RICCI CURVATURE OF GRAPHS

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Abstract

We modify the definition of Ricci curvature of Ollivier of Markov chains on graphs to study the properties of the Ricci curvature of general graphs, Cartesian product of graphs, random graphs, and some special class of graphs.

1 Introduction

The Ricci curvature plays a very important role on geometric analysis on Riemannian manifolds. Many results are established on manifolds with non-negative Ricci curvature or on manifolds with Ricci curvature bounded below.

The definition of the Ricci curvature on metric spaces was first from the well-known Bakry and Emery notation. Bakry and Emery[1] found a way to define the "lower Ricci curvature bound" through the heat semigroup $(P_t)_{t\geq 0}$ on a metric measure space M. There are some recent works on giving a good notion for a metric measure space to have a "lower Ricci curvature bound", see [21], [18] and [19]. Those notations of Ricci curvature work on so called length spaces. In 2009, Ollivier [20] gave a notion of coarse Ricci curvature of Markov chains valid on arbitrary metric spaces, such as graphs.

Graphs and manifolds are quite different in their nature. But they do share some similar properties through Laplace operators, heat kernels, and random walks, etc. Many pioneering works were done by Chung, Yau, and their coauthors [3, 4, 5, 8, 9, 10, 12, 13, 14, 16, 15].

A graph $G = (V, E)$ is a pair of the vertex-set V and the edge-set E. Each edge is an unordered pair of two vertices. Unless otherwise specified, we always assume a graph G is

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simple (no loops and no multi-edges) and connected. It may have infinite but countable number of vertices. For each vertex v, the degree d_v is always bounded. Starting from a vertex v_1 we select a vertex v_2 in the neighborhood of v_1 at random and move to v_2 then we select a vertex v_3 in the neighborhood of v_2 at random and move to v_3, etc. The random sequence of vertices selected this way is a random walk on the graph. Ollivier [20]'s definition of the coarse Ricci curvature of Markov chains on metric space can be naturally defined over such graphs.

The first definition of Ricci curvature on graphs was introduced by Fan Chung and Yau in 1996 [8]. In the course of obtaining a good log-Sobolev inequality, they found the following definition of Ricci curvature to be useful:

We say that a regular graph G has a local k-frame at a vertex x if there exist injective mappings $\eta_1, \ldots, \eta_k$ from a neighborhood of x into V so that

(1) x is adjacent to $\eta_i x$ for $1 \le i \le k$;

(2) $\eta_i\, x \neq \eta_j\, x \quad \text{if } i \neq j$.

The graph G is said to be Ricci-flat at x if there is a local k-frame in a neighborhood of x so that for all i ,

$$\bigcup_j (\eta_i \eta_j)\, x = \bigcup_j (\eta_j \eta_i)\, x \ .$$

For a more general definition of Ricci curvature, in [17], Lin and Yau give a generalization of lower Ricci curvature bound in the framework of graphs. We first define the Laplace operator on graphs without loops and multiple edges. The description in the following can be used for weighted graphs. But for simplicity, we set all weights here equal to 1.

Let $V^{\mathbb{R}} = \{f; f : V \to \mathbb{R}\}$. The Laplace operator Δ of a graph G is

$$\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)]$$

for all $f \in V^{\mathbb{R}}$. For graphs, we have

$$|\nabla f(x)|^2 = \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)]^2 \ .$$

We first introduce a bilinear operator $\Gamma : V^{\mathbb{R}} \times V^{\mathbb{R}} \to V^{\mathbb{R}}$ by

$$\Gamma(f,g)(x) = \frac{1}{2}\{\Delta(f(x)\cdot g(x)) - f(x)\Delta g(x) - g(x)\Delta f(x)\} \ .$$

The Ricci curvature operator Γ_2 is defined by iterating the Γ:

$$\Gamma_2(f,g)(x) = \frac{1}{2}\{\Delta\Gamma(f,g)(x) - \Gamma(f,\Delta g)(x) - \Gamma(g,\Delta f)(x)\} \ .$$

The Laplace operator Δ on graphs satisfies the curvature-dimension type inequality $CD(m, K)$ ($m \in (1, +\infty]$)(the notation is from Bakry and Emery[1]) if

$$\Gamma_2(f, f)(x) \geq \frac{1}{m}(\Delta(f(x)))^2 + k(x) \cdot \Gamma(f, f)(x)$$

We call m the dimension of the operator Δ and $k(x)$ the lower bound of the Ricci curvature of the operator Δ.

In the paper [17], Lin and Yau proved that the Ricci curvature for a locally finite graph in the sense of Bakry and Emery is bounded below. The Ricci flat graph in the sense of Fan Chung and Yau was proved to be a graph with Ricci curvature bounded below by zero. In the same paper, Lin and Yau also showed that the Ricci curvature in the sence of Ollivier for simple random walk on graphs is bounded below. For non-negative Ricci curvature graphs, Fan Chung, Lin and Yau can prove some Harnack inequalities and Log-Harnack inequalities (see [7]).

In this paper, we will modify Ollivier's definition of Ricci curvature for Markov chains on graphs.

The definition of Ricci curvatures of graphs is given at Section 2. We proved a theorem on the Ricci curvatures of the Cartesian product graphs. For graphs with positive curvatures, we established the upper bounds for diameters and the number of vertices. We also proved a lower bound on the first nonzero Laplacian eigenvalue. Ricci curvatures of random graphs $G(n, p)$ are considered in the last section. Here $G(n, p)$ denotes Erdős-Renyi's random graph on n vertices and with probability $p \in [0, 1]$. (For each unordered pair of vertices $\{x, y\}$, xy is an edge (or non-edge) of $G(n, p)$ with probability p (or with probability $1 - p$ respectively) independently to other pairs.) We proved that the Ricci curvature of $G(n, p)$ is $(1 + o(1))p$ if $p \gg \sqrt[3]{(\ln n)/n}$. It is almost about 0, -1, and -2, when $\sqrt[3]{(\ln n)/n} > p \geq 2\sqrt{(\ln n)/n}$, $n^{-1/2} \gg p \gg \sqrt[3]{(\ln n)/n^2}$, or $n^{-2/3} \gg p \gg (\ln n)/n$, respectively.

2 Notations

We will use similar notations as in [20]. A probability distribution (over the vertex-set $V(G)$) is a mapping $m\colon V \to [0, 1]$ satisfying $\sum_{x \in V} m(x) = 1$. Suppose two probability distributions m_1 and m_2 have finite support. A coupling between m_1 and m_2 is a mapping $A\colon V \times V \to [0, 1]$ with finite support so that

$$\sum_{y \in V} A(x, y) = m_1(x) \text{ and } \sum_{x \in V} A(x, y) = m_2(y).$$

Let $d(x, y)$ be the graph distance between two vertices x and y. The transportation distance between two probability distributions m_1 and m_2 is defined as follows.

$$W(m_1, m_2) = \inf_A \sum_{x,y \in V} A(x, y) d(x, y), \tag{1}$$

where the infimum is taken over all coupling A between m_1 and m_2. A function f over G is c-Lipschitz if

$$|f(x) - f(y)| \leq c d(x, y)$$

for all $x, y \in V$. By the duality theorem of a linear optimization problem, the transportation distance can also be written as follows.

$$W(m_1, m_2) = \sup_f \sum_{x \in V} f(x)[m_1(x) - m_2(x)], \tag{2}$$

where the supremum is taken over all 1-Lipschitz function f.

Remark: Any c-Lipschitz function f over a metric subspace can be extended to a c-Lipschitz function over the whole metric space. The $W(m_1, m_2)$ only depends on distances among vertices in $\mathrm{supp}(m_1) \cup \mathrm{supp}(m_2)$.

For any vertex x, let $\Gamma(x)$ denote the set of neighborhood of x, i.e.,

$$\Gamma(x) = \{v; vx \in E(G)\}.$$

Let $N(x) = \Gamma(x) \cup \{x\}$.

For any $\alpha \in [0, 1]$ and any vertex x, the probability measure m_x^α is defined as

$$m_x^\alpha(v) = \begin{cases} \alpha & \text{if } v = x, \\ (1-\alpha)/d_x & \text{if } v \in \Gamma(x), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For any $x, y \in V$, we define α-Ricci-curvature κ_α to be

$$\kappa_\alpha(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)}. \tag{4}$$

Note that $W(m_x^1, m_y^1) = d(x, y)$, so $\kappa_1(x, y) = 0$ holds for any $x, y \in V(G)$.

