# Loose Laplacian spectra of random hypergraphs

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**Abstract**

Let $H = (V, E)$ be an $r$-uniform hypergraph with the vertex set $V$ and the edge set $E$. For $1 \leq s \leq r/2$, we define a weighted graph $G^{(s)}$ on the vertex set $\binom{V}{s}$ as follows. Every pair of $s$-sets $I$ and $J$ is associated with a weight $w(I, J)$, which is the number of edges in $H$ passing through $I$ and $J$ if $I \cap J = \emptyset$, and 0 if $I \cap J \neq \emptyset$. The $s$-th Laplacian $\mathcal{L}^{(s)}$ of $H$ is defined to be the normalized Laplacian of $G^{(s)}$. The eigenvalues of $\mathcal{L}^{(s)}$ are listed as $\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}-1}^{(s)}$ in non-decreasing order. Let $\bar{\lambda}^{(s)}(H) = \max_{i\neq 0}\{|1 - \lambda_i^{(s)}|\}$. The parameters $\bar{\lambda}^{(s)}(H)$ and $\lambda_1^{(s)}(H)$, which were introduced in our previous paper [26], have a number of connections to the mixing rate of high-ordered random walks, the generalized distances/diameters, and the edge expansions.

For $0 < p < 1$, let $H^r(n, p)$ be a random $r$-uniform hypergraph over $[n] := \{1, 2, \ldots, n\}$, where each $r$-set of $[n]$ has probability $p$ to be an edge independently. For $1 \leq s \leq r/2$, $p(1 - p) \gg \frac{\log^4 n}{n^{r-s}}$, and $1 - p \gg \frac{\log n}{n^2}$, we prove that almost surely

$$\bar{\lambda}^{(s)}(H^r(n, p)) \leq \frac{s}{n - s} + \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1)\right)\sqrt{\frac{1 - p}{\binom{n-s}{r-s}p}}.$$

We also prove that the empirical distribution of the eigenvalues of $\mathcal{L}^{(s)}$ for $H^r(n, p)$ follows the Semicircle Law if $p(1 - p) \gg \frac{\log n}{n^{r-s}}$.

## 1 Introduction

The spectrum of the adjacency matrix (and/or the Laplacian matrix) of a random graph was well-studied in the literature [1, 10, 11, 13, 14, 15, 17, 18, 21]. Given a graph $G$, let $\mu_1(G), \ldots, \mu_n(G)$ be the eigenvalues of the adjacency matrix of $G$ in the non-decreasing order, and $\lambda_0(G), \ldots, \lambda_{n-1}(G)$ be the eigenvalues of (normalized) Laplacian matrix of $G$ respectively. Let $G(n, p)$ be the Edős-Rényi random graph model. Füredi and Komlós [21] showed that if $np(1 - p) \gg \log^6 n$ then almost surely $\mu_n = (1 + o(1))np$ and $\max\{-\mu_1, \mu_{n-1}\} \leq (2 + o(1))\sqrt{np(1 - p)}$. The results are extended to sparse random graphs [17, 25] and general random matrices [15, 21]. Alon-Krivelevich-Vu [1] proved the concentration of the $s$-th largest eigenvalue of a random symmetric matrix with independent random entries of absolute value at most 1. Friedman (in a series of papers [18, 19, 20]) proved that the second largest eigenvalue of random $d$-regular graphs is almost surely $(2+o(1))\sqrt{d - 1}$ for any $d \geq 4$. Chung-Lu-Vu [11] studied the Laplacian eigenvalues of random graphs with given expected

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degrees; their results were supplemented by Coja-Oghlan [13, 14] for much sparser random graphs.

In this paper, we study the spectra of the Laplacians of random hypergraphs. Laplacians for regular hypergraphs was first introduced by Chung [5] in 1993 using homology approach. Rodríguez [28, 29] treated a hypergraph as a multi-edge graph and then defined its Laplacian to be the Laplacian of the corresponding multi-edge graph. Inspired by these work, we [26] introduced the generalized Laplacian eigenvalues of hypergraphs through high-ordered random walks. Let $H = (V, E)$ be an $r$-uniform hypergraph on $n$ vertices. We can associate $r-1$ Laplacians $\mathcal{L}^{(s)}$ $(1 \leq s \leq r-1)$ to $H$; roughly speaking, $\mathcal{L}^{(s)}$ captures the incidence relations between $s$-sets and edges in $H$. Our definition of the Laplacian at the spacial case $s = 1$ is the same as the Laplacian considered by Rodríguez [28, 29]. The $s$-th Laplacian is *loose* if $1 \leq s \leq r/2$, and is *tight* if $r/2 < s \leq r-1$. Here we only consider the spectra of loose Laplacians.

For $1 \leq s \leq r/2$, we consider an auxiliary weighted graph $G^{(s)}$ defined as follows: the vertex set of $G^{(s)}$ is $\binom{V}{s}$ while the weighted function $W\colon \binom{V}{s} \times \binom{V}{s} \to \mathbb{Z}$ is defined as

$$W(S,T) = \begin{cases} |\{F \in E(H)\colon S \cup T \subset F\}| & \text{if } S \cap T = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The *$s$-th Laplacian* of $H$, denoted by $\mathcal{L}^{(s)}$, is the normalized Laplacian of $G^{(s)}$. For any $s$-set $S$, let $d_S$ be the number of edges in $H$ passing through $S$; the degree of $S$ in $G^{(s)}$ is $\binom{r-s}{s} d_S$. Let $D$ be the diagonal matrix of the degrees $\{d_S\}$ and $W$ be the weight matrix $\{w(S,T)\}$. Note that $T := \binom{r-s}{s} D$ is the diagonal matrix of degrees in $G^{(s)}$. We have

$$\mathcal{L}^{(s)} = I - T^{-1/2} W T^{-1/2}. \tag{2}$$

The eigenvalues of $\mathcal{L}^{(s)}$ are listed as $\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}-1}^{(s)}$ in non-decreasing order. We have

$$0 = \lambda_0^{(s)} \leq \lambda_1^{(s)} \leq \cdots \leq \lambda_{\binom{n}{s}-1}^{(s)} \leq 2. \tag{3}$$

The first non-trivial eigenvalue $\lambda_1^{(s)} > 0$ if and only if $G^{(s)}$ is connected. When this occurs, we say $H$ is *$s$-connected*. The diameter of $G^{(s)}$ is called the *$s$-th diameter* of $H$. The largest eigenvalue $\lambda_{\binom{n}{s}-1}^{(s)}$ is also denoted by $\lambda_{max}^{(s)}$. The (Laplacian) spectral radius, denoted by $\bar{\lambda}^{(s)}$, is the maximum of $1 - \lambda_1^{(s)}$ and $\lambda_{max}^{(s)} - 1$.

This definition differs slightly with the one in [26], where the vertex set of the auxiliary graph (denoted by $G^{(s)'}$) is the set of all distinct $s$-tuples instead. Note that $G^{(s)'}$ is the blow-up of $G^{(s)}$. Their Laplacian spectra differ only by the multiplicity of 1's. Therefore, two different definitions give the same values of $\lambda_1^{(s)}$, $\lambda_{max}^{(s)}$, and $\bar{\lambda}^{(s)}$.

For different $s$, the following inequalities were proved in [26].

$$\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \ldots \geq \lambda_1^{(\lfloor r/2 \rfloor)}; \tag{4}$$
$$\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \ldots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}. \tag{5}$$

The $s$-th Laplacian has a number of connections to the mixing rate of high-ordered random walks, the generalized distances/diameters, and the edge expansions. Here we list some applications, which are similar to results in [26], and results for graphs [4, 6, 7, 8, 9, 12].

**Random $s$-Walks:** The mixing rate of the random $s$-walk on $H$ is at most $\bar{\lambda}^{(s)}$.

**The $s$-Diameter:** The $s$-diameter of $H$ is at most

$$\left\lceil \frac{\log \frac{|E(H)|\binom{r}{s}}{\delta}}{\log \frac{\lambda^{(s)}_{\max}+\lambda^{(s)}_1}{\lambda^{(s)}_{\max}-\lambda^{(s)}_1}} \right\rceil.$$

Here $\delta = \min_{S\in\binom{V}{s}} d_S$ is the minimum degree among all $s$-sets.

**Edge expansion:** For $1 \le t \le s \le \frac{r}{2}$, $\mathcal{S} \subset \binom{V}{t}$, and $\mathcal{T} \subset \binom{V}{t}$, define

$$E(\mathcal{S},\mathcal{T}) = \{F \in E(H) \colon \exists S \in \mathcal{S}, \exists T \in \mathcal{T} \text{ such that } S \cap T = \emptyset, \text{ and } S \cup T \subset F\},$$

$$e(\mathcal{S},\mathcal{T}) = \frac{|E(\mathcal{S},\mathcal{T})|}{\left|E(\binom{V}{s},\binom{V}{t})\right|},$$

$$e(\mathcal{S}) = \frac{\sum_{S\in\mathcal{S}} d_S}{\sum_{S\in\binom{V}{s}} d_S},$$

$$e(\mathcal{T}) = \frac{\sum_{T\in\mathcal{T}} d_T}{\sum_{T\in\binom{V}{t}} d_T}.$$

Then we have

$$|e(\mathcal{S},\mathcal{T}) - e(\mathcal{S})e(\mathcal{T})| \le \bar{\lambda}^{(s)}\sqrt{e(\mathcal{S})e(\mathcal{T})e(\bar{\mathcal{S}})e(\bar{\mathcal{T}})}.$$

The proofs of these claims are very similar to those in [26] and are omitted here.

