The Fractional Chromatic Number of Triangle-free Graphs with $\Delta \leq 3$

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Abstract

Let G be any triangle-free graph with maximum degree $\Delta \leq 3$. Staton proved that the independence number of G is at least $\frac{5}{14}n$. Heckman and Thomas conjectured that Staton's result can be strengthened into a bound on the fractional chromatic number of G, namely $\chi_f(G) \leq \frac{14}{5}$. Recently, Hatami and Zhu proved $\chi_f(G) \leq 3 - \frac{3}{64}$. In this paper, we prove $\chi_f(G) \leq 3 - \frac{3}{43}$.

1 Introduction

This paper investigates the fractional chromatic number of a triangle free graph with maximum degree at most 3. For a simple (finite) graph $G = (V, E)$, the fractional chromatic number of G is the linear programming relaxation of the chromatic number of G. Let $\mathcal{I}(G)$ be the family of independent sets of G. A mapping $f\colon \mathcal{I}(G) \to [0, 1]$ is called an r-fractional coloring of G if $\sum_{S \in \mathcal{I}(G)} f(S) \leq r$ and $\sum_{v \in S, S \in \mathcal{I}(G)} f(S) \geq 1$. The *fractional chromatic number* $\chi_f(G)$ of G is the least r for which G has an r-fractional coloring.

Alternatively, the fractional chromatic number can also be defined through multiple colorings. A *b-fold coloring* of G assigns a set of b colors to each vertex so that any two adjacent vertices receive disjoint sets of colors. We say a graph G is *$a : b$-colorable* if there is a *b-fold coloring* of G in which each color is drawn from a palette of a colors. We refer to such a coloring as an $a : b$-coloring. The *b-fold coloring number*, denoted as $\chi_b(G)$, is the smallest integer a so that G has a $a : b$-coloring. Note that $\chi_1(G) = \chi(G)$. It is known, in [10], that $\chi_b(G)$ (as a function of b) is sub-additive and so the infimum of $\frac{\chi_b(G)}{b}$ always exists, which turns out to be an alternative definition of $\chi_f(G)$. (Moreover, $\chi_f(G)$ is a rational number and the infimum can be replaced by minimum.)

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Let $\chi(G)$ be the chromatic number of G and $\omega(G)$ be the clique number of G. A simple relation holds:

$$\omega(G) \leq \chi_f(G) \leq \chi(G). \tag{1}$$

Now we consider a graph G with maximum degree Δ at most three. If G is not K_4, then by Brooks' theorem, G is 3-colorable. If G contains a triangle, then $\chi_f(G) \geq \omega(G) = 3$. Equation (1) implies $\chi_f(G) = 3$. One may ask what is the possible value of $\chi_f(G)$ if G is triangle-free with $\Delta \leq 3$. This problem is motivated by a well-known, solved problem of determining the maximum independence number $\alpha(G)$ for such graphs. Staton, in [11], showed that

$$\alpha(G) \geq 5n/14 \tag{2}$$

for any triangle-free graph G on n vertices with maximum degree at most 3. Actually, Staton's bound is the best possible since the generalized Peterson graph $P(7, 2)$ has 14 vertices and independence number 5 as noticed by Fajtlowicz, in [2]. Griggs and Murphy, in [4], designed a linear-time algorithm to find an independent set in G of size at least $5(n-k)/14$, where k is the number of components of G that are 3-regular. Heckman and Thomas, in [6], gave a simpler proof of Staton's bound and designed a linear-time algorithm to find an independent set in G with size at least $5n/14$.

In the same paper [6], Heckman and Thomas conjectured

$$\chi_f(G) \leq \frac{14}{5} \tag{3}$$

for every triangle-free graph with maximum degree at most 3. Note that, in [10],

$$\chi_f(G) = \frac{n}{\alpha(G)}, \tag{4}$$

provided G is vertex transitive. It implies that the generalized Peterson graph $P(7, 2)$ has the fractional chromatic number 2.8. Thus, the conjecture is tight if it holds.

Recently, Hatami and Zhu, in [5], proved $\chi_f(G) \leq 3 - \frac{3}{64}$, provided G is triangle free with maximum degree at most three. Inspired by their method, we introduce the concept of "admissible sets" and prove the following theorem.

Theorem 1 *If G is triangle-free and has maximum degree at most 3, then $\chi_f(G) \leq 3 - \frac{3}{43}$.*

The rest of the paper is organized as follows. In section two, we will study the convex structure of fractional colorings and the fractionally-critical graphs. In section three, we will prove several key lemmas. In section four, we will show that G can be partitioned into 42 admissible sets and present the proof of the main theorem. We will give some concluding remarks at the end.

2 Lemmas and Notations

In this section, we introduce an alternative definition of "fractional colorings." The new definition highlights the convex structure of the set of all fractional colorings. The extreme points of these "fractional colorings" play a central role in our proofs and seem to have independent interest. Our approach is analogous to define rational numbers from integers.

2.1 Convex structures of fractional colorings

In this paper, we use bold letter $\mathbf{c}$ to represent the colorings. Recall that a b-fold coloring of a graph G assigns a set of b colors to each vertex such that any two adjacent vertices receive disjoint sets of colors. Given a b-fold coloring $\mathbf{c}$, let $A(\mathbf{c}) = \cup_{v\in V(G)}\mathbf{c}(v)$ be the set of all colors used in $\mathbf{c}$. Two b-fold colorings $\mathbf{c}_1$ and $\mathbf{c}_2$ are *isomorphic* if there is a bijection $\phi\colon A(\mathbf{c}_1) \to A(\mathbf{c}_2)$ so that $\phi \circ \mathbf{c}_1 = \mathbf{c}_2$. In this case we write $\mathbf{c}_1 \cong \mathbf{c}_2$. The isomorphic relation $\cong$ is an equivalence relation. We denote $\bar{\mathbf{c}}$ be the isomorphic class where $\mathbf{c}$ belongs. Whenever clear under the context, we will not distinguish a b-fold coloring $\mathbf{c}$ and its isomorphic class $\bar{\mathbf{c}}$.

For a graph G and a positive integer b, let $\mathcal{C}_b(G)$ be the set of all (isomorphic classes of) b-fold colorings of G. For $\mathbf{c}_1 \in \mathcal{C}_{b_1}(G)$ and $\mathbf{c}_2 \in \mathcal{C}_{b_2}(G)$, we can define $\mathbf{c}_1 + \mathbf{c}_2 \in \mathcal{C}_{b_1+b_2}(G)$ as follows: for any $v \in V(G)$,

$$(\mathbf{c}_1 + \mathbf{c}_2)(v) = \mathbf{c}_1(v) \sqcup \mathbf{c}_2(v),$$

i.e., $\mathbf{c}_1 + \mathbf{c}_2$ assigns v the disjoint union of $\mathbf{c}_1(v)$ and $\mathbf{c}_2(v)$.