LEMMA 2.1. *For two vertices x, y, κ_α is concave in $\alpha \in [0, 1]$.*

Proof: For $0 \leq \alpha < \beta < \gamma \leq 1$, let $\lambda = (\gamma - \beta)/(\gamma - \alpha)$. Then $\beta = \lambda\alpha + (1 - \lambda)\gamma$. The concavity of κ_α means

$$\kappa_\beta \geq \lambda \kappa_\alpha + (1 - \lambda)\kappa_\gamma.$$

Let A be the coupling between m_x^α and m_y^α achieving the infimum in the definition of $W(m_x^\alpha, m_y^\alpha)$. Let B be the coupling between m_x^γ and m_y^γ achieving the infimum in the definition of $W(m_x^\gamma, m_y^\gamma)$. We have

$$\begin{aligned} W(m_x^\alpha, m_y^\alpha) &= \sum_{u,v\in V} A(u,v)d(u,v), \\ W(m_x^\gamma, m_y^\gamma) &= \sum_{u,v\in V} B(u,v)d(u,v). \end{aligned}$$

Let $C = \lambda A + (1-\lambda)B$. Here we verify that C is a coupling between m_x^β and m_y^β. We have

$$\begin{aligned} \sum_{u\in V} C(u,v) &= \sum_{u\in V} \lambda A(u,v) + (1-\lambda)B(u,v) \\ &= \lambda m_y^\alpha(v) + (1-\lambda) m_y^\gamma(v) \\ &= m_y^\beta(v). \end{aligned}$$

The last equality is verified case by case. If $v = y$, we have

$$\begin{aligned} \lambda m_y^\alpha(v) + (1-\lambda)m_y^\gamma(v) &= \lambda\alpha + (1-\lambda)\gamma \\ &= \beta \\ &= m_y^\beta(v). \end{aligned}$$

If $v \in \Gamma(y)$, then

$$\begin{aligned} \lambda m_y^\alpha(v) + (1-\lambda)m_y^\gamma(v) &= \lambda\frac{1-\alpha}{d_y} + (1-\lambda)\frac{1-\gamma}{d_y} \\ &= \frac{1-\beta}{d_y} \\ &= m_y^\beta(v). \end{aligned}$$

If $v \neq y$ and v is not in the neighborhood of y, then the inequality holds trivially since $m_y^\alpha(v) = m_y^\beta(v) = m_y^\gamma(v) = 0$.

Similarly, we can show

$$\sum_{v\in V} C(u,v) = m_x^\beta(u).$$

Thus C is a coupling between m_x^β and m_y^β. We have

$$\begin{aligned} W(m_x^\beta, m_y^\beta) &\leq \sum_{u,v\in V} C(u,v)d(u,v) \\ &= \lambda \sum_{u,v\in V} A(u,v)d(u,v) + (1-\lambda)\sum_{u,v\in V} B(u,v)d(u,v) \\ &= \lambda W(m_x^\alpha, m_y^\alpha) + (1-\lambda) W(m_x^\gamma, m_y^\gamma). \end{aligned}$$

Therefore, we have

$$\begin{aligned}\kappa_\beta(x,y) &= 1-\frac{W(m_x^\beta, m_y^\beta)}{d(x,y)}\\ &\geq \lambda\left[1-\frac{W(m_x^\alpha, m_y^\alpha)}{d(x,y)}\right]+(1-\lambda)\left[1-\frac{W(m_x^\gamma, m_y^\gamma)}{d(x,y)}\right]\\ &= \lambda\kappa_\alpha(x,y)+(1-\lambda)\kappa_\gamma(x,y).\end{aligned}$$

□

LEMMA 2.2. *For any* $\alpha \in [0,1]$ *and any two vertices* x *and* y*, we have*

$$\kappa_\alpha(x,y) \leq (1-\alpha)\frac{2}{d(x,y)}.$$

Proof: Define $\delta_x(v)=1$ if $v=x$ and 0 otherwise. We have

$$\begin{aligned}W(m_x^\alpha, m_y^\alpha) &\geq W(\delta_x,\delta_y)-W(\delta_x,m_x^\alpha)-W(\delta_y,m_y^\alpha)\\ &= d(x,y)-2(1-\alpha).\end{aligned}$$

Thus

$$\begin{aligned}\kappa_\alpha(x,y) &= 1-\frac{W(m_x^\alpha, m_y^\alpha)}{d(x,y)}\\ &\leq (1-\alpha)\frac{2}{d(x,y)}.\end{aligned}$$

□

Lemma 2.1 implies the function $h(\alpha)=\kappa_\alpha(x,y)/(1-\alpha)$ is an increasing function on α over $[0,1)$. Lemma 2.2 says $h(\alpha)$ is bounded. Thus, the limit $\lim_{\alpha\to 1}\kappa_\alpha(x,y)/(1-\alpha)$ exists. This limit, denoted by $\kappa(x,y)$, is called the Ricci curvature at (x,y) in the graph G.

Remark: This definition of Ricci curvature κ is slightly different from those in [20], where Ollivier considered κ_0 and $\kappa_{1/2}$ instead.

Although the Ricci curvature $\kappa(x,y)$ is defined for all pairs (x,y), it makes more sense to consider only $\kappa(x,y)$ for $xy\in E(G)$. The following lemma is similar to [20, Proposition 19].

LEMMA 2.3. *If* $\kappa(x,y)\geq\kappa_0$ *for any edge* $xy\in E(G)$*, then* $\kappa(x,y)\geq\kappa_0$ *for any pair of vertices* (x,y).

Proof: Suppose $d(x,y)=s$ and x and y are connected by a path P of length s. Denote the vertices of P by $x=x_0,x_1,\ldots,x_{s-1},x_s=y$ so that x_{i-1} and x_i are adjacent for $1\leq i\leq s$.

For $\alpha \in [0,1)$, we have

$$\begin{aligned} \frac{\kappa_\alpha(x,y)}{1-\alpha} &= \frac{1}{1-\alpha}\left[1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x,y)}\right] \\ &\geq \frac{1}{1-\alpha}\left[1 - \frac{\sum_{i=1}^s W(m_{x_{i-1}}^\alpha, m_{x_i}^\alpha)}{s}\right] \\ &= \frac{\sum_{i=1}^s \kappa_\alpha(x_{i-1}, x_i)}{s(1-\alpha)}. \end{aligned}$$

Take the limit of both hand sides as $\alpha \to 1$. We get

$$\kappa(x,y) \geq \frac{\sum_{i=1}^s \kappa(x_{i-1}, x_i)}{s} \geq \kappa_0.$$

□

We say G has a constant Ricci curvature r if for any edge xy of G, we have $\kappa(x,y) = r$. We write $\kappa(G) = r$, for short.

Example 1: The complete graph K_n has a constant Ricci curvature $n/(n-1)$. This is the only graph with a constant Ricci curvature greater than 1.

Example 2: The cycle C_n for $n \geq 6$ has a constant Ricci curvature 0. For small cycles C_3, C_4, and C_5, we have

$$\begin{aligned} \kappa(C_3) &= \frac{3}{2}, \\ \kappa(C_4) &= 1, \\ \kappa(C_5) &= \frac{1}{2}. \end{aligned}$$

Example 3: The hypercube Q^n has a constant Ricci curvature $2/n$. Moreover, we can show for any edge xy,

$$\kappa_\alpha(x,y) = \begin{cases} 2\alpha & \text{if } 0 \leq \alpha \leq 1/(n+1), \\ 2(1-\alpha)/n & \text{if } 1/(n+1) \leq \alpha \leq 1. \end{cases}$$

3 Ricci curvatures of Cartesian product graphs

Given two graphs G and H, the Cartesian product (denoted by $G \square H$) is a graph over the vertex set $V(G) \times V(H)$, where two pairs (u_1, v_1) and (u_2, v_2) are connected if "$u_1 = u_2$ and $v_1 v_2 \in E(H)$" or "$u_1 u_2 \in E(G)$ and $v_1 = v_2$". If both G and H are regular graphs, then the Ricci Curvature of $G \square H$ can be computed by the following theorem.

Theorem 3.1. *Suppose G is d_G-regular and H is d_H-regular. Then the Ricci curvature of $G\square H$ is given by*

$$\kappa^{G\square H}((u_1,v),(u_2,v)) = \frac{d_G}{d_G+d_H}\kappa^G(u_1,u_2) \tag{5}$$

$$\kappa^{G\square H}((u,v_1),(u,v_2)) = \frac{d_H}{d_G+d_H}\kappa^H(v_1,v_2). \tag{6}$$

Here $u \in V(G)$, $v \in V(H)$, $u_1u_2 \in E(G)$, and $v_1v_2 \in E(H)$.