Our first result is the eigenvalues of the $s$-th Laplacian of the complete $r$-uniform hypergraph $K_n^r$.

**Theorem 1** *Let $K_n^r$ be the complete $r$-uniform hypergraph on $n$ vertices. For $1 \le s \le r/2$, the eigenvalues of $s$-th Laplacian of $K_n^r$ are given by*

$$1 - \frac{(-1)^i\binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1} \text{ for } 0 \le i \le s.$$

Here we point out an application of this theorem to the celebrated Erdős-Ko-Rado Theorem, which states "if the $n \ge 2s$, then the size of the maximum intersecting family of $s$-sets in $[n]$ is at most $\binom{n-1}{s-1}$." (The theorem was originally proved by Erdős-Ko-Rado [16] for sufficiently large $n$; the simplest proof was due to Katona [24].) Here we present a proof adapted from Calderbank-Frankl [2], where they use the eigenvalues of Kneser graph instead. (The relation between $\mathcal{L}^{(s)}(K_n^r)$ and the Laplacian of the Kneser graph is explained in section 2.)

It suffices to show for any intersecting family $U$ of $s$-sets, $|U| \le \binom{n-1}{s-1}$. Note that $U$ is an independent set of $G^{(s)}(K_n^r)$. Restricting to $U$, $\mathcal{L}^{(s)}(K_n^r)$ became an identity matrix; where all eigenvalues are equal to 1. By Cauchy's interlace theorem, we have

$$\lambda_k^{(s)} \le 1 \le \lambda^{(s)}_{\binom{n}{s}-|U|+k} \tag{6}$$

for $0 \le k \le |U|-1$. Let $N^+$ (or $N^-$) be the number of eigenvalues of $\mathcal{L}^{(s)}(K_n^r)$ which is $\ge 1$ (or $\le 1$) respectively. Inequality (6) implies that $|U| \le N^+$ and $|U| \le N^-$. By Theorem 1, $N^+ = \sum_{i=0}^{\lfloor (s-1)/2 \rfloor}\left(\binom{n}{2i+1} - \binom{n}{2i}\right)$ and $N^- = \sum_{i=0}^{\lfloor s/2 \rfloor}\left(\binom{n}{2i} - \binom{n}{2i-1}\right)$. We have

$$|U| \le \min\{N^+, N^-\} = \sum_{i=0}^{s-1}(-1)^{s-1-i}\binom{n}{i} = \binom{n-1}{s-1}.$$

For $0 < p < 1$, let $H^r(n,p)$ be a random $r$-uniform hypergraph over $[n] = \{1, 2, \ldots, n\}$, where each $r$-set of $[n]$ has probability $p$ to be an edge independently. We can estimate the Laplacian spectrum of $H^r(n,p)$ using the Laplacian spectrum of $K_n^r$ as follows.

**Theorem 2** *Let $H^r(n,p)$ be a random $r$-uniform hypergraph. For $1 \le s \le r/2$, if $p(1-p) \gg \frac{\log^4 n}{n^{r-s}}$ and $1 - p \gg \frac{\log n}{n^2}$, then almost surely the $s$-th spectral radius $\bar{\lambda}^{(s)}(H^r(n,p))$ satisfies*

$$\bar{\lambda}^{(s)}(H^r(n,p)) \le \frac{s}{n-s} + \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}. \tag{7}$$

*Moreover, for $1 \le k \le \binom{n}{s} - 1$ almost surely we have*

$$|\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)| \le \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}. \tag{8}$$

Note that $G(n,p)$ is a special case of $H^r(n,p)$ with $r = 2$. By choosing $s = 1$, Theorem 2 implies that

$$\bar{\lambda}(G(n,p)) \le (3 + o(1)) \sqrt{\frac{1-p}{(n-1)p}} \quad \text{for } p(1-p) \gg \frac{\log^4 n}{n}. \tag{9}$$

Chung-Lu-Vu's result[11], when restricted to $G(n,p)$, implies

$$\bar{\lambda}(G(n,p)) \le (4 + o(1)) \frac{1}{\sqrt{np}} \quad \text{for } 1 - \epsilon \ge p \gg \frac{\log^6 n}{n}. \tag{10}$$

Inequality 9 has a smaller constant and works for a larger range of $p$ than inequality 10.

Füredi and Komlós [21] proved the empirical distribution of the eigenvalues of $G(n,p)$ follows the Semicircle Law. Chung, Lu, and Vu [11] proved a similar result for the random graphs with given expected degrees. Here we prove a similar result for random hypergraphs.

**Theorem 3** *For $1 \le s \le r/2$, if $p(1-p) \gg \frac{\log n}{n^{r-s}}$, then almost surely the empirical distribution of eigenvalues of the $s$-th Laplacian of $H^r(n,p)$ follows the Semicircle Law centered at 1 and with radius $(2 + o(1)) \sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s} p}}$.*

**Remark 1** *The proof of Theorem 3 actually implies the eigenvalues of $\mathcal{L}^{(s)}(H^r(n,p)) - \mathcal{L}^{(s)}(K_n^r)$ follows the Semicircle Law centered at 0 and with radius $(2 + o(1)) \sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s} p}}$. Thus we have*

$$\max_{1 \le k \le \binom{n}{s} - 1} |\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)| \ge \left( \frac{2}{\sqrt{\binom{r-s}{s}}} + o(1) \right) \sqrt{\frac{1-p}{\binom{n-s}{r-s} p}}. \tag{11}$$

*This shows that the upper bound of $|\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)|$ in inequality (8) in Theorem 2 is best up to a constant factor.*

The rest of the paper is organized as follows. In section 2, we introduce the notation and prove some basic lemmas. We will prove Theorem 1 in section 3 and Theorem 2 in section 4.

# 2 Notation and Lemmas

## 2.1 Laplacian eigenvalues of hypergraphs

Let $H = (V, E)$ be an $r$-uniform hypergraph. For any subset $S$ ($|S| < r$), the degree of $S$, denoted by $d_S$, is the number of edges passing through $S$. For each $1 \leq s \leq r/2$, we associate a weighted graph $G^{(s)}$ on the vertex set $\binom{V}{s}$ to $H$ as follows. Every pair of $s$-sets $S$ and $T$ is associated with a weight $w(S, T)$, which is given by

$$w(S,T) = \begin{cases} d_{S\cup T} & \text{if } S \cap T = \emptyset, \\ 0 & \text{otherwise .} \end{cases}$$

The $s$-th Laplacian $\mathcal{L}^{(s)}$ of $H$ is defined to be the normalized Laplacian of $G^{(s)}$. The degree of $S$ in $G^{(s)}$ is $\sum_T w(S,T) = \binom{r-s}{s} d_S$.

We assume that the $s$-sets in $\binom{V}{s}$ are ordered alphabetically. Let $N := \binom{n}{s}$; all square matrices considered in the paper have the dimension $N \times N$ and all vectors have dimension $N$. Let $W := (W(S,T))$ be the weight matrix, $D$ be the diagonal matrix with diagonal entries $D(S,S) = d_S$, $\mathbf{d}$ be the column vector with entries $d_S$ at position $S \in \binom{V}{S}$, $J$ be the square matrix of all 1's, and $\mathbf{1}$ be the column vector of all 1's. Let $T := \binom{r-s}{s} D$; here $T$ is the diagonal matrix of degrees in $G^{(s)}$. Then, we have

$$\mathcal{L}^{(s)} = I - T^{-1/2} W T^{-1/2}.$$

We list the eigenvalues of $\mathcal{L}^{(s)}$ as

$$0 = \lambda_0^{(s)} \leq \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}-1}^{(s)} \leq 2.$$

We aim to compute the spectral radius $\bar{\lambda}^{(s)}(H) = \max_{i\neq 0} |1 - \lambda_i^{(s)}|$. Let $\mathrm{vol}^{(s)}(H) := \sum_{S\in\binom{V}{s}} d_s$ and $\phi_0 := \frac{1}{\sqrt{\mathrm{vol}^{(s)}(H)}} D^{1/2}\mathbf{1}$. Note that $\phi_0$ is the unit eigenvector corresponding to the trivial eigenvalue 0 of $\mathcal{L}^{(s)}$.

We are ready to prove theorem 1.