Let $\mathcal{C}(G) = \cup_{b=0}^{\infty}\mathcal{C}_b(G)$. It is easy to check that "+" is commutative and associative. Under this addition, $\mathcal{C}(G)$ forms a commutative monoid with the unique 0-fold coloring (denoted by 0, for short) as the identity. For a positive integer t and $\mathbf{c} \in \mathcal{C}_b(G)$, we define

$$t \cdot \mathbf{c} = \overbrace{\mathbf{c} + \cdots + \mathbf{c}}^{t}$$

to be the new tb-fold coloring by duplicating each color t times.

For $\mathbf{c}_1 \in \mathcal{C}_{b_1}(G)$ and $\mathbf{c}_2 \in \mathcal{C}_{b_2}(G)$, we say $\mathbf{c}_1$ and $\mathbf{c}_2$ are *equivalent*, denoted as $\mathbf{c}_1 \sim \mathbf{c}_2$, if there exists a positive integer s so that $sb_2 \cdot \mathbf{c}_1 \cong sb_1 \cdot \mathbf{c}_2$. (This is an analogy of the classical definition of rational numbers.)

Lemma 1 *The binary relation $\sim$ is an equivalence relation over $\mathcal{C}(G)$.*

Proof: It is easy to check $\sim$ is reflexive and symmetric. Now we prove $\sim$ is transitive. Let $\mathbf{c}_i \in \mathcal{C}_{b_i}(G)$ for $i = 1, 2, 3$. Suppose $\mathbf{c}_1 \sim \mathbf{c}_2$ and $\mathbf{c}_2 \sim \mathbf{c}_3$. We need to prove $\mathbf{c}_1 \sim \mathbf{c}_3$.

Since $\mathbf{c}_1 \sim \mathbf{c}_2$, there is a positive integer s so that

$$sb_2 \cdot \mathbf{c}_1 \cong sb_1 \cdot \mathbf{c}_2.$$

Since $\mathbf{c}_2 \sim \mathbf{c}_3$, there is a positive integer t so that

$$tb_3 \cdot \mathbf{c}_2 \cong tb_2 \cdot \mathbf{c}_3.$$

We have

$$\begin{aligned}
stb_2b_3 \cdot \mathbf{c}_1 &\cong tb_3 \cdot (sb_2 \cdot \mathbf{c}_1) \\
&\cong tb_3 \cdot (sb_1 \cdot \mathbf{c}_2) \\
&= sb_1 \cdot (tb_3 \cdot \mathbf{c}_2) \\
&\cong sb_1 \cdot (tb_2 \cdot \mathbf{c}_3) \\
&= stb_2b_1 \cdot \mathbf{c}_3.
\end{aligned}$$

Thus, $\mathbf{c}_1 \sim \mathbf{c}_3$ by definition. □

Let $\mathcal{F}(G) = \mathcal{C}(G)/\sim$ be the set of all equivalence classes. Each equivalence class is called a fractional coloring of G. For any $\mathbf{c} \in \mathcal{C}_b(G)$, the equivalence class of $\mathbf{c}$ under $\sim$ is denoted by $\pi(\mathbf{c}) = \frac{\mathbf{c}}{b}$.

Remark: The notation $\frac{\mathbf{c}}{b}$ makes sense only when $\mathbf{c}$ is a b-fold coloring.

Given any $\lambda = \frac{q}{p} \in [0,1]$ and two fractional colorings (two equivalence classes) $\frac{\mathbf{c}_1}{b_1}$ and $\frac{\mathbf{c}_2}{b_2}$, we define the linear combination as

$$\lambda \mathbf{c}_1 + (1-\lambda)\mathbf{c}_2 = \frac{qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2}{pb_1b_2}.$$

The following lemma shows this definition is independent of the choices of $\mathbf{c}_1$ and $\mathbf{c}_2$, and so $\lambda\mathbf{c}_1 + (1-\lambda)\mathbf{c}_2$ is a fractional coloring depending only on λ, $\frac{\mathbf{c}_1}{b_1}$, and $\frac{\mathbf{c}_2}{b_2}$.

Lemma 2 *Assume $\mathbf{c}_i \in \mathcal{C}_{b_i}(G)$ for $i = 1,2,3,4$ satisfying $\mathbf{c}_1 \sim \mathbf{c}_3$ and $\mathbf{c}_2 \sim \mathbf{c}_4$. For any non-negative integers p, q, p', and q' satisfying $\frac{q}{p} = \frac{q'}{p'} \in [0,1]$, we have $qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2 \sim q'b_4 \cdot \mathbf{c}_3 + (p'-q')b_3 \cdot \mathbf{c}_4$.*

Proof: There exist two positive integers s and t satisfying

$$\begin{aligned}
sb_3 \cdot \mathbf{c}_1 &\cong sb_1 \cdot \mathbf{c}_3; \\
tb_4 \cdot \mathbf{c}_2 &\cong tb_2 \cdot \mathbf{c}_4.
\end{aligned}$$

We have

$$\begin{aligned}
p'stb_3b_4 \cdot (qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2) &= p'qstb_2b_3b_4 \cdot \mathbf{c}_1 + p'(p-q)stb_1b_3b_4 \cdot \mathbf{c}_2 \\
&= p'qtb_2b_4 \cdot (sb_3 \cdot \mathbf{c}_1) + p'(p-q)sb_1b_3 \cdot (tb_4 \cdot \mathbf{c}_2) \\
&\cong p'qtb_2b_4 \cdot (sb_1 \cdot \mathbf{c}_3) + p'(p-q)sb_1b_3 \cdot (tb_2 \cdot \mathbf{c}_4) \\
&= p'qstb_1b_2b_4 \cdot \mathbf{c}_3 + p'(p-q)stb_1b_2b_3 \cdot \mathbf{c}_4 \\
&= pq'stb_1b_2b_4 \cdot \mathbf{c}_3 + p(p'-q')stb_1b_2b_3 \cdot \mathbf{c}_4 \\
&= pstb_1b_2 \cdot (q'b_4 \cdot \mathbf{c}_3 + (p'-q')b_3 \cdot \mathbf{c}_4).
\end{aligned}$$

Note $qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2$ is a pb_1b_2-fold coloring while $q'b_4 \cdot \mathbf{c}_3 + (p'-q')b_3 \cdot \mathbf{c}_4$ is a $p'b_3b_4$-fold coloring. The above equality shows that $qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2 \sim q'b_4 \cdot \mathbf{c}_3 + (p'-q')b_3 \cdot \mathbf{c}_4$. $\square$

Define a function $g\colon \mathcal{F}(G) \to \mathbb{Q}$ as $g_G(\frac{\mathbf{c}}{b}) = \frac{|A(\mathbf{c})|}{b}$. If the graph G is clear under context, then we write it as $g(\frac{\mathbf{c}}{b})$ for short. It is easy to check that g does not depend on the choice of $\mathbf{c}$ and so g is well-defined. For any $\tau > 0$, we define

$$\mathcal{F}_\tau(G) = \{\frac{\mathbf{c}}{b} \in \mathcal{F}(G) | g(\frac{\mathbf{c}}{b}) \leq \tau\}.$$

Theorem 2 *For any graph G on n vertices, there is an embedding $\phi\colon \mathcal{F}(G) \to \mathbb{Q}^{2^n-1}$ such that ϕ keeps convex structure. Moreover, for any rational number τ, $\mathcal{F}_\tau(G)$ is convex with finite extreme fractional colorings.*

Proof: We would like to define $\phi\colon \mathcal{F}(G) \to \mathbb{Q}^{2^n-1}$ as follows.