Remark: Similar relation does not hold if we replace κ by κ_α. Only one directional inequality can hold for κ_α (see inequality (7)). This is one of reasons why we define the Ricci curvature κ as $\lim_{\alpha\to 1}\kappa(x,y)/(1-\alpha)$.

Cartesian product is an effective way to construct graphs with positive constant Ricci curvature. The following corollary can be derived from Theorem 3.1 using induction on n. We omit its proof here.

Corollary 3.2. *Suppose G is regular and has constant curvature κ. Let G^n denote the n-th power of the Cartesian product of G. Then G has constant curvature κ/n.*

Note $Q^1 = K_2$ and $\kappa(K_2) = 2$. By the corollary, we have

$$\kappa(Q^n) = \kappa((K_2)^n) = \frac{2}{n}.$$

Proof of Theorem 3.1: By symmetry, it is sufficient to prove equality (5). We claim the following two inequalities on $\kappa_\alpha^{G\square H}$.

Claim 1: We claim

$$\kappa_\alpha^{G\square H}((u_1,v),(u_2,v)) \geq \frac{d_G}{d_G+d_H}\kappa_\alpha^G(u_1,u_2) \tag{7}$$

for $0 \leq \alpha \leq 1$ and $u_1u_2 \in E(G)$.

Claim 2: We claim

$$\kappa_\alpha^{G\square H}((u_1,v),(u_2,v)) \leq \frac{d_G+\alpha d_H}{d_G+d_H}\kappa_{\alpha'}^G(u_1,u_2) \tag{8}$$

for any $u_1u_2 \in E(G)$. Here $\alpha' = \alpha(d_G+d_H)/(d_G+\alpha d_H)$.

Claim 1 is corresponding to Ollivier's result for the case $\alpha = 0$ or $1/2$ (Proposition 27 of [20]), where he uses the word "L1-tensorization" instead of "Cartesian product". Claim 2 is new.

From Claim 1, divide both hand sides of inequality (7) by $1-\alpha$ and then take the limit as $\alpha \to 1$. We get

$$\kappa^{G\square H}((u_1,v),(u_2,v)) \geq \frac{d_G}{d_G+d_H}\kappa^G(u_1,u_2).$$

From the definition of α', we have

$$\frac{1-\alpha'}{1-\alpha} = \frac{d_G}{d_G + \alpha d_H}.$$

Thus

$$\begin{aligned}
\kappa^{G\Box H}((u_1,v),(u_2,v)) &= \lim_{\alpha\to 1} \frac{\kappa_\alpha^{G\Box H}((u_1,v),(u_2,v))}{1-\alpha} \\
&\leq \lim_{\alpha'\to 1} \frac{d_G}{d_G+d_H}\frac{\kappa^G_{\alpha'}(u_1,u_2)}{1-\alpha'} \\
&= \frac{d_G}{d_G+d_H}\kappa^G(u_1,u_2).
\end{aligned}$$

It suffices to prove two claims. It is not pleasant to read with all superscript $G, H, G\Box H$ on every parameters. We use the following conventions. We use letters x, x_1, x_2, u, u_1, u_2 to denote a vertex of G and use letters y, y_2, y_2, v, v_1, v_2 to denote a vertex of H. We use the pairs such as (x, y), (u_1, v) to denote a vertex of $G\Box H$. For example, both $m^\alpha_{u_1}$ and $m^\alpha_{u_1,v}$ describe the probability distribution of α-lazy random walks, but the first one is on the graph G while the second one is on the graph $G\Box H$.

Proof of Claim 1: Let A be a coupling between $m^\alpha_{u_1}$ and $m^\alpha_{u_2}$ which reaches the infimum in the definition of $W_G(m^\alpha_{u_1}, m^\alpha_{u_2})$.

We have

$$W_G(m^\alpha_{u_1}, m^\alpha_{u_2}) = \sum_{x_1,x_2\in V(G)} A(x_1,x_2)d(x_1,x_2).$$

We define a coupling $B\colon V(G\Box H)\times V(G\Box H)\to [0,1]$ as follows.

$$\begin{aligned}
&B((x_1,y_1),(x_2,y_2)) \\
&\quad = \begin{cases}
d_G/(d_G+d_H)\cdot A(u_1,u_2) + d_H/(d_G+d_H)\cdot \alpha & \text{if } x_1=u_1, x_2=u_2, y_1=y_2=v, \\
d_G/(d_G+d_H)\cdot A(x_1,x_2) & \text{if } y_1=y_2=v; (x_1,x_2)\neq(u_1,u_2), \\
d_G/(d_G+d_H)\cdot m^\alpha_v(y_1) & \text{if } x_1=u_1, x_2=u_2, \\
 & \quad y_1=y_2=y\in\Gamma_H(v), \\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

Now we verify that B is a coupling between $m^\alpha_{(u_1,v)}$ and $m^\alpha_{(u_2,v)}$.

$$\begin{aligned}
&\sum_{(x_2,y_2)\in V(G\Box H)} B((x_1,y_1),(x_2,y_2)) \\
&\quad = \sum_{x_2\in V(G)} B((x_1,v),(x_2,v))\delta_v(y_1) \\
&\qquad + \sum_{y_2\in \Gamma_H(v)} B((u_1,y_1),(u_2,y_2))\delta_{u_1}(x_1)(1-\delta_v(y_1)) \\
&\quad = \sum_{x_2\in V(G)} \frac{d_G}{d_G+d_H}A(x_1,x_2)\delta_v(y_1) + \frac{d_H}{d_G+d_H}\alpha\delta_{u_1}(x_1)\delta_v(y_1) \\
&\qquad + \frac{d_H}{d_G+d_H} m_v^\alpha(y_1)\delta_{u_1}(x_1)(1-\delta_v(y_1)) \\
&\quad = \frac{d_G}{d_G+d_H} m_{u_1}^\alpha(x_1)\delta_v(y_1) + \frac{d_H}{d_G+d_H} m_v^\alpha(y_1)\delta_{u_1}(x_1) \\
&\quad = m_{(u_1,v)}^\alpha(x_1,y_1).
\end{aligned}$$

Similarly, we have

$$\sum_{(x_1,y_1)\in V(G\Box H)} B((x_1,y_1),(x_2,y_2)) = m_{(u_2,v)}^\alpha(x_2,y_2).$$

Thus, we have

$$\begin{aligned}
W(m_{(u_1,v)}^\alpha, m_{(u_2,v)}^\alpha) &\leq \sum_{(x_1,y_1),(x_2,y_2)} B((x_1,y_1),(x_2,y_2))d((x_1,y_1),(x_2,y_2)) \\
&= \sum_{x_1,x_2\in V(G)} B((x_1,v),(x_2,v))d((x_1,v),(x_2,v)) \\
&\quad + \sum_{y_1\in\Gamma_H(v)} B((x_1,y_1),(x_2,y_1))d((x_1,y_1),(x_2,y_1)) \\
&= \sum_{x_1,x_2\in V(G)} \frac{d_G}{d_G+d_H}A(x_1,x_2)d(x_1,x_2) + \alpha\frac{d_H}{d_G+d_H} \\
&\quad + \sum_{y_1\in\Gamma_H(v)} \frac{d_H}{d_G+d_H} m_v^\alpha(y_1) \\
&= \frac{d_G}{d_G+d_H} W(m_{u_1}^\alpha, m_{u_2}^\alpha) + \frac{d_H}{d_G+d_H}.
\end{aligned}$$

We get

$$\begin{aligned}
\kappa_\alpha^{G\Box H}((u_1,v),(u_2,v)) &= 1 - W(m_{(u_1,v)}^\alpha, m_{(u_2,v)}^\alpha) \\
&\geq 1 - \frac{d_G}{d_G+d_H}W(m_{u_1}^\alpha, m_{u_2}^\alpha) - \frac{d_H}{d_G+d_H} \\
&= \frac{d_G}{d_G+d_H}(1 - W(m_{u_1}^\alpha, m_{u_2}^\alpha)) \\
&= \frac{d_G}{d_G+d_H}\kappa_\alpha^G(u_1,u_2).
\end{aligned}$$