**Proof of Theorem 1:** We can write down $\mathcal{L}^{(s)}(K_n^r)$ using the following notation. The Kneser graph $K(n, s)$ is a graph over the vertex set $\binom{[n]}{s}$; two $s$-sets $S$ and $T$ form an edge of $K(n, s)$ if and only if $S \cap T = 0$. Let $K$ be the adjacency matrix of $K(n, s)$; the eigenvalues of $K$ are $(-1)^i \binom{n-s-i}{s-i}$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$ for $0 \leq i \leq s$ (see [22]). Note that $K(n, s)$ is a regular graph; so the Laplacian eigenvalues can be determined from the eigenvalues of its adjacency matrix. We observe that the associated weighted graph $G^{(s)}$ for the complete $r$-uniform hypergraph $K_n^r$ is essentially the Kneser graph with each edge associated with a weight $\binom{n-2s}{r-2s}$. Note that the multiplicative factor $\binom{n-2s}{r-2s}$ is canceled after normalization. The $\mathcal{L}^{(s)}$ (for $K_n^r$) is exactly the Laplacian of Kneser graph. Hence,

$$\mathcal{L}^{(s)}(K_n^r) = I - \frac{1}{\binom{n-s}{s}} K.$$

Thus, the eigenvalues of $s$-th Laplacian of $K_n^r$ are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{s-i}}{\binom{n-s}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1} \text{ for } 0 \leq i \leq s.$$

□

**Remark 2** *For $1 \le s \le r/2$, we have*

$$
\begin{aligned}
\lambda_1^{(s)}(K_n^r) &= 1 - \frac{s(s-1)}{(n-s)(n-s-1)}, &(12)\\
\lambda_{max}^{(s)}(K_n^r) &= 1 + \frac{s}{n-s}, &(13)\\
\bar{\lambda}^{(s)}(K_n^r) &= \frac{s}{n-s}. &(14)
\end{aligned}
$$

## 2.2 Random hypergraphs

Let $H^r(n,p)$ be a random $r$-uniform hypergraph over the vertex set $V = [n]$ and each $r$-set has probability $p$ to be an edge independently. We would like to bound the spectral radius of the $s$-th Laplacian of $H^r(n,p)$ for $1 \le s \le r/2$.

For any $F \in \binom{V}{r}$, let $X_F$ be the random indicator variable for $F$ being an edge in $H^r(n,p)$; all $X_F$'s are independent to each other. For any $S, T \in \binom{V}{s}$, we have

$$
W(S,T) = \begin{cases} \sum_{\substack{F \in \binom{n}{r} \\ S \cup T \subset F}} X_F & \text{if } S \cap T = \emptyset; \\ 0 & \text{otherwise.} \end{cases}
$$

Thus,

$$
\mathrm{E}(W(S,T)) = \begin{cases} \binom{n-2s}{r-2s} p & \text{if } S \cap T = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \tag{15}
$$

The degree $d_S = \sum_{S \subset F \in \binom{V}{r}} X_F$; we have $\mathrm{E}(d_S) = \binom{n-s}{r-s} p$. For simplicity, let $d := \binom{n-s}{r-s} p$.

We use the following Lemma to compare the eigenvalues of two matrices.

**Lemma 1** *Given any two $(N \times N)$-Hermitian matrices $A$ and $B$, for $1 \le k \le N$, let $\mu_k(A)$ (or $\mu_k(B)$) be the $k$-th eigenvalues of $A$ (or $B$) in the increasing order. We have*

$$
|\mu_k(A) - \mu_k(B)| \le \|A - B\|.
$$

**Proof:** By the Min-Max Theorem (see [27]), we have

$$
\begin{aligned}
\mu_k(A) &= \min_{S_k} \max_{x \in S_k, \|x\|=1} x'Ax, \\
\mu_k(B) &= \min_{S_k} \max_{x \in S_k, \|x\|=1} x'Bx.
\end{aligned}
$$

where the minimum is taken over all $k$-th dimensional subspace $S_k \subset \mathbb{R}^N$. We have

$$
\begin{aligned}
\mu_k(A) &= \min_{S_k} \max_{x \in S_k, \|x\|=1} x'Ax \\
&= \min_{S_k} \max_{x \in S_k, \|x\|=1} (x'Bx + x'(A-B)x) \\
&\le \min_{S_k} \max_{x \in S_k, \|x\|=1} (x'Bx + \|A-B\|) \\
&= \mu_k(B) + \|A - B\|.
\end{aligned}
$$

Similarly, we can show $\mu_k(A) \ge \mu_k(B) - \|A - B\|$. The proof of the Lemma is finished. $\square$

Our idea is to bound the spectral norm of the difference of $\mathcal{L}^{(s)}(H^r(n,p))$ and $\mathcal{L}^{(s)}(K_n^r)$. Let $M := \mathcal{L}^{(s)}(K_n^r) - \mathcal{L}^{(s)}(H^r(n,p)) = T^{-1/2} W T^{-1/2} - \frac{1}{\binom{n-s}{s}} K$. We write $M = M_1 + M_2 +$

$M_3 + M_4$, where

$$\begin{aligned}
M_1 &= \frac{1}{\binom{r-s}{s}}\left(D^{-1/2}(W - \mathrm{E}(W))D^{-1/2} - d^{-1}(W - \mathrm{E}(W))\right), \\
M_2 &= \frac{1}{\binom{r-s}{s}d}(W - \mathrm{E}(W)), \\
M_3 &= \frac{1}{\binom{r-s}{s}}D^{-1/2}\mathrm{E}(W)D^{-1/2} - \frac{d}{\binom{n}{s}}D^{-1/2}JD^{-1/2} - \frac{1}{\binom{n-s}{s}}K + \frac{1}{\binom{n}{s}}J, \\
M_4 &= \frac{1}{\binom{n}{s}}(dD^{-1/2}JD^{-1/2} - J).
\end{aligned}$$

By the triangular inequality of matrix norms, we have

$$\|M\| \le \|M_1\| + \|M_2\| + \|M_3\| + \|M_4\|.$$

Through this paper, the norm of any square matrix is the spectral norm. We would like to bound $\|M_i\|$ for $i = 1, 2, 3, 4$. We use the following Chernoff inequality.

**Theorem 4** *[3] Let $X_1, \ldots, X_n$ be independent random variables with*

$$\Pr(X_i = 1) = p, \qquad \Pr(X_i = 0) = 1 - p.$$

*We consider the sum $X = \sum_{i=1}^n X_i$, with expectation $\mathrm{E}(X) = np$. Then we have*

$$\begin{aligned}
&\textit{(Lower tail)} & \Pr(X \le \mathrm{E}(X) - \lambda) &\le e^{-\lambda^2/2\mathrm{E}(X)}, \\
&\textit{(Upper tail)} & \Pr(X \ge \mathrm{E}(X) + \lambda) &\le e^{-\frac{\lambda^2}{2(\mathrm{E}(X)+\lambda/3)}}.
\end{aligned}$$

**Lemma 2** *Suppose $d \ge \log N$. With probability at least $1 - \frac{1}{N^3}$, for any $S \in \binom{V}{s}$, we have $d_S \in (d - 3\sqrt{d\log N}, d + 3\sqrt{d \log N})$.*

**Proof:** Note $d_s = \sum_{F: S \subset F} X_F$ and $\mathrm{E}(d_S) = d$. Applying the lower tail of Chernoff's inequality with $\lambda = 3\sqrt{\mathrm{E}(X)\log N}$, we have

$$\Pr\left(X - \mathrm{E}(X) \le -\lambda\right) \le e^{-\lambda^2/2\mathrm{E}(X)} = \frac{1}{N^{9/2}}.$$

Applying the upper tail of Chernoff's inequality with $\lambda = 3\sqrt{\mathrm{E}(X)\log N}$, we have

$$\Pr\left(X - \mathrm{E}(X) \ge \lambda\right) \le e^{-\frac{\lambda^2}{2(\mathrm{E}(X)+\lambda/3)}} \le \frac{1}{N^{27/8}}.$$

$\square$

For convenience, let $d_{min} := d - 3\sqrt{d\log N}$, $d_{max} := d + 3\sqrt{d \log N}$; almost surely we have $d_{min} \le d_S \le d_{max}$ for all $S$.