Given a fractional coloring $\frac{\mathbf{c}}{b}$, we can fill these colors into the regions of the general Venn Diagram on n-sets. For $i = 1, 2, \ldots, 2^n - 1$, we can write i as a binary string $a_1a_2\cdots a_n$ such that $a_v \in \{0,1\}$ for all $1 \leq v \leq n$. Write $\mathbf{c}^1(v) = \mathbf{c}(v)$ and $\mathbf{c}^0(v) = \overline{\mathbf{c}(v)}$ (the complement set of $c(v)$). Then the number of colors in i-th region of Venn Diagram can be written as

$$h_i(\mathbf{c}) = \left|\cap_{v=1}^n \mathbf{c}^{a_v}(v)\right|.$$

By the definition, h_i is additive, i.e.

$$h_i(\mathbf{c}_1 + \mathbf{c}_2) = h_i(\mathbf{c}_1) + h_i(\mathbf{c}_2).$$

Thus $\frac{h_i(\mathbf{c})}{b}$ depends only on the fractional coloring $\frac{\mathbf{c}}{b}$ but not on $\mathbf{c}$ itself.

The i-th coordinate of $\phi(\frac{\mathbf{c}}{b})$ is defined to be

$$\phi_i(\frac{\mathbf{c}}{b}) = \frac{h_i(\mathbf{c})}{b}.$$

It is easy to check that ϕ_i is a well-defined function on $\mathcal{F}(G)$. Moreover, for any $\lambda = \frac{q}{p} \in [0,1]$ and any two fractional colorings $\frac{\mathbf{c}_1}{b_1}$ and $\frac{\mathbf{c}_2}{b_2}$, we have

$$\begin{aligned}
\phi_i\left(\lambda\frac{\mathbf{c}_1}{b_1} + (1-\lambda)\frac{\mathbf{c}_2}{b_2}\right) &= \phi_i\left(\frac{qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2}{pb_1b_2}\right) \\
&= \frac{h_i(qb_2 \cdot \mathbf{c}_1 + (p-q)b_1 \cdot \mathbf{c}_2)}{pb_1b_2} \\
&= \frac{qb_2h_i(\mathbf{c}_1) + (p-q)b_1h_i(\mathbf{c}_2)}{pb_1b_2} \\
&= \frac{q}{p}\frac{h_i(\mathbf{c}_1)}{b_1} + (1-\frac{q}{p})\frac{h_i(\mathbf{c}_2)}{b_2} \\
&= \lambda\phi_i(\frac{\mathbf{c}_1}{b_1}) + (1-\lambda)\phi_i(\frac{\mathbf{c}_2}{b_2}).
\end{aligned}$$

Thus ϕ keeps the convex structure.

It remains to show ϕ is a $1-1$ mapping. Suppose $\phi(\frac{\mathbf{c}_1}{b_1}) = \phi(\frac{\mathbf{c}_2}{b_2})$. We need to show $\mathbf{c}_1 \sim \mathbf{c}_2$. Let $\mathbf{c}'_1 = b_2 \cdot \mathbf{c}_1$ and $\mathbf{c}'_2 = b_1 \cdot \mathbf{c}_2$. Both $\mathbf{c}'_1$ and $\mathbf{c}'_2$ are b_1b_2-fold coloring. Note $\phi(\frac{\mathbf{c}'_1}{b_1b_2}) = \phi(\frac{\mathbf{c}'_2}{b_1b_2})$. For $j = 1, 2$ and $1 \leq i \leq 2^n - 1$, we denote the set of colors in the i-th Venn Diagram region of $A(\mathbf{c}'_j)$ by $B_i(\mathbf{c}'_j)$. Since $\phi(\frac{\mathbf{c}'_1}{b_1b_2}) = \phi(\frac{\mathbf{c}'_2}{b_1b_2})$, we have

$$|B_i(\mathbf{c}'_1)| = |B_i(\mathbf{c}'_2)|.$$

There is a bijection ψ_i from $B_i(\mathbf{c}'_1)$ to $B_i(\mathbf{c}'_2)$. Note that for $j = 1, 2$, we have a partition of $A(\mathbf{c}'_j)$ as the following

$$A(\mathbf{c}'_j) = \sqcup_{i=1}^{2^n-1} B_i(\mathbf{c}'_j).$$

Define a bijection ψ from $A(\mathbf{c}'_1)$ to $A(\mathbf{c}'_2)$ be the union of all ψ_i $(1 \leq i \leq 2^n - 1)$. We have

$$\psi \circ \mathbf{c}'_1 = \mathbf{c}'_2.$$

Thus $\mathbf{c}'_1 \cong \mathbf{c}'_2$. Note that $\mathbf{c}_1 \sim \mathbf{c}'_1$ and $\mathbf{c}_2 \sim \mathbf{c}'_2$. We conclude that $\mathbf{c}_1 \sim \mathbf{c}_2$.

Under the embedding, $\phi(\mathcal{F}_\tau(G))$ consists of all rational points in a polytope defined by the intersection of finite half spaces. Note that all coefficients of the equations of hyperplanes are rational. The extreme points are rational as well. They are finite. □

Remark: It is well-known that for any graph G there is a $a : b$-coloring of G with $\frac{a}{b} = \chi_f(G)$. In our terminology, we have $\chi_f(G) = \min\{g(\frac{\mathbf{c}}{b})|$ for any $\frac{\mathbf{c}}{b} \in \mathcal{F}(G)\}$.