Proof of Claim 2: Let f be a 1-Lipschitz function which achieves the supremum in the duality theorem of $W(m_{u_1}^{\alpha'}, m_{u_2}^{\alpha'})$, i.e.,

$$W(m_{u_1}^{\alpha'}, m_{u_2}^{\alpha'}) = \sum_{x \in N(u_1)} f(x) m_{u_1}^{\alpha'}(x) - \sum_{y \in N(u_2)} f(y) m_{u_2}^{\alpha'}(y).$$

We define a function $F \colon N((u_1, v)) \cup N((u_2, v)) \to \mathbb{R}$ as

$$F(x,y) = \begin{cases} f(x) & \text{if } y = v, \\ (f(u_1) + f(u_2) + 1)/2 & \text{if } x = u_1, y \neq v, \\ (f(u_1) + f(u_2) - 1)/2 & \text{if } x = u_2, y \neq v. \end{cases}$$

It is easy to check F is an 1-Lipschitz function over $N((u_1, v)) \cup N((u_2, v))$ so that F can be extended to an 1-Lipschitz function over $V(G \Box H)$. Thus, we have

$$\begin{aligned} W(m_{(u_1,v)}^{\alpha}, m_{(u_2,v)}^{\alpha}) &\geq \sum_{(x,y) \in N((u_1,v))} F(x,y) m_{(u_1,v)}^{\alpha}(x,y) \\ &\quad - \sum_{(x,y) \in N((u_2,v))} F(x,y) m_{(u_2,v)}^{\alpha}(x,y) \\ &= \sum_{x \in N(u_1)} f(x) m_{(u_1,v)}^{\alpha}(x,v) - \sum_{x \in N(u_2)} f(x) m_{(u_2,v)}^{\alpha}(x,v) \\ &\quad + \frac{f(u_1) + f(u_2) + 1}{2}(1-\alpha)\frac{d_H}{d_G + d_H} \\ &\quad - \frac{f(u_1) + f(u_2) - 1}{2}(1-\alpha)\frac{d_H}{d_G + d_H} \\ &= \frac{d_G + \alpha d_H}{d_G + d_H}\left[\sum_{x \in N(u_1)} f(x) m_{u_1}^{\alpha'}(x) - \sum_{y \in N(u_2)} f(y) m_{u_2}^{\alpha'}(y)\right] \\ &\quad + (1-\alpha)\frac{d_H}{d_G + d_H} \\ &= \frac{d_G + \alpha d_H}{d_G + d_H} W(m_{u_1}^{\alpha'}, m_{u_2}^{\alpha'}) + (1-\alpha)\frac{d_H}{d_G + d_H}. \end{aligned}$$

We get

$$\begin{aligned} \kappa_{\alpha}^{G \Box H}((u_1, v), (u_2, v)) &= 1 - W(m_{(u_1,v)}^{\alpha}, m_{(u_2,v)}^{\alpha}) \\ &\leq 1 - \frac{d_G + \alpha d_H}{d_G + d_H} W(m_{u_1}^{\alpha'}, m_{u_2}^{\alpha'}) - (1-\alpha)\frac{d_H}{d_G + d_H} \\ &= \frac{d_G + \alpha d_H}{d_G + d_H}(1 - W(m_{u_1}^{\alpha'}, m_{u_2}^{\alpha'})) \\ &= \frac{d_G + \alpha d_H}{d_G + d_H} \kappa_{\alpha'}^{G}(u_1, u_2). \end{aligned}$$

The proof of Theorem 3.1 is finished. □

4 Graphs with positive Ricci curvatures

Here is a Bonnet-Myers type theorem on graphs, which is corresponding to Ollivier's [20, Proposition 23].

THEOREM 4.1. *For any $x, y \in V(G)$, if $\kappa(x,y) > 0$, then*

$$d(x,y) \leq \left\lfloor \frac{2}{\kappa(x,y)} \right\rfloor.$$

Moreover, if for any edge xy, $\kappa(x,y) \geq \kappa > 0$, then the diameter of graph G is bounded as follows:

$$diam(G) \leq \frac{2}{\kappa}.$$

Proof: By Lemma 2.2, we have

$$\frac{\kappa_\alpha(x,y)}{1-\alpha} \leq \frac{2}{d(x,y)}.$$

Take the limit as $\alpha \to 1$. We have

$$\kappa(x,y) \leq \frac{2}{d(x,y)}.$$

Since $\kappa(x,y) > 0$, we have

$$d(x,y) \leq \frac{2}{\kappa(x,y)}.$$

If for any edge xy, $\kappa(x,y) \geq \kappa$. By Lemma 2.3, for any $x, y \in V(G)$, we have $\kappa(x,y) \geq \kappa$. Thus, the diameter of G is at most $2/\kappa$. □

Now we assume G is a finite graph on n vertices. Let A be the adjacent matrix of the graph G and $D = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of degrees. The normalized Laplacian is the matrix $L = I - D^{-1/2}AD^{-1/2}$. The eigenvalues of L are called the Laplacian Eigenvalues of G, which is listed as

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}.$$

The following theorem is slightly stronger than Ollivier's [20, Proposition 30].

THEOREM 4.2. *Suppose G is a finite graph and λ_1 is the first nonzero Laplacian eigenvalue of G. If for any edge xy, $\kappa(x,y) \geq \kappa > 0$, then $\lambda_1 \geq \kappa$.*

Proof: Since G is finite, $\lim_{\alpha\to 1} \kappa_\alpha(x,y)/(1-\alpha)$ converges uniformly for all $x, y \in V(G)$. For any $\epsilon > 0$ small enough, there exists an $\alpha_0 \in [0,1)$ such that for any $\alpha \in (\alpha_0, 1)$ and for any $x, y \in V(G)$, we have

$$\frac{\kappa_\alpha(x,y)}{1-\alpha} > (1-\epsilon)\kappa > 0.$$

Let M_α be the average operator associated to the α-lazy random walk, i.e., for any function $f\colon V(G) \to \mathbb{R}$, $M_\alpha(f)$ is a function defined as follows:

$$M_\alpha(f)(x) = \sum_{z \in V(G)} f(z) m_x^\alpha(z).$$

If f is k-Lipschitz, then we have

$$\begin{aligned} |M_\alpha(f)(x) - M_\alpha(f)(y)| &= |\sum_{z \in V(G)} f(z)(m_x^\alpha(z) - m_y^\alpha(z))| \\ &\leq kW(m_x^\alpha, m_y^\alpha) \\ &\leq k(1-(1-\epsilon)(1-\alpha)\kappa)d(x,y). \end{aligned}$$

$M_\alpha(f)$ is a $k(1-(1-\epsilon)(1-\alpha)\kappa)$-Lipschitz function. The mixing rate of M_α is at most $1-(1-\epsilon)(1-\alpha)\kappa$. On the other hand, M_α can be written as an $n \times n$-matrix

$$M_\alpha = \alpha I + (1-\alpha) D^{-1} A.$$

It has eigenvalues $1, 1-(1-\alpha)\lambda_1, 1-(1-\alpha)\lambda_2, \ldots, 1-(1-\alpha)\lambda_{n-1}$. As $\alpha \to 1$, the mixing rate of M_α is exactly $1-(1-\alpha)\lambda_1$. We have

$$1-(1-\alpha)\lambda_1 \leq 1-(1-\epsilon)(1-\alpha)\kappa.$$

Or

$$\lambda_1 \geq (1-\epsilon)\kappa.$$

Let $\epsilon \to 0$. We get

$$\lambda_1 \geq \kappa.$$

□

Remark: Theorems 4.1 and 4.2 are tight for K_n, Q^n, and C_3.

Given the maximum degree Δ and the diameter D, the number of vertices of a graph G can not exceed the Moore bound:

(9) $$n \leq 1 + \sum_{k=1}^{D} \Delta(\Delta-1)^{k-1}.$$

The following theorem bounds the number of vertices in a graph with positive Ricci curvature. It is much smaller than the Moore bound.

Theorem 4.3. *Suppose that for any $xy \in E(G)$, the Ricci curvature $\kappa(x,y) \geq \kappa > 0$. Let Δ be the maximum degree of G. Then the number of vertices is at most*

$$n \leq 1 + \sum_{k=1}^{\lfloor 2/\kappa \rfloor} \Delta^k \prod_{i=1}^{k-1}(1 - i\frac{\kappa}{2}).$$

For any two distinct x, y, the neighborhood of y can be partitioned into three sets according to their distance to x. Namely

$$\Gamma_x^+(y) = \{v; v \in \Gamma(y), d(x,v) = d(x,y)+1\}; \tag{10}$$

$$\Gamma_x^0(y) = \{v; v \in \Gamma(y), d(x,v) = d(x,y)\}; \tag{11}$$

$$\Gamma_x^-(y) = \{v; v \in \Gamma(y), d(x,v) = d(x,y)-1\}; \tag{12}$$

The following lemma improves Lemma 2.2.