**Lemma 3** *If $d \ge \log N$, then almost surely $\|M_3\| = O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right)$.*

**Proof:** Note $\mathrm{E}(W) = \binom{n-2s}{r-2s}pK$, where $K$ is the adjacency matrix of the Kneser graph $K(n, s)$. Let $M_0 := \frac{1}{\binom{n-s}{s}}K - \frac{1}{\binom{n}{s}}J$. We can rewrite $M_3$ as

$$M_3 = dD^{-1/2}M_0D^{-1/2} - M_0.$$

Note $\|M_0\| = \bar{\lambda}^{(s)}(K_n^r) = \frac{s}{n-s}$. We have

$$\begin{aligned}
\|M_3\| &= \|dD^{-1/2}M_0D^{-1/2} - M_0\| \\
&\leq \|(dD^{-1/2} - d^{1/2}I)M_0D^{-1/2}\| + \|M_0(d^{1/2}D^{-1/2} - I)\| \\
&\leq \|(d^{1/2}I - dD^{-1/2})\|\|M_0\|\|D^{-1/2}\| + \|M_0\|\|(d^{1/2}D^{-1/2} - I)\| \\
&\leq \left|d^{1/2} - dd_{min}^{-1/2}\right| \frac{s}{n-s} d_{min}^{-1/2} + \frac{s}{n-s}\left|d^{1/2}d_{min}^{-1/2} - 1\right| \\
&= O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right).
\end{aligned}$$

$\square$

**Lemma 4** *If $p(1-p) \gg \frac{\log n}{n^{r-s}}$, then almost surely*

$$\sum_{S \in \binom{V}{s}} (d_S - d)^2 = (1 + o(1))\binom{n}{s} d(1-p).$$

**Proof:** For $S \in \binom{V}{s}$, let $X_S = (d_S - d)^2$. We have

$$\mathrm{E}(X_S) = \mathrm{E}((d_S - d)^2) = \mathrm{Var}(d_S) = \binom{n-s}{r-s} p(1-p) = d(1-p).$$

We use the second moment method to prove that $\sum_S X_s$ concentrates around its expectation $\binom{n}{s}d(1-p)$. For any $S, T \in \binom{V}{s}$, the covariance can be calculated as follows.

$$\begin{aligned}
\mathrm{Cov}(X_S, X_T) &= \mathrm{E}(X_S X_T) - \mathrm{E}(X_S)\mathrm{E}(X_T) \\
&= \mathrm{E}((d_S - d)^2(d_T - d)^2) - d^2(1-p)^2.
\end{aligned}$$

For $F \in \binom{V}{r}$, let $Y_F = X_F - \mathrm{E}(X_F)$. Then we have $d_S - d = \sum_{S \subset F} Y_F$.

$$\mathrm{E}((d_S - d)^2(d_T - d)^2) = \sum_{\substack{F_1, F_2:\ S \subset F_1 \cap F_2 \\ F_3, F_4:\ T \subset F_3 \cap F_4}} \mathrm{E}(Y_{F_1}Y_{F_2}Y_{F_3}Y_{F_4}).$$

Since $\mathrm{E}(Y_{F_i}) = 0$, the non-zero terms occur only if

1. $F_1 = F_2 = F_3 = F_4$. In this case, we have

   $$\mathrm{E}(Y_{F_1}Y_{F_2}Y_{F_3}Y_{F_4}) = \mathrm{E}(Y_{F_1}^4) = (1-p)^4 p + (-p)^4(1-p) = p(1-p)(1 - 3p + 3p^2).$$

   The number of choices is $\binom{n-|S\cup T|}{r-|S\cup T|}$.

2. $F_1 = F_2 \neq F_3 = F_4$. In this case, we have

   $$\mathrm{E}(Y_{F_1}Y_{F_2}Y_{F_3}Y_{F_4}) = \mathrm{E}(Y_{F_1}^2)\mathrm{E}(Y_{F_3}^2) = p^2(1-p)^2.$$

   The number of choices is $\binom{n-s}{r-|S|}\binom{n-s}{r-|T|} - \binom{n-|S\cup T|}{r-|S\cup T|}$.

3. $F_1 = F_3 \neq F_2 = F_4$. In this case, we have

   $$\mathrm{E}(Y_{F_1}Y_{F_2}Y_{F_3}Y_{F_4}) = \mathrm{E}(Y_{F_1}^2)\mathrm{E}(Y_{F_2}^2) = p^2(1-p)^2.$$

   The number of choices is $\binom{n-|S\cup T|}{r-|S\cup T|}^2 - \binom{n-|S\cup T|}{r-|S\cup T|}$.

4. $F_1 = F_4 \neq F_2 = F_3$. This is the same as item 3.

Thus, we have

$$
\begin{aligned}
\mathrm{E}(X_S X_T) &= \binom{n-|S\cup T|}{r-|S\cup T|} p(1-p)(1-3p+3p^2) \\
&\quad + \left( \binom{n-s}{r-s}^2 + 2\binom{n-|S\cup T|}{r-|S\cup T|}^2 - 3\binom{n-|S\cup T|}{r-|S\cup T|} \right) p^2(1-p)^2. \\
&= \binom{n-|S\cup T|}{r-|S\cup T|} p(1-p)(1-6p+6p^2) + \left( \binom{n-s}{r-s}^2 + 2\binom{n-|S\cup T|}{r-|S\cup T|}^2 \right) p^2(1-p)^2.
\end{aligned}
$$

This expression on the right depends only on the size of $S \cup T$. Putting together, we get

$$
\begin{aligned}
\mathrm{Var}\left( \sum_{S\in\binom{V}{s}} X_S \right) &= \sum_{S,T\in\binom{V}{s}} \mathrm{Cov}(X_S, X_T) \\
&= \sum_{S,T\in\binom{V}{s}} (\mathrm{E}(X_S X_T) - d^2(1-p)^2) \\
&= \sum_{S,T\in\binom{V}{s}} \left( \mathrm{E}(X_S X_T) - \binom{n-s}{r-s}^2 p^2(1-p)^2 \right) \\
&= \sum_{i=s}^{2s} \sum_{|S\cup T|=i} \left( \binom{n-i}{r-i} p(1-p)(1-6p+6p^2) + 2\binom{n-i}{r-i}^2 p^2(1-p)^2 \right) \\
&\leq \sum_{i=s}^{2s} \sum_{|S\cup T|=i} \binom{n-i}{r-i} p(1-p) \left( 1-6p+6p^2 + 2\binom{n-s}{r-s} p(1-p) \right) \\
&\leq \sum_{i=s}^{2s} \sum_{|S\cup T|=i} \binom{n-i}{r-i} 3dp(1-p)^2 \\
&= \binom{n}{r} 3dp(1-p)^2 \sum_{i=s}^{2s} \frac{r!}{(i-s)!^2(2s-i)!(r-i)!} \\
&< 3\cdot 4^r \binom{n}{r} dp(1-p)^2 \\
&= O\left( \binom{n}{s} d^2(1-p)^2 \right).
\end{aligned}
$$

Let $X = \sum_S X_S$. We have $\mathrm{E}[X] = \binom{n}{s} d(1-p)$ and $\mathrm{Var}(X) = O\left(\binom{n}{s} d^2(1-p)^2\right)$. Applying Chebyshev's inequality to $X = \sum_{S\in\binom{V}{s}}$, we have

$$
\Pr\left( |X - \mathrm{E}(X)| \geq \log n \sqrt{\mathrm{Var}(X)} \right) \leq \frac{1}{\log^2 n}.
$$

Thus, almost surely $X = \mathrm{E}(X) + O(\log n\sqrt{\mathrm{Var}(X)}) = (1+o(1))\binom{n}{s} d(1-p)$. □

**Lemma 5** *If $p(1-p) \gg \frac{\log n}{n^{r-s}}$, then almost surely $\|M_4\| \leq (1+o(1))\sqrt{\frac{1-p}{d}}$.*

**Proof:** We can rewrite $M_4$ as

$$\begin{aligned}
M_4 &= \frac{1}{\binom{n}{s}}(dD^{-1/2}JD^{-1/2} - J) \\
&= \frac{1}{\binom{n}{s}}\left(\left(d^{1/2}D^{-1/2} - I\right)JD^{-1/2}d^{1/2} + J\left(d^{1/2}D^{-1/2} - I\right)\right) \\
&= \frac{1}{\binom{n}{s}}\left(\alpha\mathbf{1}'D^{-1/2}d^{1/2} + \mathbf{1}\alpha'\right).
\end{aligned}$$

Here $\alpha := d^{1/2}D^{-1/2}\mathbf{1} - \mathbf{1}$. Note that the spectral norm of a vector is the same as the $L_2$-norm. We have

$$\begin{aligned}
\|\alpha\| &= \|d^{1/2}D^{-1/2}\mathbf{1} - \mathbf{1}\| \\
&= \sqrt{\sum_{S\in\binom{V}{s}}\left(\frac{\sqrt{d}}{\sqrt{d_S}} - 1\right)^2} \\
&= \sqrt{\sum_{S\in\binom{V}{s}}\frac{(d_S - d)^2}{d_S(\sqrt{d} + \sqrt{d_S})^2}} \\
&\leq \frac{\sqrt{\sum_{S\in\binom{V}{s}}(d_S - d)^2}}{\sqrt{d_{min}}(\sqrt{d} + \sqrt{d_{min}})} \\
&= (\frac{1}{2} + o(1))\sqrt{\frac{(1-p)\binom{n}{s}}{d}}.
\end{aligned}$$

In the last step, we applied Lemma 4. Therefore, we have

$$\begin{aligned}
\|M_4\| &= \left\|\frac{1}{\binom{n}{s}}\left(\alpha\mathbf{1}'D^{-1/2}d^{1/2} + \mathbf{1}\alpha'\right)\right\| \\
&= \frac{1}{\binom{n}{s}}\left(\left\|\alpha\mathbf{1}'D^{-1/2}d^{1/2}\right\| + \|\mathbf{1}\alpha'\|\right) \\
&\leq \frac{1}{\binom{n}{s}}\|\alpha\|\left(\|\mathbf{1}'D^{-1/2}d^{1/2}\| + \|\mathbf{1}\|\right) \\
&= \frac{1}{\binom{n}{s}}\|\alpha\|\left(\sqrt{\sum_{S\in\binom{n}{s}}\frac{d}{d_S}} + \sqrt{\binom{n}{s}}\right) \\
&\leq \frac{1}{\binom{n}{s}}\left(\frac{1}{2} + o(1)\right)\sqrt{\frac{(1-p)\binom{n}{s}}{d}}(2 + o(1))\left(\sqrt{\binom{n}{s}}\right) \\
&= (1 + o(1))\sqrt{\frac{1-p}{d}}.
\end{aligned}$$

# 3 Proof of Theorem 2

To estimate the spectral norm of $M_1$ and $M_2$, we need consider the matrix $C := W - \mathrm{E}(W)$. We estimate the trace of $C^t$ as follows.