2.2 Coloring restriction and extension

Let H be a subgraph of G. Any b-fold coloring of G is naturally a b-fold coloring of H. This *restriction* operation induces a mapping $i_G^H \colon \mathcal{F}(G) \to \mathcal{F}(H)$. It is easy to check that i_G^H keeps convex structure, i.e., for any $\frac{\mathbf{c}_1}{b_1}, \frac{\mathbf{c}_2}{b_2} \in \mathcal{F}(G)$ and $\lambda \in [0, 1] \cap \mathbb{Q}$, we have

$$i_G^H \left(\lambda \frac{\mathbf{c}_1}{b_1} + (1-\lambda) \frac{\mathbf{c}_2}{b_2} \right) = \lambda i_G^H \left(\frac{\mathbf{c}_1}{b_1} \right) + (1-\lambda) i_G^H \left(\frac{\mathbf{c}_2}{b_2} \right).$$

It is also trivial that

$$g_H \left(i_G^H \left(\frac{\mathbf{c}}{b} \right) \right) \leq g_G \left(\frac{\mathbf{c}}{b} \right).$$

Now we consider a reverse operation. We say a fractional coloring $\frac{\mathbf{c}_1}{b_1} \in \mathcal{F}(H)$ is *extensible* in $\mathcal{F}_t(G)$ if there is a fractional coloring $\frac{\mathbf{c}}{b} \in \mathcal{F}_t(G)$ satisfying

$$i_G^H \left(\frac{\mathbf{c}}{b} \right) = \frac{\mathbf{c}_1}{b_1}.$$

We say a fractional coloring $\frac{\mathbf{c}_1}{b_1} \in \mathcal{F}(H)$ is *fully extensible* in $\mathcal{F}(G)$ if it is extensible in $\mathcal{F}_t(G)$, where $t = g_H(\frac{\mathbf{c}_1}{b_1})$. (It also implies that $\frac{\mathbf{c}_1}{b_1}$ is extensible in $\mathcal{F}_t(G)$ for all $t \geq g_H(\frac{\mathbf{c}_1}{b_1})$.)

Lemma 3 *Let H be a subgraph of G and $\frac{\mathbf{c}_1}{b_1}, \frac{\mathbf{c}_2}{b_2} \in \mathcal{F}(H)$. Suppose that for $i = 1, 2$, $\frac{\mathbf{c}_i}{b_i} \in \mathcal{F}(H)$ are fully extensible in $\mathcal{F}(G)$. Then for any $\lambda \in \mathbb{Q} \cap [0, 1]$, $\lambda\frac{\mathbf{c}_1}{b_1} + (1-\lambda)\frac{\mathbf{c}_2}{b_2}$ is fully extensible in $\mathcal{F}(G)$.*

Proof: Let $t_i = g_H(\frac{\mathbf{c}_i}{b_i})$ for $i = 1, 2$. Note that there are fractional colorings $\frac{\mathbf{c}'_i}{b'_i} \in \mathcal{F}_{t_i}(G)$ such that $i_G^H\left(\frac{\mathbf{c}'_i}{b'_i}\right) = \frac{\mathbf{c}_i}{b_i}$ for $i = 1, 2$. Let $\frac{\mathbf{c}}{b} = \lambda\frac{\mathbf{c}'_1}{b'_1} + (1-\lambda)\frac{\mathbf{c}'_2}{b'_2}$. Then we have $g_G(\frac{\mathbf{c}}{b}) = \lambda t_1 + (1-\lambda)t_2$ and,

$$i_G^H\left(\frac{\mathbf{c}}{b}\right) = \lambda i_G^H\left(\frac{\mathbf{c}'_1}{b'_1}\right) + (1-\lambda) i_G^H\left(\frac{\mathbf{c}'_2}{b'_2}\right) = \lambda\frac{\mathbf{c}_1}{b_1} + (1-\lambda)\frac{\mathbf{c}_2}{b_2}.$$

Note that

$$g_H\left(\lambda\frac{\mathbf{c}_1}{b_1} + (1-\lambda)\frac{\mathbf{c}_2}{b_2}\right) = \lambda t_1 + (1-\lambda)t_2.$$

Therefore, $\lambda\frac{\mathbf{c}_1}{b_1} + (1-\lambda)\frac{\mathbf{c}_2}{b_2}$ is fully extensible in $\mathcal{F}(G)$ by definition. $\square$

We say $G = G_1 \cup G_2$ if $V(G) = V(G_1) \cup V(G_2)$ and $\mathbf{E}(G) = \mathbf{E}(G_1) \cup \mathbf{E}(G_2)$. Similarly, we say $H = G_1 \cap G_2$ if $V(H) = V(G_1) \cap V(G_2)$ and $\mathbf{E}(H) = \mathbf{E}(G_1) \cap \mathbf{E}(G_2)$.

Lemma 4 *Let G be a graph. Suppose that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = H$. Suppose that two fractional colorings $\frac{\mathbf{c}_1}{b_1} \in \mathcal{F}(G_1)$ and $\frac{\mathbf{c}_2}{b_2} \in \mathcal{F}(G_2)$ satisfy $i_{G_1}^H(\frac{\mathbf{c}_1}{b_1}) = i_{G_2}^H(\frac{\mathbf{c}_2}{b_2})$. Then there exists a fractional coloring $\frac{\mathbf{c}}{b} \in \mathcal{F}(G)$ satisfying*

$$i_G^{G_i}(\frac{\mathbf{c}}{b}) = \frac{\mathbf{c}_i}{b_i}$$

and $g_G(\frac{\mathbf{c}}{b}) = \max\{g_{G_1}(\frac{\mathbf{c}_1}{b_1}), g_{G_2}(\frac{\mathbf{c}_2}{b_2})\}$ for $i = 1, 2$.

Proof: Without loss of generality, we can assume $b_1 = b_2 = b$. We also assume $g_{G_1}(\frac{\mathbf{c}_1}{b_1}) \leq g_{G_2}(\frac{\mathbf{c}_2}{b_2})$ so that $|A_{G_1}(\mathbf{c}_1)| \leq |A_{G_2}(\mathbf{c}_2)|$. Since $i_{G_1}^H(\frac{\mathbf{c}_1}{b_1}) = i_{G_2}^H(\frac{\mathbf{c}_2}{b_2})$, there is a bijection ϕ from $\cup_{v\in V(H)}\mathbf{c}_1(v)$ to $\cup_{v\in V(H)}\mathbf{c}_2(v)$. Extend ϕ as an 1-1 mapping from $A_{G_1}(\mathbf{c}_1)$ to $A_{G_2}(\mathbf{c}_2)$ in an arbitrary way. Now we define a b-fold coloring $\mathbf{c}$ of G as follows.

$$\mathbf{c}(v) = \begin{cases} \phi(\mathbf{c}_1(v)) & \text{if } v \in V(G_1), \\ \mathbf{c}_2(v) & \text{if } v \in V(G_2). \end{cases}$$

Since G_1 and G_2 cover all edges of G, $\mathbf{c}$ is well-defined. Note $\mathbf{c}|_{V(G_1)} = \phi \circ \mathbf{c}_1 \cong \mathbf{c}_1$ and $\mathbf{c}|_{V(G_2)} = \mathbf{c}_2$. Thus for $i = 1, 2$,

$$i_G^{G_i}\left(\frac{\mathbf{c}}{b}\right) = \frac{\mathbf{c}_i}{b_i}.$$

We also have $g_G\left(\frac{\mathbf{c}}{b}\right) = \frac{|A_{G_2}(\mathbf{c}_2)|}{b} = g_{G_2}(\frac{\mathbf{c}_2}{b_2})$. $\square$

Theorem 3 *Let G be a graph. Suppose that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = H$. Suppose that $\chi_f(G_1) \leq t$ and any extreme fractional colorings in $\mathcal{F}_t(H)$ are extensible in $\mathcal{F}_t(G_2)$. Then $\chi_f(G) \leq t$.*