Lemma 4.4. *For any two distinct vertices x and y, we have*

$$\kappa(x,y) \le \frac{1 + (|\Gamma_x^-(y)| - |\Gamma_x^+(y)|)/d_y}{d(x,y)}.$$

Proof: For any $0 \le \alpha < 1$, the function $f(z) = d(x,z)$ is clearly a 1-Lipschitz function. Thus,

$$\begin{aligned}
W(m_x^\alpha, m_y^\alpha) &\ge \sum_{z \in V(G)} d(x,z)(m_y^\alpha(z) - m_x^\alpha(z)) \\
&= \alpha d(x,y) + \frac{1-\alpha}{d_y}|\Gamma_x^-(y)|(d(x,y)-1) + \frac{1-\alpha}{d_y}|\Gamma_x^0(y)|d(x,y) \\
&\quad + \frac{1-\alpha}{d_y}|\Gamma_x^+(y)|(d(x,y)+1) - (1-\alpha). \\
&= d(x,y) - \frac{(1-\alpha)(|\Gamma_x^-(y)| - |\Gamma_x^+(y)|)}{d_y} - (1-\alpha).
\end{aligned}$$

We have

$$\begin{aligned}
\kappa(x,y) &= \lim_{\alpha \to 1} \frac{\kappa_\alpha(x,y)}{1-\alpha} \\
&= \lim_{\alpha \to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)/d(x,y)}{1-\alpha} \\
&\le \frac{1}{d(x,y)}\left[1 + \frac{(|\Gamma_x^-(y)| - |\Gamma_x^+(y)|)}{d_y}\right].
\end{aligned}$$

The proof of this Lemma is finished. □

Proof of Theorem 4.3: Theorem 4.1 states that the diameter is at most $\lfloor 2/\kappa \rfloor$. Pick any vertex x, for $1 \le i \le \lfloor 2/\kappa \rfloor$, and let $\Gamma_i(x) = \{y; d(x,y) = i\}$. For any $y \in \Gamma_i(x)$, by Lemma 4.4, we have

$$\begin{aligned}
2 - i\kappa &\ge 2 - d(x,y)\kappa(x,y) \\
&\ge 1 - \frac{|\Gamma_x^-(y)| - |\Gamma_x^+(y)|}{d_y} \\
&\ge \frac{2|\Gamma_x^+(y)|}{d_y}.
\end{aligned}$$

Thus,

$$|\Gamma_x^+(y)| \leq (1 - \frac{i\kappa}{2})d_y \leq (1 - \frac{i\kappa}{2})\Delta.$$

We have

$$\begin{aligned} |\Gamma_{i+1}(x)| &\leq \sum_{y\in\Gamma_i(x)} |\Gamma_x^+(y)| \\ &\leq \sum_{y\in\Gamma_i(x)} (1 - \frac{i\kappa}{2})\Delta \\ &= |\Gamma_i(x)|(1 - \frac{i\kappa}{2})\Delta. \end{aligned}$$

By induction on k, we have

$$|\Gamma_k(x)| \leq \Delta^k \prod_{i=1}^{k-1}(1 - \frac{i\kappa}{2}).$$

We have

$$\begin{aligned} n &= 1 + \sum_{k=1}^{\lfloor 2/\kappa \rfloor} |\Gamma_k(x)| \\ &\leq 1 + \sum_{k=1}^{\lfloor 2/\kappa \rfloor} \Delta^k \prod_{i=1}^{k-1}(1 - \frac{i\kappa}{2}). \end{aligned}$$

□

5 Ricci curvature of random graphs

In this section we will examine the Ricci Curvature of the classical Erdős-Renyi random graphs $G(n,p)$. Here $G(n,p)$ is a random graph on n vertices in which a pair of vertices appear as an edge of $G(n,p)$ with probability p independently. We say a graph property P is *almost surely* satisfied if the limit of the probability that P holds goes to 1 as n goes to infinity. We say $f(n) \gg g(n)$ if $\lim_{n\to\infty} g(n)/f(n) = 0$.

THEOREM 5.1. *Suppose that xy is an edge of the random graph $G(n,p)$. The following statements hold for the curvature $\kappa(x,y)$.*

1. *If $p \geq \sqrt[3]{(\ln n)/n}$, almost surely, we have*

$$\kappa(x,y) = p + O\left(\sqrt{\frac{\ln n}{np}}\right).$$

In particular, if $p \gg \sqrt[3]{(\ln n)/n}$, almost surely, we have $\kappa(x,y) = (1+o(1))p$.

2. If $\sqrt[3]{(\ln n)/n} > p \geq 2\sqrt{(\ln n)/n}$*, almost surely, we have*

$$\kappa(x,y) = O\left(\frac{\ln n}{np^2}\right).$$

3. If $1/\sqrt{n} \gg p \gg \sqrt[3]{(\ln n)/n^2}$*, almost surely, we have*

$$\kappa(x,y) = -1 + O(np^2) + O(\frac{\ln n}{n^2p^3}).$$

4. If $\sqrt[3]{1/n^2} \gg p \gg \frac{\ln n}{n}$*, almost surely, we have*

$$\kappa(x,y) = -2 + O(n^2p^3) + O\left(\sqrt{\frac{\ln n}{np}}\right).$$

5.1 Lemmas

We will use the following Chernoff's inequality.

Lemma 5.2. [2] *Let* $X_1, \ldots, X_n$ *be independent random variables with*

$$\Pr(X_i = 1) = p_i, \qquad \Pr(X_i = 0) = 1 - p_i.$$

We consider the sum $X = \sum_{i=1}^n X_i$*, with expectation* $\mathrm{E}(X) = \sum_{i=1}^n p_i$*. Then we have*

$$\begin{array}{lrcl} (\textit{Lower tail}) & \Pr(X \leq E(X) - \lambda) & \leq & e^{-\lambda^2/2\mathrm{E}(X)}, \\ (\textit{Upper tail}) & \Pr(X \geq E(X) + \lambda) & \leq & e^{-\lambda^2/(2\mathrm{E}(X)+2\lambda/3)}. \end{array}$$

Before we prove our theorem, we need a few lemmas.

Lemma 5.3. *If* $p \geq (8\ln n)/(3n)$*, then with probability at least* $1 - 2/n$*, all degrees of* $G(n,p)$ *fall in the range* $((n-1)p - \sqrt{4np\ln n}, (n-1)p + \sqrt{6np\ln n})$*.*

Proof: For each vertex v, the degree d_v is the sum of $n-1$ independent random variables $X_1, \ldots, X_{n-1}$ with identical distribution

$$\Pr(X_i = 1) = p, \qquad \Pr(X_i = 0) = 1 - p.$$

Note $\mathrm{E}(d_v) = (n-1)p$. Applying Chernoff's inequality with the lower tail $\lambda = \sqrt{4np\ln n}$, we have

$$\Pr(d_v - (n-1)p < -\sqrt{4np\ln n}) \leq e^{-(4np\ln n)/(2(n-1)p)} < \frac{1}{n^2}.$$

Applying Chernoff's inequality with the upper tail $\lambda = \sqrt{6np\ln n}$, we have

$$\Pr(d_v - (n-1)p > \sqrt{6np\ln n}) \leq e^{-(6np\ln n)/(2(n-1)p+2/3\sqrt{6np\ln n})} < \frac{1}{n^2}.$$

In the last step, we used the assumption $p \geq (8 \ln n)/(3n)$.

The probability that there is a vertex v so that $d_v \notin ((n-1)p - \sqrt{4np\ln n}, (n-1)p + \sqrt{6np\ln n})$ is at most

$$n\left(\frac{1}{n^2} + \frac{1}{n^2}\right) = \frac{2}{n}.$$

□

The co-degree d_{xy} of a pair of vertices (x, y) is the cardinality of the common neighborhood of x and y. Roughly speaking, when $p \gg \sqrt{(\ln n)/n}$, d_{xy} follows the binomial distribution $B(n-2, p^2)$; when $p \ll \sqrt{(\ln n)/n}$ it follows the Poisson distribution with mean $(n-2)p^2$. We can expect that all co-degrees are concentrated around a small interval if $p = \Omega(\sqrt{(\ln n)/n})$ and are bounded by $O(\ln n)$ if p is $p = O(\sqrt{(\ln n)/n})$. The transition occurs around $p = O(\sqrt{(\ln n)/n})$. We have the following Lemma, where the constant "2" is not signigicant.