**Lemma 6** *For any $k \ll (n^{r-s}p(1-p))^{1/4}$, we have*

$$\mathrm{E}\left(\mathrm{Trace}(C^{2k})\right) = (1+o(1))\frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k, \tag{16}$$

$$\mathrm{E}\left(\mathrm{Trace}(C^{2k+1})\right) = O\left(\frac{k(2k+1)n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k\right). \tag{17}$$

The proof of this technical Lemma is quite long. We will delay its proof until the end of this section.

**Lemma 7** *Suppose $p(1-p) \gg \frac{\log^4 n}{n^{r-s}}$. Almost surely, we have $\|C\| \le (2+o(1))\sqrt{\binom{r-s}{s}d(1-p)}$.*

**Proof:** By Lemma 6, we have $\mathrm{E}(\mathrm{Trace}(C^{2k})) = (1+o(1))\frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k$. As $\mathrm{E}(\|C\|^{2k}) \le \mathrm{E}(\mathrm{Trace}(C^{2k}))$, we have

$$\mathrm{E}(\|C\|^{2k}) \le (1+o(1))\frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k.$$

Let $U := \frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k$. By Markov's inequality,

$$\begin{aligned}
\Pr\left(\|C\| \ge (1+\epsilon)\sqrt[2k]{U}\right) &= \Pr\left(\|C\|^{2k} \ge (1+\epsilon)^{2k}U\right)\\
&\le \frac{\mathrm{E}(\|C\|^{2k})}{(1+\epsilon)^{2k}U}\\
&\le \frac{(1+o(1))U}{(1+\epsilon)^{2k}U}\\
&= \frac{1+o(1)}{(1+\epsilon)^{2k}}.
\end{aligned}$$

Let $g(n)$ be a slowly growing function such that $g(n) \to \infty$ as $n$ approaches the infinity and $g(n) \ll \frac{(n^{r-s}p(1-p))^{1/4}}{s\log n}$. This is possible because $n^{r-s}p(1-p) \gg \log^4 n$. Choose $k = sg(n)\log n$ and $\epsilon = 1/g(n)$. We have $k \ll (n^{r-s}p(1-p))^{1/4}$ and $\epsilon \to 0$. Then we have $(1+o(1))/(1+\epsilon)^{2k} = O(n^{-s})$, which implies that almost surely

$$\begin{aligned}
|\|C\| &\le (1+o(1))\sqrt[2k]{U}\\
&= (1+o(1))\left(\frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k\right)^{\frac{1}{2k}}\\
&< n^{\frac{s}{2k}}2\sqrt{\frac{n^{r-s}p(1-p)}{s!(r-2s)!}}\\
&= (2+o(1))\sqrt{\binom{r-s}{s}d(1-p)}.
\end{aligned}$$

□

Recall $M_2 = \frac{1}{\binom{r-s}{s}d}C$. We have

**Lemma 8** *Suppose $p(1-p) \gg \frac{\log^4 n}{n^{r-s}}$. Almost surely, we have $\|M_2\| \le \frac{(2+o(1))\sqrt{1-p}}{\sqrt{\binom{r-s}{s}d}}$.*

**Lemma 9** *Suppose $p(1-p) \gg \frac{\log^4 n}{n^{r-s}}$. Almost surely, we have $\|M_1\| = O\left(\frac{\sqrt{(1-p)\log N}}{d}\right)$.*

**Proof:** We have

$$\begin{aligned} M_1 &= \frac{1}{\binom{r-s}{s}}\left(D^{-1/2}CD^{-1/2} - d^{-1}C\right) \\ &= \frac{1}{\binom{r-s}{s}}\left((D^{-1/2} - d^{-1/2}I)CD^{-1/2} + d^{-1/2}C(D^{-1/2} - d^{-1/2}I)\right). \end{aligned}$$

Note $\|D^{-1/2} - d^{-1/2}I\| \leq |d_{min}^{-1/2} - d^{-1/2}| = O(\frac{\sqrt{\log N}}{d})$, $\|D^{-1/2}\| \leq d_{min}^{-1/2} = (1 + o(1))d^{-1/2}$, and $\|C\| = (2+o(1))\sqrt{\binom{r-s}{s}d(1-p)}$. We have

$$\begin{aligned} \|M_1\| &= \frac{1}{\binom{r-s}{s}}\left\|(D^{-1/2} - d^{-1/2}I)CD^{-1/2} + d^{-1/2}C(D^{-1/2} - d^{-1/2}I)\right\| \\ &= O\left(\frac{\sqrt{(1-p)\log N}}{d}\right). \end{aligned}$$

$\square$

**Proof of Theorem 2:** Combining Lemmas 3, 5, 8, and 9, we have

$$\begin{aligned} \|M\| &= \|M_1 + M_2 + M_3 + M_4\| \\ &\leq \|M_1\| + \|M_2\| + \|M_3\| + \|M_4\| \\ &\leq O\left(\frac{\sqrt{(1-p)\log N}}{d}\right) + \frac{(2+o(1))\sqrt{1-p}}{\sqrt{\binom{r-s}{s}d}} + O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) + (1+o(1))\sqrt{\frac{1-p}{d}} \\ &= \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1)\right)\sqrt{\frac{1-p}{d}}. \end{aligned}$$

In the last step, we use the fact $\frac{\sqrt{\log N}}{n\sqrt{d}} = o\left(\sqrt{\frac{1-p}{d}}\right)$ since $1-p \gg \frac{\log n}{n^2}$.

By Lemma 1, for $1 \leq k \leq \binom{n}{s} - 1$, we have

$$|\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^r)| \leq \|M\| \leq \left(\frac{2}{\sqrt{\binom{r-s}{s}}} + 1 + o(1)\right)\sqrt{\frac{1-p}{d}}.$$

$\square$

**Proof of Lemma 6:** For any fixed positive integer $t$, the terms in Trace$(C^t)$ are of the form

$$c_{S_1S_2}c_{S_2S_3}\ldots c_{S_tS_{S_1}}.$$

Here $c_{ST} = W(S,T) - \mathrm{E}(W(S,T)) = \sum_{\substack{F\in\binom{V}{r} \\ S\cup T\subset F}}(X_F - \mathrm{E}(X_F))$ if $S\cap T = \emptyset$; $c_{ST} = 0$ otherwise.

Note $c_{S_iS_j} = 0$ if $S_i \cap S_j \neq \emptyset$. Thus we need only to consider the sequence $S_1S_2\ldots S_tS_1$ such that $S_i \cap S_{i+1} = \emptyset$ for each $1 \leq i \leq t$, here $t+1 = 1$.

For $F \in \binom{V}{r}$ and $S, T \in \binom{V}{s}$, we define a random variable $c_{ST}^F$ as follows.

$$c_{ST}^F = \begin{cases} X_F - \mathrm{E}(X_F) & \text{if } S\cap T = \emptyset \text{ and } S\cup T \subseteq F; \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $S_1F_1S_2F_2S_3\ldots S_tF_tS_1$ is called a closed $s$-walk of length $t$ if

1. $S_1, \ldots, S_t \in \binom{V}{s}$,
2. $F_1, \ldots, F_t \in \binom{V}{r}$,
3. $S_i \cap S_{i+1} = \emptyset$, for $i = 1, 2, \ldots, t$,
4. $S_i \cup S_{i+1} \subset F_i$, for $i = 1, 2, \ldots, t$.

Here we use the convention $S_{t+1} = S_1$. Those $r$-sets $F_i$'s are referred as edges while those $s$-sets $S_i$'s are referred as stops.

Using the notation above, we rewrite the trace as

$$\mathrm{Trace}(C^t) = \sum_{\text{closed } s\text{-walks}} c_{S_1 S_2}^{F_1} c_{S_2 S_3}^{F_2} \ldots c_{S_t S_1}^{F_t},$$

where the summation is over all possible closed $s$-walk of length $t$.

Taking the expectation on both sides, we get

$$\mathrm{E}(\mathrm{Trace}(C^t)) = \sum_{\text{closed } s\text{-walks}} \mathrm{E}(c_{S_1 S_2}^{F_1} c_{S_2 S_3}^{F_2} \ldots c_{S_t S_1}^{F_t}).$$

The terms in the product above can be regrouped according to the values of $F_i$'s; those terms with distinct $F$'s are independent to each other. Since $\mathrm{E}(c_{S,T}^F) = 0$, the contribution of a closed walk is 0 if some $F$ appears just once. Thus we need only to consider the set of closed walks where each edge appears at least twice or do not occur; we call these closed walks as *good* closed walks. A good closed walk can contain at most $\lfloor \frac{t}{2} \rfloor$ distinct edges.