Proof: There is a fractional coloring $\frac{\mathbf{c}}{b} \in \mathcal{F}(G_1)$ satisfying $g_{G_1}(\frac{\mathbf{c}_1}{b_1}) = t$. Since every extreme fractional coloring of $\mathcal{F}_t(H)$ is extensible in $\mathcal{F}_t(G_2)$, so there exist fractional colorings $\frac{\mathbf{c}_1}{b_1}, \frac{\mathbf{c}_2}{b_2}, \cdots, \frac{\mathbf{c}_r}{b_r} \in \mathcal{F}_t(G_2)$ so that $i_{G_2}^H(\frac{\mathbf{c}_1}{b_1}), i_{G_2}^H(\frac{\mathbf{c}_2}{b_2}), \cdots, i_{G_2}^H(\frac{\mathbf{c}_r}{b_r})$ are all extreme fractional colorings in $\mathcal{F}_t(H)$. The fractional coloring $i_{G_1}^H(\frac{\mathbf{c}}{b})$ can be written as a linear combination of the extreme ones. Hence, there exist $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{Q} \cap [0,1]$ such that $\sum_{i=1}^r \lambda_i = 1$ and

$$i_{G_1}^H(\frac{\mathbf{c}}{b}) = \sum_{i=1}^r \lambda_i i_{G_2}^H(\frac{\mathbf{c}_i}{b_i}).$$

Let $\frac{\mathbf{c}'}{b'} = \sum_{i=1}^r \lambda_i \frac{\mathbf{c}_i}{b_i} \in \mathcal{F}_t(G_2)$. We have

$$i_{G_1}^H(\frac{\mathbf{c}}{b}) = i_{G_2}^H(\frac{\mathbf{c}'}{b'}).$$

Applying Lemma 4, there exists a fractional coloring $\frac{\mathbf{c}''}{b''} \in \mathcal{F}_t(G)$. Therefore, $\chi_f(G) \leq t$. $\square$

Corollary 1 *Let G be a graph. Suppose that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_r$ for some positive integer r. Then*

$$\chi_f(G) = \max\{\chi_f(G_1), \chi_f(G_2)\}.$$

Proof: Without loss of generality, we assume $\chi_f(G_1) \geq \chi_f(G_2)$. Let $t = \chi_f(G_1)$. We have $t \geq r$ as G_1 contains K_r. Because $\chi_f(G_2) \leq t$, so we have $\mathcal{F}_t(G_2) \neq \emptyset$. Since H is a complete graph, then $\mathcal{F}_t(H)$ contains only one fractional coloring, namely, color all vertices of H using distinct colors. It is trivial that any extreme fractional colorings in $\mathcal{F}_t(H)$ is extensible in G_2. Applying Theorem 3, we have $\chi_f(G) \leq t$. The other direction is trivial. $\square$

Let uv be a non-edge of a graph G_2. We denote G_2+uv the supergraph of G_2 by adding the edge uv and denote G_2/uv be the quotient graph by identifying the vertex u and the vertex v.

Lemma 5 *Let G be a graph. Suppose that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{u, v\}$.*

1. *If uv is an edge of G, then*

$$\chi_f(G) = \max\{\chi_f(G_1), \chi_f(G_2)\}.$$

2. *If uv is not an edge of G, then*

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\}.$$

Proof: Part 1 is a simple application of Corollary 1. For the proof of part 2, let $t = \max\{\chi_f(G_1), \chi_f(G_2+uv), \chi_f(G_2/uv)\}$. Note $t \geq 2$. All fractional colorings of $\mathcal{F}_t(\{uv\})$ can be represented by the following weighted Venn Diagram.

There are two extreme fractional colorings $s = 0$ and $s = 1$. The fractional coloring corresponding to $s = 0$ is extensible in $\mathcal{F}_t(G_2)$ since $\chi_f(G + uv) \leq t$. The fractional coloring corresponding to $s = 1$ is extensible in $\mathcal{F}_t(G_2)$ since $\chi_f(G/uv) \leq t$. Applying Theorem 3, we have $\chi_f(G) \leq t$. Part 2 is proved. $\square$

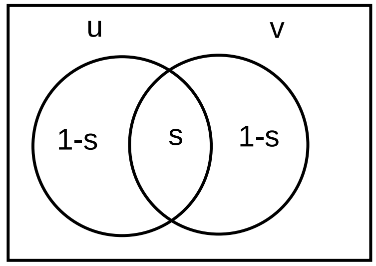


Figure 1: Fractional colorings on the set of two vertices u and v are represented as a weighted Venn Diagram.

2.3 Fractionally-critical graphs

In this subsection, we will apply our mechanism to triangle-free graphs with maximum degree at most 3.

Recall that a graph G is *k-critical* (for a positive integer k) if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph H of G. For any rational number $t \geq 2$, a graph G is called to be *t-fractionally-critical* if $\chi(G) = t$ and $\chi(H) < t$ for any proper subgraph H of G. For simplicity, we say G is *fractionally-critical* if G is $\chi_f(G)$-fractionally-critical.

We will study the properties of fractionally-critical graphs. The following lemma is a consequence of Corollary 1.

Lemma 6 *Suppose that G is a fractionally-critical graph with $\chi_f(G) \geq 2$. Then G is 2-connected. Moreover, if G has a vertex-cut $\{u, v\}$, then uv is not an edge of G.*

For any vertex u of a graph G and a positive integer i, let $N_G^i(u) = \{v \in V : v \neq u$ and there is a path of length i connecting u and $v\}$.

Lemma 7 *Suppose that G is a fractionally-critical triangle-free graph with $\Delta \leq 3$ and $\frac{11}{4} < \chi_f(G) < 3$. Then for any vertex x and any 5-cycle C, we have $|V(C) \cap N_G^2(x)| \leq 3$.*

Proof: Let $t = \chi_f(G)$. We have $\frac{11}{4} < t < 3$. We will prove the statement by contradiction. Suppose that there is a vertex x and a 5-cycle C satisfying $|V(C) \cap N_G^2(x)| \geq 4$. Combined with the fact G is triangle-free, we have the following cases.

1. $|V(C) \cap N_G^2(x)| = 5$. It is easy to check that G contains the following subgraph G_9 as shown in Figure 2. Since G is 2-connected, G_9 is the entire graph (in [5]). Thus $\chi_f(G) \leq \frac{8}{3} < t$. Contradiction!

2. $|V(C) \cap N_G^2(x)| = 4$ and $|V(C) \cap N_G^1(x)| = 1$. It is easy to check G is the unique graph G_8 on 8 vertices as shown in Figure 3. Thus $\chi_f(G) \leq \frac{8}{3} < t$. Contradiction!

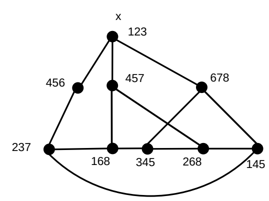


Figure 2: An 8:3-coloring of G_9, where $|V(C) \cap N_G^2(x)| = 5$.