Lemma 5.4. *If $p \geq 2\sqrt{(\ln n)/n}$, then with probability at least $1 - 1/n$, all co-degrees of $G(n,p)$ fall in the range $((n-2)p^2 - \sqrt{6np^2\ln n}, (n-2)p^2 + \sqrt{9np^2\ln n})$.*

If $p \leq 2\sqrt{(\ln n)/n}$, then with probability at least $1 - 1/n$, all co-degrees of $G(n,p)$ are at most $6\ln n$.

Proof: For a pair of vertices x and y, the codegree $|\Gamma(x) \cap \Gamma(y)|$ is the sum of $n-2$ independent random variables $X_1, \ldots, X_{n-1}$ with identical distribution

$$\Pr(X_i = 1) = p^2, \qquad \Pr(X_i = 0) = 1 - p^2.$$

Note $\mathrm{E}(|\Gamma(x) \cap \Gamma(y)|) = (n-2)p^2$. Applying Chernoff's inequality with the lower tail $\lambda = \sqrt{6np^2\ln n}$, we have

$$\Pr(|\Gamma(x) \cap \Gamma(y)| - (n-2)p^2 < -\sqrt{6np^2\ln n}) \leq e^{-(6np^2\ln n)/(2(n-2)p^2)} < \frac{1}{n^3}.$$

If $p \geq 2\sqrt{(\ln n)/n}$, we apply Chernoff's inequality with the upper tail $\lambda = \sqrt{9np^2\ln n}$.

$$\Pr(|\Gamma(x) \cap \Gamma(y)| - (n-2)p^2 > \sqrt{9np^2\ln n}) \leq e^{-(9np^2\ln n)/(2(n-2)p^2 + 2/3\sqrt{9np^2\ln n})} < \frac{1}{n^3}.$$

If $p \leq 2\sqrt{(\ln n)/n}$, we apply Chernoff's inequality with the upper tail $\lambda = 6\ln n$.

$$\Pr(|\Gamma(x) \cap \Gamma(y)| - (n-2)p^2 > 6\ln n) \leq e^{-(6\ln n)^2/(2(n-2)p^2 + 2/3 \cdot 6\ln n)} < \frac{1}{n^3}.$$

Now the number of pairs is at most $\binom{n}{2} < n^2/2$. The sum of the probabilities of small events is at most

$$\frac{n^2}{2}\left(\frac{1}{n^3} + \frac{1}{n^3}\right) = \frac{1}{n}.$$

□

The following lemma holds for general graphs.

LEMMA 5.5. *Suppose that* $\phi\colon \Gamma(x)\setminus N(y) \to \Gamma(y)\setminus N(x)$ *is an injective mapping. Then we have*

$$\kappa(x,y) \geq 1 - \frac{1}{d_y} \sum_{u\in\Gamma(x)\setminus N(y)} d(u,\phi(u)) + \frac{1}{d_x} - \frac{3(d_y-d_x)}{d_y}.$$

Proof: We denote the codegree of xy by $d_{xy} = |\Gamma(x)\cap\Gamma(y)|$. Let $R = \Gamma(x)\setminus N(y)$ and $r = |R|$. We have $r = d_x - 1 - d_{xy}$. Let $T = \Gamma(y)\setminus (N(x)\cup \phi(R))$ and $t = |T|$. We have $t = d_y - d_x$. We define a coupling A between m_x^α and m_y^α as follows.

$$A(u,v) = \begin{cases} (1-\alpha)/d_y & \text{if } u\in R \text{ and } v=\phi(u),\\ \alpha-(1-\alpha)/d_x & \text{if } u=x, v=y,\\ (1-\alpha)(1/d_x-1/d_y)\cdot 1/t & \text{if } u\in N(x)\setminus\{y\}, v\in T;\\ (1-\alpha)/d_x & \text{if } u=v=y,\\ (1-\alpha)/d_y & \text{if } u=v=x,\\ (1-\alpha)/d_y & \text{if } u=v\in\Gamma(x)\cap\Gamma(y),\\ 0 & \text{otherwise .}\end{cases}$$

We have

$$\begin{aligned} W(m_x^\alpha, m_y^\alpha) &\leq \sum_{u,v} A(u,v)d(u,v)\\ &= A(x,y) + \sum_{u\in R} A(u,\phi(u))d(u,\phi(u)) + \sum_{u\in N(x)\setminus\{y\}, v\in T} A(u,v)d(u,v)\\ &\leq \left(\alpha - \frac{1-\alpha}{d_x}\right) + \frac{1-\alpha}{d_y}\sum_{u\in R} d(u,\phi(u)) + 3(1-\alpha)\left(\frac{1}{d_x}-\frac{1}{d_y}\right)d_x. \end{aligned}$$

We have

$$\begin{aligned} \kappa(x,y) &= \lim_{\alpha\to 1} \frac{1-W(m_x^\alpha, m_y^\alpha)}{1-\alpha}\\ &\geq 1 - \frac{1}{d_y}\sum_{u\in R} d(u,\phi(u)) + \frac{1}{d_x} - \frac{3(d_y-d_x)}{d_y}. \end{aligned}$$

□

5.2 Proof of Theorem 5.1

Proof of Theorem 5.1: First let us prove item 1 and 2.

Without loss of generality, we assume $d_x \leq d_y$. For any edge xy of $G(n,p)$, we define an 1-Lipschitz function f over $N(x)\cup N(y)$ as follows.

$$f(v) = \begin{cases} 0 & \text{if } v\in N(y), v\neq x,\\ 1 & \text{otherwise.}\end{cases}$$

We have

$$\begin{aligned} W(m_x^\alpha, m_y^\alpha) &\geq \sum_v f(v)(m_x^\alpha(v) - m_y^\alpha(v)) \\ &= \alpha - \frac{1-\alpha}{d_y} + (1-\alpha)\frac{d_x - d_{xy} - 1}{d_x} \\ &= 1 - (1-\alpha)\left(\frac{1}{d_y} + \frac{d_{xy}+1}{d_x}\right) \end{aligned}$$

Thus

$$\begin{aligned} \kappa(x,y) &= \lim_{\alpha\to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)}{1-\alpha} \\ &\leq \frac{d_{xy}+1}{d_x} + \frac{1}{d_y}. \end{aligned}$$

By Lemma 5.3 and 5.4, with probability at least $1 - 3/n$, for any edge xy, we have

$$\kappa(x,y) \leq \frac{(n-2)p^2 + \sqrt{9np^2\ln n} + 2}{(n-1)p - \sqrt{4np\ln n}} = p + O\left(\sqrt{\frac{\ln n}{n}}\right).$$

For the lower bound, we will construct a matching M from $\Gamma(x)\setminus N(y)$ to $\Gamma(y)\setminus N(x)$ as follows. Let $U_0 = \Gamma(x)\setminus N(y)$ and $V_0 = \Gamma(y)\setminus N(x)$. Pick up a vertex $u_1 \in U_0$. Reveal the neighborhood of u_1 in V_0. Pick a vertex in the neighborhood, and denote it by v_1. Let $U_1 = U_0 \setminus \{u_1\}$ and $V_1 = V_0 \setminus \{v_1\}$ and continue this process. The process ends when $\Gamma(u_{i+1}) \cap V_i = \emptyset$. The probability that the maximum matching between U_0 and V_0 is at most k is less than

$$\sum_{i=1}^{k} (1-p)^{|V_0|-i} < \frac{1}{p}(1-p)^{|V_0|-k} \leq \sqrt{\frac{n}{\ln n}} e^{-p(|V_0|-k)} \leq ne^{-p(|V_0|-k)}.$$

Choose $k = \lfloor |V_0| - (3\ln n)/p \rfloor$. With probability at least $1 - 1/n^2$, there is a Matching M of size k between $\Gamma(x)\setminus N(y)$ and $\Gamma(y)\setminus N(x)$.