Let $\mathcal{G}$ be the set of good closed walks. For $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$, let $\mathcal{G}_i^j$ be the set of good closed walks with exactly $i$ distinct edges and $j$ distinct vertices; and let $\mathcal{G}_i := \cup_j \mathcal{G}_i^j$.

We consider a good closed walk in $\mathcal{G}_i$. When a new edge comes in the walk, it can visit at most $(r-s)$ new vertices. Thus such a good closed walk covers at most $m_i := s + i(r-s)$ vertices. Any walk contains at least one edge. Hence, the number of vertices in a walk from $\mathcal{G}_i$ is in the interval $[r, m_i]$.

Let $a_i^j := \sum_{S_1 F_1 S_2 \ldots S_t S_1 \in \mathcal{G}_i^j} \mathrm{E}(c_{S_1 S_2}^{F_1} c_{S_2 S_3}^{F_2} \ldots c_{S_t S_1}^{F_t})$ and $a_i := \sum_{j=r}^{m_i} a_i^j$. We have

$$\mathrm{E}(\mathrm{Trace}(C^t)) = \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} a_i = \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=r}^{m_i} a_i^j. \tag{18}$$

Assume that an edge $F$ occurs $l$ times in a good closed walk and $T := \{i : 1 \leq i \leq t \text{ and } F_i = F\}$. We have $\Pr\left(\Pi_{i \in T} c_{S_i S_{i+1}}^F = (1-p)^l\right) = p$ and $\Pr\left(\Pi_{i \in T} c_{S_i S_{i+1}}^F = (-p)^l\right) = 1 - p$. Thus, for each positive integer $l \geq 2$, we have

$$\mathrm{E}\left(\Pi_{i \in T} c_{S_i S_{i+1}}^F\right) = (1-p)^l p + (-p)^l (1-p) \leq p(1-p).$$

The equality holds if $l = 2$.

Pick a good closed walk $S_1 F_1 S_2 F_2 S_3 \ldots S_t F_t S_1$ in $\mathcal{G}_q$. Say, it contains $q$ distinct edges $F^1, F^2, \ldots, F^q$. For each $1 \leq i \leq q$, let $T_i := \{1 \leq j \leq t : F_j = F^i\}$; then $\sum_{i=1}^q |T_i| = t$. We have

$$\mathrm{E}(c_{S_1 S_2}^{F_1} c_{S_2 S_3}^{F_2} \ldots c_{S_t S_1}^{F_t}) = \Pi_{i=1}^q \Pi_{j \in T_i} \mathrm{E}(c_{S_j S_{j+1}}^{F^i}) \leq \Pi_{i=1}^q p(1-p) = p^q (1-p)^q.$$

This implies

$$a_i^j \leq \left| \mathcal{G}_i^j \right| p^i (1-p)^i \tag{19}$$

for all $1 \le i \le \lfloor \frac{t}{2} \rfloor$ and $r \le j \le m_i$. In particular, the equality holds when $t = 2i$.

**Claim a:** For $1 \le i \le \lfloor \frac{t}{2} \rfloor$, we have

$$|\mathcal{G}_i| = (1 + o(1))|\mathcal{G}_i^{m_i}|. \tag{20}$$

**Proof:** Let $\mathcal{B}_i^j := \mathcal{G}_i^j(K_j^r)$ be the set of good closed walks (of length $t$) with $i$ distinct edges and $j$ distinct vertices on $K_j^r$. We have

$$|\mathcal{G}_i^j| = \binom{n}{j} \left|\mathcal{B}_i^j\right|. \tag{21}$$

We define a map $\phi_i \colon \mathcal{B}_i^j \to \mathcal{B}_i^{m_i}$ as follows. For any good closed walk $S_1F_1S_2F_2S_3 \ldots S_tF_tS_1 \in \mathcal{B}_i^j$, we scan the walk from left to right. Suppose that an edge $F$ appears in the walk for the first time, say $F = F_l$. If $|F \cap (\cup_{x<l} F_x)| > |S_l|$, then we replace the vertices in $F \cap (\cup_{x<l} F_x) \setminus S_l$ by next available vertices in $[m_i] \setminus [j]$. We keep the procedure for all distinct edges. At the end, the resulted walk has the following property "Any new edge visits $r - s$ new vertices." Observe the resulted walk is in $\mathcal{B}_i^{m_i}$. It is possible that different walks in $B_i^j$ be mapped into the same walk in $B_i^{m_i}$; there is at most $j^{m_i-j}$ sequences from $\mathcal{B}_i^j$ with the same image. We have

$$|\mathcal{B}_i^j| \le |\mathcal{B}_i^{m_i}| j^{m_i-j}. \tag{22}$$

Combining equations (21) and (22), we get

$$\begin{aligned}
|\mathcal{G}_i| &= \sum_{j=r}^{m_i} |\mathcal{G}_i^j| \\
&= \sum_{j=r}^{m_i} \binom{n}{j} \left|\mathcal{B}_i^j\right| \\
&\le \sum_{j=r}^{m_i} \binom{n}{j} |\mathcal{B}_i^{m_i}| j^{m_i-j} \\
&= \binom{n}{m_i} |\mathcal{B}_i^{m_i}| \sum_{j=r}^{m_i} \frac{\binom{n}{j}}{\binom{n}{m_i}} j^{m_i-j} \\
&< |\mathcal{G}_i^{m_i}| \sum_{j=r}^{m_i} \left( \frac{m_i j}{n - m_i + 1} \right)^{m_i-j} \\
&< |\mathcal{G}_i^{m_i}| \sum_{j=r}^{m_i} \left( \frac{m_i^2}{n - m_i + 1} \right)^{m_i-j} \\
&< |\mathcal{G}_i^{m_i}| \frac{1}{1 - \frac{m_i^2}{n-m_i+1}} \\
&= (1 + o(1))|\mathcal{G}_i^{m_i}|.
\end{aligned}$$

It is enough to estimate $|\mathcal{G}_i^{m_i}|$ for $1 \le i \le \lfloor \frac{t}{2} \rfloor$. Given a walk $w := S_1F_1S_2F_2S_3 \ldots S_tF_tS_1 \in \mathcal{B}_i^{m_i}$, let $\mathcal{S}$ be the set of distinct stops in $w$ and $\mathcal{F}$ be the set of distinct edges in $w$. List the edges in $\mathcal{F}$ as $\{F^1, F^2, \ldots, F^i\}$ with the indices in an increasing order. We define an auxiliary graph $T_w$ with the vertex set $\mathcal{S} \cup \mathcal{T}$ and the edge set $\{SF\colon \text{ if } S \in \mathcal{S}, F \in \mathcal{F}, \text{ and } S \subset F\}$.

**Claim b:** The graph $T_w$ is a tree.

**Proof:** A closed walk $w$ induces a closed walk on $T_w$. Thus $T_w$ is connected. Suppose that $T_w$ is not a tree, then there is a cycle $C$ in $T_w$. Let $F_j$ be the edge in $C$ with the highest index. When $F_j$ is first created, $F_j$ brings in $r-s$ new vertices; thus, $|\mathcal{F}_j \cap (\cup_{l\leq j} F_l)| = s$. This contradicts the fact that $F_j$ contains two different stops in $C$. Hence, $T_w$ is a tree.

For $1 \leq j \leq i$, let $S^j$ be the stop right after the first occurrence of $F^j$ in the walk $w$ and $T^j$ be the stop right before the first occurrence of $F^j$ in the walk $w$, and $E^j = F^j \setminus (S^j \cup T^j)$. We also let $S^0 := S_1$ be the first stop of $w$. Observe that

$$[m_i] = \left(\cup_{j=0}^{i} S^j\right) \cup \left(\cup_{j=1}^{i} E_j\right)$$

is a partition of $[m_i]$; each $S^j$ is an $s$-set while each $E_j$ is an $(r-2s)$-set. The number of choices of such partition is

$$\binom{m_i}{s, \ldots, s, r-2s, \ldots, r-2s} = \frac{m_i!}{(s!)^{i+1}((r-2s)!)^i}.$$

We can associate a walk $w$ with a code of length $t$ consisting of symbols '(', ')', and '$*$'. We read the walk $w$ from left to right. If $w$ visit the stop $S^j$ through the edge $F^j$ for the first time, we encode it by an open parenthesis; if $w$ visit a stop from $S^j$ through the edge $F^j$ for the first time, we use a close parenthesis; else we use '$*$'. A walk $w$ can be viewed as a walk on $T_w$; an open parenthesis means the walk passing through an edge $F^j S^j$ (for some $j$) while a closed parenthesis means the walk passing through an edge $S^j F^j$ (for some $j$). Since $\{S^j F_j\}_{j=1,\ldots,i}$ is a matching of the tree $T_w$, the resulted parenthesis sequence is valid; where valid means that each open parenthesis can be matched to a closed parenthesis. There are exactly $i$ pairs of parentheses and $t-2i$ '$*$'s; the number of ways to choose the positions of $*$'s is $\binom{t}{2i}$. The number of ways to arrange the parentheses is the Catalan number $\frac{1}{i+1}\binom{2i}{i}$. At each of position '$*$', there is at most $i$ ways to choose an existed edge and $\binom{r-s}{s}$ ways to choose the next stop in the edge. Putting together, we have

$$|\mathcal{B}_i^{m_i}| \leq \frac{m_i!}{(s!)^{i+1}(r-2s)^i}\binom{t}{2i}\frac{1}{i+1}\binom{2i}{i}\left(i\binom{r-s}{s}\right)^{t-2i}. \tag{23}$$

**Case 1:** $t = 2k$ is even. We would like to show the inequality (23) is tight for $i = k$. In this case, each edge appears exactly twice in any walk $w$ of $\mathcal{B}_k^{m_k}$. The structure of $T_w$ is more clear in this case.