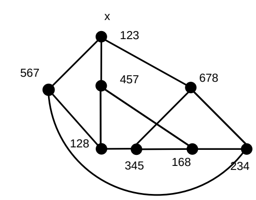


Figure 3: An 8 : 3-coloring of G_8, where $|V(C) \cap N_G^2(x)| = 4$ and $|V(C) \cap N_G^1(x)| = 1$.

3. $|V(C) \cap N_G^2(x)| = 4$ and there exists one vertex on C having the distance of 3 to x. Hatami and Zhu, in [5], showed that G contains one of the five graphs in Figure 4 as a subgraph. (Note that some of the marked vertices u, v, and w may be missing or overlapped. These degenerated cases result a smaller vertex-cut, and can be covered in a similar but easier way. We will discuss them at the end of this proof.)

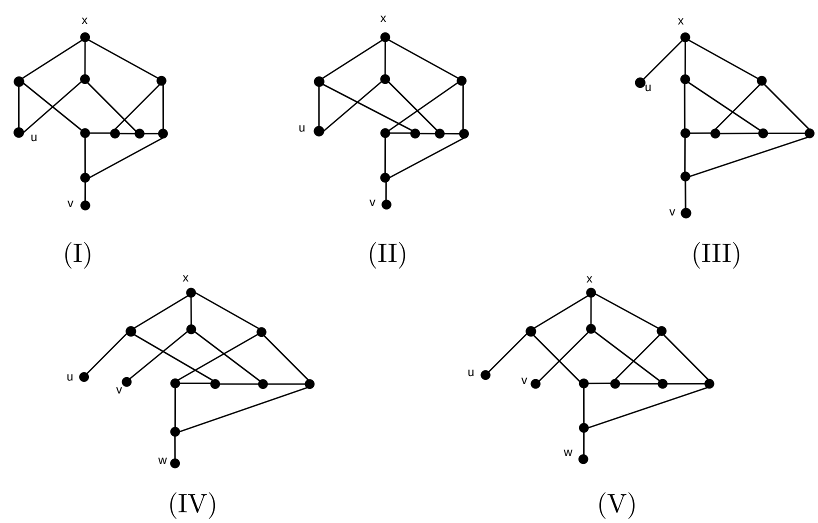
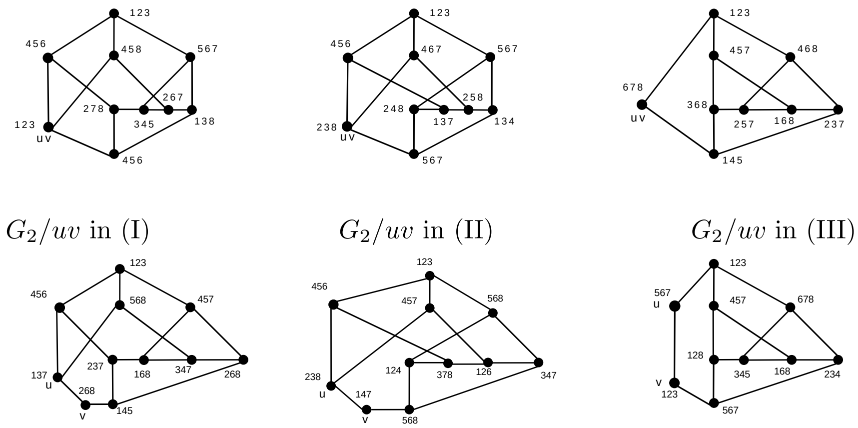


Figure 4: All possible cases of $|V(C) \cap N_G^2(x)| = 4$ and $|V(C) \setminus (N_G^1(x) \cup N_G^2(x))| = 1$.

If G contains a subgraph of type (I), (II), or (III), then G has a pair of cut-vertex $\{u, v\}$. Let G_1 and G_2 be the two connected subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{u, v\}$, and $x \in G_2$. In all three cases, $G_2 + uv$ and G_2/uv are 8:3-colorable. Please see Figure 5.



G_2/uv in (I) G_2/uv in (II) G_2/uv in (III)

G_2+uv in (I) G_2+uv in (II) G_2+uv in (III)

Figure 5: G_2+uv and G_2/uv are all $8\!:\!3$-colorable in cases (I), (II), and (III).

Applying Lemma 5, we have

$$\begin{aligned}\chi_f(G) &\leq \max\{\chi_f(G_1), \chi_f(G_2/uv), \chi_f(G_2+uv)\} \\ &\leq \max\{\chi_f(G_1), \frac{8}{3}\}.\end{aligned}$$

Since $\chi_f(G) \geq t > \frac{8}{3}$, then we must have $\chi_f(G_1) \geq \chi_f(G) = t$. This is a contradiction to the assumption that G is fractionally-critical.

If G contains one of the subgraphs (IV) and (V), then G has a vertex-cut set $H = \{u, v, w\}$ as shown in Figure 4. Let G_1 and G_2 be the two connected subgraphs of G such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u, v, w\}$, and $x \in G_2$. We shall show $\chi_f(G_1) = t = \chi_f(G)$. Suppose not. Then we can assume $\chi_f(G_1) \leq t_0 < t$, where $\frac{8}{3} < \frac{11}{4} < t_0 < t < 3$. The fractional colorings $\mathcal{F}_{t_0}(H)$ can be described as

$$\mathcal{F}_{t_0}(H) = \left\{ (x,y,z,s) \in \mathbb{Q}^4 \;\middle|\; \begin{array}{c} x+y+s \leq 1 \\ x+z+s \leq 1 \\ y+z+s \leq 1 \\ 3-x-y-z-2s \leq t_0 \\ x,y,z,s \geq 0 \end{array} \right\}.$$

See the weighted Venn Diagram in Figure 6.

The extreme fractional colorings of $\mathcal{F}_{t_0}(H)$ are:

(a) $x = y = z = 0$ and $s = 1$.

(b) $x = 1$ and $y = z = s = 0$.

(c) $y = 1$ and $x = z = s = 0$.

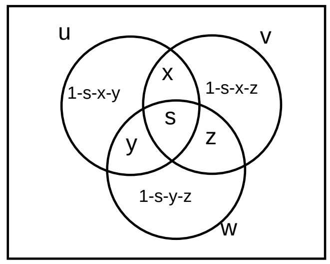


Figure 6: The general fractional colorings on the vertices u, v, and w.

(d) $z = 1$ and $x = y = s = 0$.

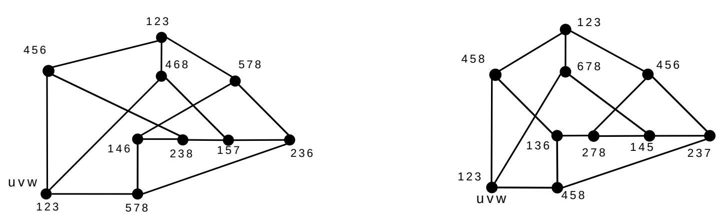
(e) $x = 3 - t_0$ and $y = z = s = 0$.