Now we extend the matching M to an injective mapping $\phi\colon \Gamma(x)\setminus N(y) \to \Gamma(y)\setminus N(x)$ arbitrarily. Applying Lemma 5.3, 5.4, and 5.5, with probability at least $1 - 4/n$, we have

$$\begin{aligned} \kappa(x,y) &\geq 1 - \frac{1}{d_y} \sum_{u\in\Gamma(x)\setminus N(y)} d(u,\phi(u)) + \frac{1}{d_x} - \frac{3(d_y - d_x)}{d_y} \\ &\geq 1 - \frac{1}{d_y}(k + 3(|V_0| - k)) + \frac{1}{d_x} - \frac{3(d_y - d_x)}{d_y} \\ &\geq \frac{d_{xy}}{d_y} - \frac{2(3\ln n/p)}{d_y} - \frac{3(d_y - d_x)}{d_y} \\ &\geq \frac{(n-2)p^2 - \sqrt{6np^2\ln n}}{(n-1)p + \sqrt{6np\ln n}} - \frac{6(\ln n/p)}{(n-1)p - \sqrt{4np\ln n}} - \frac{6\sqrt{6np\ln n}}{(n-1)p - \sqrt{4np\ln n}} \\ &= p - O\left(\sqrt{\frac{\ln n}{np}}\right) - O\left(\frac{\ln n}{np^2}\right). \end{aligned}$$

Combining the upper bound and the lower bound, we have

$$\kappa(x,y) = p + O\left(\sqrt{\frac{\ln n}{np}}\right) + O\left(\frac{\ln n}{np^2}\right).$$

If $p > \sqrt[3]{(\ln n)/n}$, we have $p \geq \sqrt{(\ln n)/(np)} \geq (\ln n)/(np^2)$. Thus,

$$\kappa(x,y) = p + O\left(\sqrt{\frac{\ln n}{np}}\right).$$

If $\sqrt[3]{(\ln n)/n} \geq p$, we have $p \leq \sqrt{(\ln n)/(np)} \leq (\ln n)/(np^2)$. Then

$$\kappa(x,y) = O\left(\frac{\ln n}{np^2}\right).$$

Next we consider the range $\sqrt[3]{(\ln n)/n^2} \ll p \ll 1/\sqrt{n}$. The upper bound comes from the following 1-Lipschitz function f. Let

$$S = \{u \in \Gamma(x) \setminus N(y); d(u,v) \geq 2 \text{ for any } v \in \Gamma(y) \setminus \{x\}\}.$$

Define f over $N(x) \cup N(y)$ as follows.

$$f(v) = \begin{cases} 2 & \text{if } v \in S, \\ 1 & \text{if } v \in N(x) \setminus S, v \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} W(m_x^\alpha, m_y^\alpha) &\geq \sum_v f(v)(m_x^\alpha(v) - m_y^\alpha(v)) \\ &= 2|S|\frac{(1-\alpha)}{d_x} + \alpha - \frac{1-\alpha}{d_y}(1+d_{xy}) + (1-\alpha)\frac{d_x - |S| - 1}{d_x} \\ &= 1 + (1-\alpha)\left(\frac{|S|-1}{d_x} - \frac{1+d_{xy}}{d_y}\right) \end{aligned}$$

Thus

$$\begin{aligned} \kappa(x,y) &= \lim_{\alpha \to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)}{1-\alpha} \\ &\leq -\frac{|S|}{d_x} + \frac{1}{d_x} + \frac{1+d_{xy}}{d_y}. \end{aligned}$$

The value $|S|$ can be estimated as follows. First we reveal the neighborhood of y. For any $v \notin N(y)$, $v \in S$ if vx is an edge and vu is not an edge for any $u \in N(y) \setminus \{x\}$. Let X_v be the 0-1 random variable indicating whether $v \in S$. Since v is not in $N(y)$, we have

$\Pr(X_v = 1) = p(1-p)^{d_y}$ and $\Pr(X_v = 0) = 1 - p(1-p)^{d_y}$. Then $|S| = \sum_{v \notin N(y)} X_v$ is a sum of independent random variables. Apply Chernoff's inequality (Lemma 5.2) with $\lambda = 2\sqrt{(n-d_y-1)p(1-p)^{d_y} \ln n}$ and $\mathrm{E}(|S|) = (n-d_y-1)p(1-p)^{d_y}$. With probability at least $1 - 1/n^2$, we have

$$\begin{aligned} |S| &\geq \mathrm{E}(|S|) - \lambda \\ &= (n-d_y-1)p(1-p)^{d_y} - 2\sqrt{(n-d_y-1)p(1-p)^{d_y} \ln n} \\ &= np(1 - O(np^2)) - O(\sqrt{np \ln n}) \\ &= np\left(1 - O(np^2) - O\left(\sqrt{\frac{\ln n}{np}}\right)\right). \end{aligned}$$

In above calculation, we applied Lemma 5.3 to estimate d_y. Since $p \leq 2\sqrt{(\ln n)/n}$, by Lemma 5.4, with probability $1 - 1/n$, $d_{xy} \leq 6 \ln n$.

We have

$$\begin{aligned} \kappa(x,y) &\leq -\frac{|S|}{d_x} + \frac{1}{d_x} + \frac{1 + d_{xy}}{d_y} \\ &\leq -1 + O(np^2) + O\left(\sqrt{\frac{\ln n}{np}}\right). \end{aligned}$$

Now we prove the lower bound. Without loss of generality, we assume $d_x \leq d_y$. We greedily construct an injective mapping ϕ from $\Gamma(x) \setminus N(y)$ to $\Gamma(y) \setminus N(x)$ so that most pairs $(u, \phi(u))$ have distance at most 2. Let $U_0 = \Gamma(x) \setminus N(y)$, $V_0 = \Gamma(y) \setminus N(x)$, and $W_0 = V(G) \setminus (N(x) \cup N(y))$. Let $m = \min\{|\Gamma(x) \setminus N(y)|, \lfloor d_y - d_{xy} - 1 - 4(\ln n)/(np^2) \rfloor\}$. For $i = 1, 2 \ldots, m$, pick a vertex $u_i \in U_{i-1}$, explore its neighborhood in W_{i-1} and then its second neighborhood in V_{i-1}. Pick a vertex $v_i \in V_{i-1}$, which has distance 2 to u_i. Define $U_i = U_{i-1} \setminus \{u_i\}$, $V_i = V_{i-1} \setminus \{v_i\}$, and $W_i = W_{i-1} \setminus \Gamma(u_i)$. Map the remaining vertices (in U_m) to V_m in any 1-1 way. Note $np^2 = o(1)$. We have for all $1 \leq i \leq m$

$$|W_i| \geq |W_m| > n - (m+2)D > n - (D+2)D = (1 - o(1))n > \frac{2}{3}n.$$

Here D is the maximum degree of $G(n,p)$. Here we use the facts $D = (1 + o(1))np$ and $np^2 = o(1)$.

Note that $|\Gamma(u_i) \cap W_{i-1}|$ can be viewed as the sum of 0-1 independent random variables. Let $X = |\Gamma(u_i) \cap W_{i-1}|$. Then $\mathrm{E}(X) = |W_{i-1}|p > 2np/3$. Applying Chernoff inequality to X with $\lambda = 2\sqrt{\mathrm{E}(X) \ln n}$. With probability at least $1 - 1/n^2$, we have

$$\begin{aligned} |\Gamma(u_i) \cap W_{i-1}| &= X \\ &\geq \mathrm{E}(X) - 2\sqrt{\mathrm{E}(X) \ln n} \\ &> \frac{2}{3}np - 2\sqrt{np \ln n} \\ &> \frac{1}{2}np. \end{aligned}$$

Also note that $|V_{i-1}| \geq d_y - d_{xy} - 1 - m \geq 4(\ln n)/(np^2)$. Thus, the probability that there is no edge between $\Gamma(u_i) \cap W_{i-1}$ and V_{i-1} is at most

$$(1-p)^{|\Gamma(u_i)\cap W_{i-1}||V_{i-1}|} \leq e^{-p\cdot(np)/2\cdot 4(\ln n)/(np^2)} = \frac{1}{n^2}.$$

The above argument shows that with probability at least $1 - 1/n$

$$m = \lfloor d_y - d_{xy} - 1 - 4(\ln n)/(np^2) \rfloor.$$

Applying Lemma 5.3 and 5.5, with probability at least $1 - 4/n$, we have

$$\begin{aligned}
\kappa(x,y) &\geq 1 - \frac{1}{d_y} \sum_{u\in\Gamma(x)\setminus N(y)} d(u,\phi(u)) + \frac{1}{d_x} - \frac{3(d_y - d_x)}{d_y} \\
&\geq 1 - \frac{1}{d_y}(2m + 3(d_x - d_{xy} - 1 - m))) + \frac{1}{d_x} - \frac{3(d_y - d_x)}{d_y} \\
&= -2 + \frac{m + 3d_{xy} + 3}{d_y} + \frac{1}{d_x} \\
&> -1 - \frac{d_y - d_{xy} - 1 - m}{d_y} \\
&\geq -1 - \frac{4(\ln n)/(np^2)}{(n-1)p - \sqrt{4np\ln n}} \\
&= -1 - O\left(\frac{\ln n}{n^2p^3}\right) + O\left(\sqrt{\frac{\ln n}{np}}\right).
\end{aligned}$$

Combining the upper bound and the lower bound, we have

$$\begin{aligned}
\kappa(x,y) &= -1 + O(np^2) + O\left(\sqrt{\frac{\ln n}{np}}\right) + O\left(\frac{\ln n}{n^2p^3}\right) \\
&= -1 + O(np^2) + O\left(\frac{\ln n}{n^2p^3}\right).
\end{aligned}$$

In the last step, we use the fact that $\sqrt{(\ln n)/(np)} \leq \left(np^2 + (\ln n)/(n^2p^3)\right)/2$.