**Claim c:** There are exactly $k+1$ stops in $w$, namely $S^0, S^1, \ldots, S^k$.

**Proof:** Since each edge $F^j$ appears exactly twice in a closed walk, the degree of $F^j$ in $T_w$ is exactly 2. Contracting these $F^j$'s in $T_w$, (i.e., deleting $F^j$ and connecting the two neighbors of $F^j$), we get a new tree $T$; where $F^j$'s can be viewed as edge labellings of the tree $T$. Now $T$ has exactly $k$ edges; it implies hat $T$ has exactly $k+1$ vertices. Thus $|\mathcal{S}| = k+1$. Since $S^0, S^1, \ldots, S^k \in \mathcal{S}$, we must have $\mathcal{S} = \{S^0, S^1, \ldots, S^k\}$.

**Claim d:** We have

$$|B_k^{m_k}| = \frac{m_k!}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}. \tag{24}$$

**Proof:** From the proof equation (23), a walk in $\mathcal{B}_k^{m_k}$ determine a partition $[m_i] = \left(\cup_{j=0}^{i} S^j\right) \cup \left(\cup_{j=1}^{i} E_j\right)$ and a valid sequence of $k$ pairs of parentheses. (In this case, the number of '$*$'s is zero.) It suffices to recover a walk from a partition of $[m_k]$ and a sequence of valid parentheses.

Given a partition

$$[m_i] = \left(\cup_{j=0}^{i} S^j\right) \cup \left(\cup_{j=1}^{i} E_j\right)$$

and a valid sequence of $k$ pairs of parentheses, we first build a rooted tree $T$ as follows. At each time, we maintain a tree $T$, a current stop $S$, a set of unused stops $\mathcal{S}$. Initially $T$

contains nothing but the root stop $S_0$, $S := S_0$, and $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$. At each time, read a symbol from the sequence. If the symbol is an open parenthesis, then find an $S_i$ in $\mathcal{S}$ with index $i$ as small as possible, delete $S_i$ from $\mathcal{S}$, attach $S_i$ to $T$ as a child stop of $S$, and let $S := S_i$; if the symbol is ")", then let $S$ point to the the parent stop of the current $S$. Repeat this process until all symbols from the sequence are processed.

Since every closed parenthesis has a matching open parenthesis, this process never get stuck. When the precess ends, a rooted tree $T$ on the vertex set $\{S_0, \ldots, S_k\}$ is created. For $1 \leq i \leq k$, let $F_i$ be the union of $E_i$ and two ends of $i$-th edge, which created in the process. For example, for $k = 3$, if the sequence is $(())()$, then the corresponding rainbow closed walk is

$$S_1 F_1 S_2 F_2 S_3 F_2 S_2 F_1 S_1 F_3 S_4 F_3 S_1$$

where $F_1 = S_1 \cup S_2 \cup E_1$, $F_2 = S_2 \cup S_3 \cup E_2$, and $F_3 = S_4 \cup S_1 \cup E_3$.

Thus, this is a bijection from $\mathcal{B}_k^{m_k}$ to the combination of a partition of $[m_k]$and a valid sequence of parentheses.

The number of ways to choose these sets $S_0, S_1, \ldots, S_k, E_1, \ldots, E_k$ as a partition of $[m_k]$ is

$$\binom{m_k}{s, \ldots, s, r-2s, \ldots, r-2s} = \frac{m_k!}{(s!)^{k+1}((r-2s)!)^k}.$$

The number of valid sequences of $k$ pairs of parentheses is the Catalan number $\frac{1}{k+1}\binom{2k}{k}$. By taking product of these two numbers, we get equation (24).

For each $1 \leq i \leq k$, by inequality (19) and equation (20), we have

$$a_i \leq \sum_{j=r}^{m_i} a_i^j \leq \sum_{j=r}^{m_i} |\mathcal{G}_i^j| p^i (1-p)^i = (1+o(1)) |\mathcal{G}_i^{m_i}| p^i (1-p)^i.$$

By equation (21) and inequality (23), for $1 \leq i \leq k$, we have

$$\begin{aligned} a_i &\leq (1+o(1)) |\mathcal{G}_i^{m_i}| p^i (1-p)^i \\ &\leq (1+o(1)) \binom{n}{m_i} |\mathcal{B}_i^{m_i}| p^i (1-p)^i \\ &\leq (1+o(1)) \frac{m_i! p^i (1-p)^i}{(s!)^{i+1}((r-2s)!)^i} \binom{n}{m_i} \binom{2k}{2i} \frac{1}{i+1} \binom{2i}{i} \left( i \binom{r-s}{s} \right)^{2k-2i}. \end{aligned}$$

By equations (19), (20), (21), and (24), we have

$$\begin{aligned} a_k &= (1+o(1)) \binom{n}{m_k} |\mathcal{B}_k^{m_k}| p^k (1-p)^k \\ &= (1+o(1)) \frac{m_k! p^k (1-p)^k}{(s!)^{k+1}((r-2s)!)^k} \binom{n}{m_k} \frac{1}{k+1} \binom{2k}{k}. \end{aligned}$$

For each $1 \leq i \leq k-1$, we have

$$\begin{aligned} \frac{a_i}{a_k} &\leq (1+o(1)) \frac{(k+1)(k!)^2}{(i+1)(2k-2i)!(i!)^2} \left( \frac{i^2}{s!(r-2s)! n^{r-s} p(1-p)} \right)^{k-i} \\ &= (1+o(1)) \frac{\binom{2k+1}{2i+1}\binom{2i+1}{i}}{\binom{2k+1}{k}} \left( \frac{i^2}{s!(r-2s)! n^{r-s} p(1-p)} \right)^{k-i} \\ &\leq (1+o(1)) \left( \frac{9k^4}{s!(r-2s)! n^{r-s} p(1-p)} \right)^{k-i}, \end{aligned}$$

As we assume $k^4 \ll n^{r-s}p(1-p)$, then $\frac{a_i}{a_k} < \epsilon^{k-i}$ for any constant $\epsilon > 0$ and $1 \leq i \leq k-1$. Thus $a_k$ is the dominating term in $\mathrm{E}(\mathrm{Trace}(C^{2k}))$, i.e.,

$$\mathrm{E}\left(\mathrm{Trace}(C^{2k})\right) = \sum_{i=1}^{k} a_i = (1+o(1))a_k = (1+o(1))\frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k.$$

**Case 2:** $t = 2k+1$ is odd. Since each edge in a good walk appears at least twice, a good sequence $S_1F_1S_2F_2S_3\ldots S_{2k+1}F_{2k+1}S_1$ contains at most $k$ distinct edges. By equations (19), (20), (21), and (24), we have $a_i \leq (1+o(1))f(i)$, where

$$f(i) = \frac{n^{s+i(r-s)}}{(i+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k+1}{2i}\binom{2i}{i}\left(i\binom{r-s}{s}\right)^{2k+1-2i}p^i(1-p)^i.$$

Similarly, we can show $f(i) = o(f(k))$ for $1 \leq i \leq k-1$ and $\sum_{i=1}^{k} f(i) = (1+o(1))f(k)$. We have

$$\begin{aligned}
\mathrm{E}\left(C^{2k+1}\right) &\leq \sum_{i=1}^{k} f(i) \\
&= (1+o(1))f(k) \\
&= (1+o(1))\frac{k(2k+1)n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}\binom{r-s}{s}p^k(1-p)^k \\
&= O\left(\frac{k(2k+1)n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k}\binom{2k}{k}p^k(1-p)^k\right).
\end{aligned}$$

□

# 4 The semicircle law

Let us review the defintion of the Semicircle Law. Let $F(x)$ be the continuous distribution function with density $f(x)$ such that $f(x) = (2/\pi)\sqrt{1-x^2}$ when $|x| \leq 1$ and $f(x) = 0$ when $|x| > 1$. Let $A$ be a Hermitian matrix of dimension $N \times N$. The *empirical distribution* of the eigenvalues of $A$ is

$$F(A, x) := \frac{1}{N}|\{\text{ eigenvalues of } A \text{ less than } x\}|.$$

We say, the empirical distribution of the eigenvalues of $A$ asymptotically follows the Semicircle Law centered at $c$ with radius $R$ if $F(\frac{1}{R}(A-cI), x)$ tends to $F(x)$ in probability as $N$ goes to infinity. (In this case, we write $F(\frac{1}{R}(A-cI), x) \xrightarrow{p} F(x)$.) If $c$ is the center of the Semicircle Law, then any $c' = c + o(R)$ is also the center of the Semicircle Law.

