\#(S,M) : \text{perfect matching } M \text{ traverses } S}{(N-1)!!\binom{N}{s}}.$$

Count now in the ordered pairs in the numerator as follows: for all $(N-1)!!$ perfect matchings, decide which $s$ edges of the $N/2$ edges of the perfect matching have endpoint in $S$, and for those $s$ edges decide which endpoint out of the two possibilities will belong to $S$. $\square$

**Lemma 7** *Assume that $\frac{\ln^2 N}{N^{1/3}} \leq x \leq \frac{1}{4}$ and $xN \to \infty$. For any fixed set $S$ of size $xN$, the probability that $S$ is traversed is*

$$e^{-N\frac{x^2}{2}+O(Nx^3)},$$

*where $O()$ refers to $xN \to \infty$.*

**Proof:** From the Stirling formula

$$N! = \left(\sqrt{2\pi N} + O(N^{-\frac{1}{2}})\right)\frac{N^N}{e^N}$$

one easily obtains

$$\binom{N}{xN} = \frac{1+O(\frac{1}{xN})}{\sqrt{2\pi x(1-x)N}} e^{N\cdot H(x)}.$$

Here $H(x) = -x\ln x - (1-x)\ln(1-x)$ denotes the binary entropy function. Also,

$$\binom{\frac{N}{2}}{xN} = \frac{1+O(\frac{1}{xN})}{\sqrt{2\pi x(1-2x)N}} e^{\frac{1}{2}N\cdot H(2x)},$$

and finally we have

$$\begin{aligned}
\frac{2^{xN}\binom{\frac{N}{2}}{xN}}{\binom{N}{xN}} &= \left(1+O(\frac{1}{xN})\right)\frac{2^{xN}\frac{1}{\sqrt{2\pi x(1-2x)N}}e^{\frac{1}{2}N\cdot H(2x)}}{\frac{1}{\sqrt{2\pi x(1-x)N}}e^{N\cdot H(x)}} \\
&= \left(1+O(\frac{1}{xN})\right)\sqrt{\frac{1-x}{1-2x}}e^{N(\frac{1}{2}H(2x)-H(x)+x\ln 2)} \\
&= \left(1+O(\frac{1}{xN})\right)\sqrt{\frac{1-x}{1-2x}}e^{-N\left((1/2-x)\ln(1-2x)-(1-x)\ln(1-x)\right)} \\
&= e^{-Nx^2/2+O(x^3N)},
\end{aligned}$$

where the last inequality follows from $x = \Omega(\frac{\ln^2 N}{N^{1/3}})$. $\square$

**Theorem 11** *Consider the configuration model as in Theorem 9. Assume (60), $\bar{d} \geq 3$, $k < \frac{1}{2}\sqrt{\frac{\bar{d}}{\log 2}}$, and assume further that $\frac{N}{8k^2} - \left(\frac{\bar{D}}{\bar{d}}\right)^{g-1}$ goes to infinity. Then almost all graphs with degree sequence $d_1, ..., d_n$ and girth at least $g \geq 4$ are not $k$-colourable.*

Specializing to regular graphs, we get back the existence of graphs of high chromatic number and high girth, roughly in the same range where Erdős [8] obtained it.

**Proof.** Recall that (61) gave the probability that the multigraph resulting from the configuration model has girth at least $g$. Because of the $g \geq 4$ assumption, the probability that a resulting *graph* has girth at least $g$ is at least as much as (61).

Now we set an upper bound on the probability that $G$ is $k$-colorable. For a subset $A$ of $V(G)$, let the *volume* of $A$ be $\sum_{v\in A} d_G(v)$. If $G$ is $k$-colorable, then $G$ contains an independent set of volume at least $\frac{2N}{k}$. By Lemma 7, at $x = 2/k$, the probability of this event is at most

$$2^n \exp\left(-\frac{2N}{k^2} + O\Big(\frac{N}{k^3}\Big)\right) = \exp\left(\Big(-\frac{2}{k^2} - \frac{\log 2}{\bar{d}} + O\big(\frac{1}{k^3}\big)\Big)N\right). \tag{63}$$

Computing the difference of the exponents in (63) and in (61) we are at home. □

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