Now we consider the range $(\ln n)/n \ll p \ll \sqrt[3]{1/n^2}$. The lower bound is trivial since $\kappa(x,y) \geq -2$ holds for any x, y and any graph G.

The upper bound comes from the following 1-Lipschitz function f. Let

$$\begin{aligned}
S &= \{u \in \Gamma(x) \setminus N(y); d(u,v) = 3 \text{ for any } v \in \Gamma(y) \setminus \{x\}\}; \\
T &= \{v \in \Gamma(y) \setminus N(x); d(u,v) = 3 \text{ for any } u \in \Gamma(x) \setminus \{y\}\}.
\end{aligned}$$

Define f over $N(x) \cup N(y)$ as follows.

$$f(v) = \begin{cases} 2 & \text{if } v \in S, \\ 1 & \text{if } v \in N(x) \setminus S, \\ -1 & \text{if } v \in T, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$
\begin{aligned}
W(m_x^\alpha, m_y^\alpha) &\geq \sum_v f(v)(m_x^\alpha(v) - m_y^\alpha(v)) \\
&= 2|S|\frac{(1-\alpha)}{d_x} + \alpha - \frac{1-\alpha}{d_y} + (1-\alpha)\frac{d_x - |S| - 1}{d_x} + |T|\frac{(1-\alpha)}{d_y} \\
&= 1 + (1-\alpha)\left(\frac{|S|-1}{d_x} + \frac{|T|-1}{d_y}\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\kappa(x,y) &= \lim_{\alpha \to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)}{1-\alpha} \\
&\leq -\left(\frac{|S|-1}{d_x} + \frac{|T|-1}{d_y}\right).
\end{aligned}
$$

The value $|S|$ can be estimated as follows. Explore the neighborhood of y first. Then explore the neighborhood of $\Gamma(y) \setminus \{x\}$. Let U denote the set $\Gamma(\Gamma(y) \setminus \{x\}) \setminus \{x\}$. For any $v \notin U \cup N(y)$, $v \in S$ if vx is an edge and vu is not an edge for any $u \in U$. Let X_v be the 0-1 random variable indicating whether $v \in S$. Since v is not in $U \cup N(y)$, we have $\Pr(X_v = 1) = p(1-p)^{|U|}$ and $\Pr(X_v = 0) = 1 - p(1-p)^{|U|}$. Then $|S| = \sum_{v \notin U \cup N(y)} X_v$ is a sum of independent random variables. Apply Chernoff's inequality (Lemma 5.2) with $\lambda = 2\sqrt{(n - d_y - |U|)p(1-p)^{|U|}\ln n}$ and $\mathrm{E}(|S|) = (n - d_y - |U|)p(1-p)^{|U|}$. Also note $|U| \leq (d_y - 1)D \leq D^2 \leq (1+o(1))n^2p^2$. Thus with probability at least $1 - O(1/n^2)$, we have

$$
\begin{aligned}
|S| &\geq \mathrm{E}(|S|) - \lambda \\
&= (n - d_y - |U|)p(1-p)^{|U|} - 2\sqrt{(n - d_y - |U|)p(1-p)^{|U|}\ln n} \\
&= np(1 - O(n^2p^3)) - O(\sqrt{np\ln n}) \\
&= np\left(1 - O(n^2p^3) - O\left(\sqrt{\frac{\ln n}{np}}\right)\right).
\end{aligned}
$$

Here we applied Lemma 5.3 for the estimation of d_y. By symmetry, with probability at least $1 - O(1/n^2)$, we have

$$
|T| \geq np\left(1 - O(n^2p^3) - O\left(\sqrt{\frac{\ln n}{np}}\right)\right).
$$

We have

$$\begin{aligned}
\kappa(x,y) &\leq -\frac{|S|-1}{d_x} - \frac{|T|-1}{d_y} \\
&\leq -2\frac{np\left(1 - O(n^2p^3) - O\left(\sqrt{(\ln n)/(np)}\right)\right) - 1}{(n-1)p + \sqrt{6np\ln n}} \\
&\leq -2 + O(n^2p^3) + O\left(\sqrt{\frac{\ln n}{np}}\right).
\end{aligned}$$

□

Remark: For $p = c/\sqrt{n}$, the curvature drops quickly from 0 to -1 as c decreases. For $p = c/n^{2/3}$, the curvature drops quickly from -1 to -2 as c decreases. For the range that $p < (c \ln n)/n$, the degrees are not asymptotically regular. For most edge xy, xy is not in any small cycles C_3, C_4, or C_5. We have $\kappa(x,y) = -2 + 2/d_x + 2/d_y$.

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References

[1] D. Bakry and M. Emery, Diffusions hypercontractives, Séminaire de probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math. 1123, Springer, Berlin, 1985.

[2] H. Chernoff, A note on an inequality involving the normal distribution, Ann. Probab. 9 (1981), 533–535.

[3] Fan Chung and S.-T. Yau, Eigenvalues of graphs and Sobolev inequalities, Combin. Probab. Comput. 4 (1995), 11–25.

[4] Fan Chung and S.-T. Yau, A Harnack inequality for homogeneous graphs and subgraphs, Comm. Anal. Geom. 2 (1994), 627–640, also in Turkish J. Math. 19 (1995), 273–290.

[5] Fan Chung, A. Grigor'yan and S.-T. Yau, Upper bounds for eigenvalues of the discrete and continuous Laplace operators, Adv. Math. 117 (1996), 165–178.

[6] Fan Chung, R. L. Graham and S.-T. Yau, On sampling with Markov chains, Random Structures and Algorithms 9 (1996), 55–77.

[7] Fan Chung, Yong Lin and S.-T. Yau, Harnack inequalities and Logarithmic Harnack inequalities, preprint.

[8] Fan Chung and S.-T. Yau, Logarithmic Harnack inequalities, Math. Res. Lett. 3 (1996), 793–812.

[9] Fan Chung and S.-T. Yau, A combinatorial trace formula, Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991), 107–116, Int. Press, Cambridge, MA, 1997.

[10] Fan Chung, A. Grigor'yan and S.-T. Yau, Eigenvalues and diameters for manifolds and graphs, Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991), 79–105, Int. Press, Cambridge, MA, 1997.

[11] Fan Chung and S.-T. Yau, Eigenvalue inequalities for graphs and convex subgraphs, Comm. Anal. Geom. 5 (1997), 575–623.

[12] Fan Chung and S.-T. Yau, Coverings, heat kernels and spanning trees, Electron. J. Combin. 6 (1999), Research Paper 12, 21pp.

[13] Fan Chung and S.-T. Yau, Spanning trees in subgraphs of lattices, Contemp. Math. 245, 201–219, Amer. Math. Soc., Providence, R.I., 1999.

[14] Fan Chung and S.-T. Yau, A Harnack inequality for Dirichlet eigenvalues, J. Graph Theory 34 (2000), 247–257,

[15] Fan Chung, A. Grigor'yan and S.-T. Yau, Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs, Comm. Anal. Geom. 8 (2000), 969–1026.

[16] Fan Chung and S.-T. Yau, Discrete Green's functions, J. Combin. Theory Ser. A 91 (2000), 191–214.

[17] Yong Lin and S.-T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, Mathematical Research Letters 17(2010), 345–358.

[18] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. (2) 169 (2009), 903–991.

[19] S. Ohta, On measure contraction property of metric measure spaces, Comment. Math. Helv. 82(2007), 805–828.

[20] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, J. Funct. Anal. 256 (3) (2009), 810–864.

[21] K.-T. Sturm, On the geometry of metric measure spaces, (I),(II), Acta Math. 196 (2006), 65–131, 133–177.

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