(f) $y = 3 - t_0$ and $x = z = s = 0$.

(g) $z = 3 - t_0$ and $x = y = s = 0$.

We will show that all 7 extreme fractional colorings are extensible in $\mathcal{F}_{t_0}(G_2)$.

(a) Let G_2/uvw be the quotient graph by identifying u, v, and w as one vertex. The fractional coloring $(0, 0, 0, 1)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f(G_2/uvw) \leq t_0$. This is verified by Figure 7.



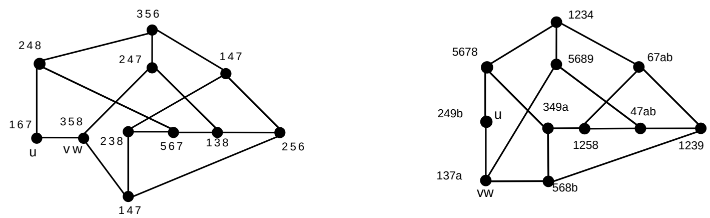
G_2/uvw in (IV) G_2/uvw in (V)

Figure 7: G_2/uvw are 8:3-colorable for subgraphs (IV) and (V).

(b) Let $(G_2/vw)+u(vw)$ be the graph obtained by first identifying v and w as one vertex vw, and then adding an edge $u(vw)$. The fractional coloring $(0, 0, 1, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f((G_2/vw)+u(vw)) \leq t_0$. This is verified by Figure 8.

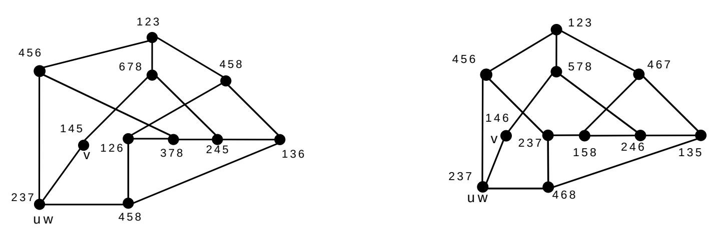
(c) Let $(G_2/uw) + v(uw)$ be the graph obtained by first identifying u and w as one vertex uw, and then adding an edge $v(uw)$. The fractional coloring $(1, 0, 0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f((G_2/uw) + v(uw)) \leq t_0$. This is verified by Figure 9.

(d) Let $(G_2/uv)+w(uv)$ be the graph obtained by first identifying u and v as one vertex uv, and then adding an edge $w(uv)$. The fractional



$(G_2/vw) + u(vw)$ in (IV) $(G_2/vw) + u(vw)$ in (V)

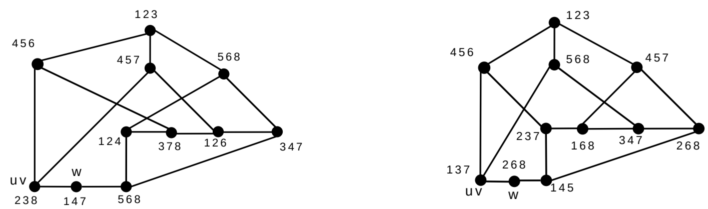
Figure 8: $(G_2/vw) + u(vw)$ in (IV) is $8\!:\!3$-colorable, while $(G_2/vw) + u(vw)$ in (V) is $11\!:\!4$-colorable.



$(G_2/uw) + v(uw)$ in (IV) $(G_2/uw) + v(uw)$ in (IV)

Figure 9: Both $(G_2/uw) + v(uw)$ in (IV) and (V) are $8\!:\!3$-colorable.

coloring $(0,0,1,0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$ if and only if $\chi_f((G_2/uv)+w(uv)) \le t_0$. This is verified by Figure 10.



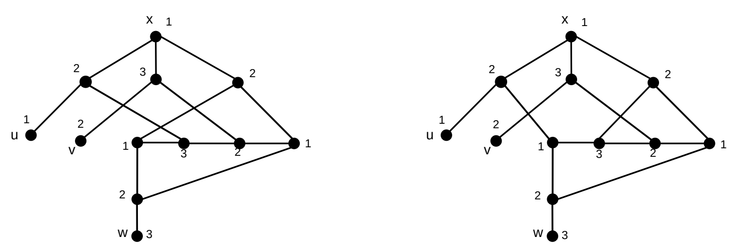
$(G_2/uv) + w(uv)$ in (IV) $(G_2/uv) + w(uv)$ in (V)

Figure 10: Both $(G_2/uv) + w(uv)$ in (IV) and (V) are 8:3-colorable.

(e) Choose $\lambda = 3t_0 - 8$. Since $\frac{8}{3} < \frac{11}{4} < t_0 < 3$, we have $0 < \lambda < 1$. Note that

$$(3 - t_0, 0, 0, 0) = \lambda(0, 0, 0, 0) + (1 - \lambda)(\frac{1}{3}, 0, 0, 0).$$

Observe that $(3 - t_0, 0, 0, 0)$ being fully extensible in $\mathcal{F}(G_2)$ implies that $(3 - t_0, 0, 0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$. To show $(3 - t_0, 0, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$, it suffices to show both $(0, 0, 0, 0)$ and $(\frac{1}{3}, 0, 0, 0)$ are fully extensible in $\mathcal{F}(G_2)$ by Lemma 3 (See Figure 11 and 12).



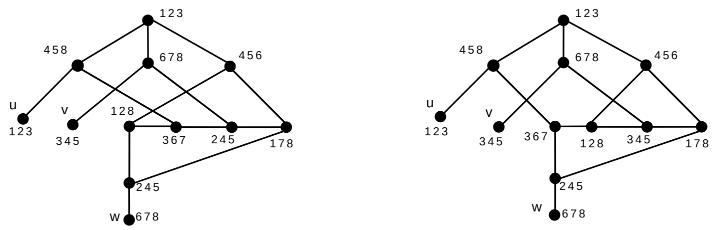
(IV) (V)

Figure 11: The fractional coloring $(0, 0, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$.

(f) Similarly, to show $(0, 3 - t_0, 0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$, it suffices to show $(0, 0, 0, 0)$ and $(0, \frac{1}{3}, 0, 0)$ are fully extensible in $\mathcal{F}(G_2)$ (See Figure 11 and 13).

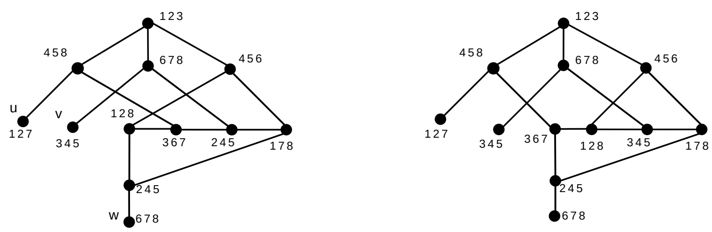
(g) Similarly, to show $(0, 0, 3 - t_0, 0)$ is extensible in $\mathcal{F}_{t_0}(G_2)$, it suffices to show $(0, 0, 0, 0)$ and $(0, 0, \frac{1}{3}, 0)$ are fully extensible in $\mathcal{F}(G_2)$ (See Figure 11 and 14).