**Theorem 5** *If $n^{r-s}p(1-p) \to \infty$, then the empirical distribution of the eigenvalues of $W - \mathrm{E}(W)$ follows the semicircle law centered at $0$ with radius $2\sqrt{\binom{r-s}{s}\binom{n-s}{r-s}p(1-p)}$.*

**Proof:** Let $R := 2\sqrt{\binom{r-s}{s}\binom{n-s}{r-s}p(1-p)}$, $C := W - \mathrm{E}(W)$, and $C_{nor} := \frac{1}{R}C$.

To prove the theorem, we need to show that for any fixed $t$, the $t$-th moment of $F(C_{nor}, x)$ (with $n$ goes to infinity) is asymptotically equal to the $t$-th moment of $F(x)$. We know the $t$-th moment of $F(C_{nor}, x)$ equals $\binom{n}{s}^{-1}\mathrm{E}(\mathrm{Trace}(C_{nor}^t))$. For even $t = 2k$, the $t$-th moment of $F(x)$ is $(2k)!/2^{2k}k!(k+1)!$. For odd $t$, the $t$-th moment of $F(x)$ is 0.

In order to prove the theorem, we need to show for any fixed $k$,

$$\frac{1}{\binom{n}{s}} \mathrm{E}(\mathrm{Trace}(C_{nor}^{2k})) = (1+o(1)) \frac{(2k)!}{2^{2k} k!(k+1)!}$$

and

$$\frac{1}{\binom{n}{s}} \mathrm{E}(\mathrm{Trace}(C_{nor}^{2k+1})) = o(1).$$

We know

$$\mathrm{E}(\mathrm{Trace}(C_{nor}^{t})) = \frac{1}{R^t} \mathrm{E}(\mathrm{Trace}(C^{t}))$$

for any $t$. By Lemma 6, we have

$$\mathrm{E}(\mathrm{Trace}(C^{2k})) = (1+o(1)) \frac{n^{s+k(r-s)}}{(k+1)(s!)^{k+1}((r-2s)!)^k} \binom{2k}{k} p^k (1-p)^k.$$

Then

$$\frac{1}{\binom{n}{s}} \mathrm{E}(\mathrm{Trace}(C_{nor}^{2k})) = (1+o(1)) \frac{(2k)!}{2^{2k} k!(k+1)!}$$

as desired.

By Lemma 6 again, we have

$$\mathrm{E}(\mathrm{Trace}(C^{2k+1})) = O\left( \frac{k(2k+1) n^{s+k(r-s)} p^k (1-p)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k} \binom{2k}{k} \right).$$

Thus

$$\begin{aligned} \frac{1}{\binom{n}{s}} \mathrm{E}(\mathrm{Trace}(C_{nor}^{2k+1})) &= O\left( \frac{(2k+1)!}{2^{2k}(k-1)!(k+1)! R} \right) \\ &= o(1). \end{aligned}$$

Here $k$ is any constant but $R \to \infty$. The theorem is proved. $\square$

The following Lemma is useful to derive the Semicircle Law from one matrix to the other.

**Lemma 10** *Let $A$ and $B$ be two $(N \times N)$-Hermitian matrices. Suppose that the empirical distribution of the eigenvalues of $A$ follows the Semicircle Law centered at $c$ with radius $R$. If either $\|B\| = o(R)$ or the rank of $B$ is $o(N)$, then the empirical distribution of the eigenvalues of $A+B$ also follows the Semicircle Law centered at $c$ with radius $R$.*

**Proof:** It suffices to show $F(\frac{1}{R}(A+B-cI), x) \xrightarrow{p} F(x)$. First we assume $\|B\| = o(R)$. By Lemma 1, for $1 \le k \le N$, we have

$$\left| \mu_k \left( \frac{1}{R}(A+B-cI) \right) - \mu_k \left( \frac{1}{R}(A-cI) \right) \right| \le \frac{\|B\|}{R} = o(1).$$

Hence

$$F\left( \frac{1}{R}(A-cI), x - \frac{\|B\|}{R} \right) \le F\left( \frac{1}{R}(A+B-cI), x \right) \le F\left( \frac{1}{R}(A-cI), x + \frac{\|B\|}{R} \right).$$

Since $\|B\| = o(R)$, we have $F\left( \frac{1}{R}(A-cI), x - \frac{\|B\|}{R} \right) \xrightarrow{p} F(x)$ and $F\left( \frac{1}{R}(A-cI), x + \frac{\|B\|}{R} \right) \xrightarrow{p} F(x)$. By the Squeeze theorem, we have $F(\frac{1}{R}(A+B-cI), x) \xrightarrow{p} F(x)$.

Now we assume $\mathrm{rank}(B) = o(N)$. Let $U$ be the kernel of $B$ (i.e. $B|_U = 0$); $U$ has co-dimension rank(B). Let $Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$. By Cauchy's interlace theorem [23], for $1 \le j \le N - \mathrm{rank}(B)$, we have

$$\begin{aligned}
\mu_j\left(\frac{1}{R}(A-cI)\right) &\le \mu_j(Z) \le \mu_{j+\mathrm{rank}(B)}\left(\frac{1}{R}(A-cI)\right),\\
\mu_j\left(\frac{1}{R}(A+B-cI)\right) &\le \mu_j(Z) \le \mu_{j+\mathrm{rank}(B)}\left(\frac{1}{R}(A+B-cI)\right).
\end{aligned}$$

Thus, for $\mathrm{rank}(B) + 1 \le j \le N - \mathrm{rank}(B)$, we have

$$\mu_{j-\mathrm{rank}(B)}\left(\frac{1}{R}(A-cI)\right) \le \mu_j\left(\frac{1}{R}(A+B-cI)\right) \le \mu_{j+\mathrm{rank}(B)}\left(\frac{1}{R}(A-cI)\right).$$

It implies

$$F\left(\frac{1}{R}(A-cI),x\right) + \frac{\mathrm{rank}(B)}{N} \le F\left(\frac{1}{R}(A+B-cI),x\right) \le F\left(\frac{1}{R}(A-cI),x\right) + \frac{\mathrm{rank}(B)}{N}.$$

Since $\mathrm{rank}(B) = o(N)$, we have $F\left(\frac{1}{R}(A-cI),x\right) \pm \frac{\mathrm{rank}(B)}{N} \xrightarrow{p} F(x)$. By the Squeeze theorem, we have $F(\frac{1}{R}(A+B-cI),x) \xrightarrow{p} F(x)$. $\square$

**Proof of Theorem 3**: Recall

$$\mathcal{L}^{(s)}(K_n^r) - \mathcal{L}^{(s)}(H^r(n,p)) = M_1 + M_2 + M_2 + M_4.$$

We can write $\mathcal{L}^{(s)}(H^r(n,p))$ as $-M_2 + \left(1 - \frac{(-1)^s}{\binom{n}{s}}\right) I + B_1 - M_3 - M_4 - M_1$, where $B_1 = \mathcal{L}^{(s)}(K_n^r) - \left(1 - \frac{(-1)^s}{\binom{n}{s}}\right) I$.

By Theorem 5, the empirical distribution of the spectrum of $W - \mathrm{E}(W)$ follows the Semicircle Law centered at 0 with radius $(2 + o(1))\sqrt{\binom{r-s}{s}\binom{n-s}{r-s}p(1-p)}$. Since $M_2 = \frac{1}{\binom{r-s}{s}d}(W - \mathrm{E}(W))$, $\left(1 - \frac{(-1)^s}{\binom{n}{s}}\right) I - M_2$ follows the Semicircle Law centered at $c := 1 - \frac{(-1)^s}{\binom{n}{s}}$ with radius $R := (2 + o(1))\sqrt{\frac{1-p}{\binom{r-s}{s}\binom{n-s}{r-s}p}}$. Note $\frac{(-1)^s}{\binom{n}{s}} = o(R)$. We can change the center to 1.

By Theorem 1, $\mathcal{L}^{(s)}(K_n^r)$ has an eigenvalue $1 - (-1)^s \frac{\binom{n-s}{s}}{\binom{n}{s}}$ with multiplicity $\binom{n}{s} - \binom{n}{s-1}$. Thus $B_1$ has rank $\binom{n}{s-1} = o\left(\binom{n}{s}\right)$. We also observe that $M_4$ has rank at most 2, $\|M_1\| = O\left(\frac{\sqrt{(1-p)\log N}}{d}\right) = o(R)$, and $\|M_3\| = O(\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) = o(R)$. Here we use the assumption $d \gg \log n$.

By Lemma 10, the matrices $B_1$, $M_1$, $M_3$, and $M_4$ will not affect the Semicircle Law. The proof of this Lemma is finished. $\square$

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