Applying Theorem 3, we have $\chi_f(G) \le t_0 < t = \chi_f(G)$. This is a contradiction and so $\chi_f(G_1) = t = \chi_f(G)$. However, G is fractionally-critical. Contradiction!



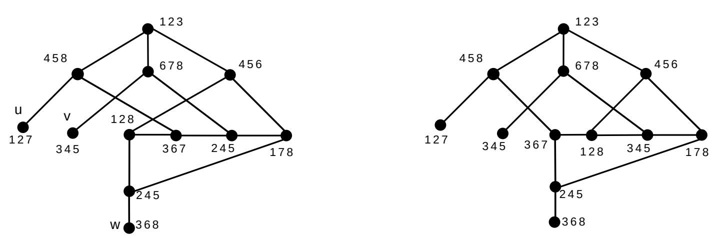
(IV) (V)

Figure 12: The fractional coloring $(\frac{1}{3}, 0, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$.



(IV) (V)

Figure 13: The fractional coloring $(0, \frac{1}{3}, 0, 0)$ is fully extensible in $\mathcal{F}(G_2)$.



(IV) (V)

Figure 14: The fractional coloring $(0, 0, \frac{1}{3}, 0)$ is fully extensible in $\mathcal{F}(G_2)$.

Now we consider the degenerated cases. For graph (I), (II), and (III), $\{u, v\}$ is degenerated into a singleton set. Since G is 2-connected, G is a subgraph of graphs listed in Figure 5. Thus G is $8:3$-colorable. Contradiction! For graph (IV) and (IV), $\{u, v, w\}$ is degenerated into a set H of size at most 2. If $|H| = 1$, then G is one of graphs in Figure 7. If $H = \{y, z\}$, G_2/yz and $G_2 + yz$ are subgraphs of graphs listed in Figures (7, 8, 9, 10). Applying Lemma 5, we get

$$\chi_f(G) \leq \max\{\chi(G_1), \frac{11}{4}\} < \chi_f(G).$$

Contradiction! Hence, Lemma 7 follows. □

Lemma 8 *Suppose that G is a fractionally-critical triangle-free graph with $\Delta \leq 3$ and $\frac{8}{3} < \chi_f(G) < 3$. Then for any vertex x and any 7-cycle C, we have $|V(C) \cap N_G^2(x)| \leq 5$.*

Proof: We prove the statement by contradiction. Suppose that there is a vertex x and a 7-cycle C satisfying $|V(C) \cap N_G^2(x)| \geq 6$. Recall that $|N_G^2(x)| \leq 6$. We have $|V(C) \cap N_G^2(x)| = 6$. Combined with the fact that G is triangle-free and 2-connected, G must be one of the following graphs which are all $8:3$-colorable, see Figure 15. Contradiction!

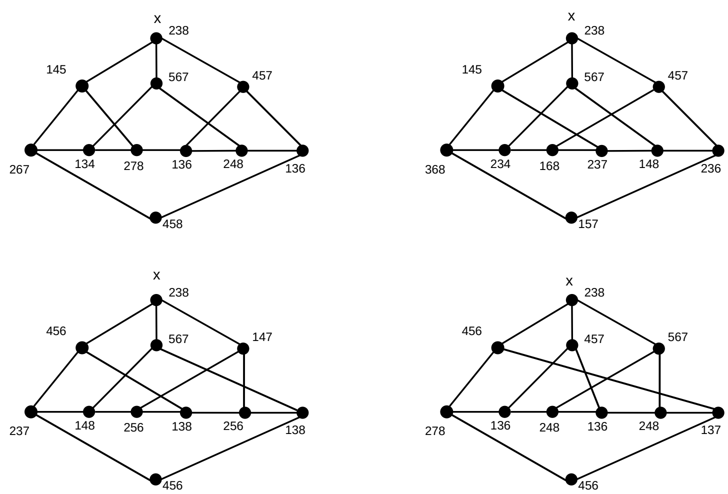


Figure 15: All possible graphs with $|V(C) \cap N_G^2(x)| = 6$.

□

3 Admissible sets and Theorem 4

The following approach is similar to the one used in [5]. A big difference is a new concept "admissible set". Basically, it replaces the independent set X (of

G^* in [5]) by three independent sets $X_1 \cup X_2 \cup X_3$.

Recall that $v \in N_G^i(u)$ if there exists a uv-path of length i in G. A set $X \subset V(G)$ is called *admissible* if X satisfies the following conditions:

1. There is a partition $X = X_1 \cup X_2 \cup X_3$.
2. If $u, v \in X_i$ for some $i \in \{1, 2, 3\}$, then $v \notin N_G^1(u) \cup N_G^3(u) \cup N_G^5(u)$.
3. If $u \in X_i$ and $v \in X_j$ for some i and j satisfying $1 \leq i \neq j \leq 3$, then $v \notin N_G^1(u) \cup N_G^2(u) \cup N_G^4(u)$.

The following key theorem links the $\chi_f(G)$ to a partition of G into admissible sets. We will prove it later.

Theorem 4 *Let G be a fractionally-critical triangle-free graph with Δ at most 3 and $\frac{11}{4} < \chi_f(G) < 3$. Suppose that $V(G)$ can be partitioned into k admissible sets, then*

$$\chi_f(G) \leq 3 - \frac{3}{k+1}. \tag{5}$$

Let $X = X_1 \cup X_2 \cup X_3$ be an admissible set. Inspired by the method used in [5], we define an auxiliary graph $G' = G'(X)$ as follows. For each $i \in \{1, 2, 3\}$, let $Y_i = \Gamma(X_i)$ be the neighbor of X_i in G. By admissible conditions, Y_1, Y_2, and Y_3 are all independent sets of G. Let G' be a graph obtained from G by deleting X; identifying each Y_i as a single vertex y_i for $1 \leq i \leq 3$; and adding three edges y_1y_2, y_2y_3, y_1y_3. We have the following lemma, which will be proved later.

Lemma 9 *Suppose that G is t-fractionally-critical triangle-free graph with Δ at most 3, where $t \in (\frac{11}{4}, 3)$. Let X be an admissible set of G and $G'(X)$ be the graph defined as above. Then $G'(X)$ is 3-colorable.*

Proof of Theorem 4: Suppose that G can be partitioned into k admissible sets, say $V(G) = \cup_{i=1}^k X_i$, where $X_i = X_i^1 \cup X_i^2 \cup X_i^3$. For each $1 \leq i \leq k$ and $1 \leq j \leq 3$, let $Y_i^j = \Gamma(X_i^j)$. From the definition of an admissible set and G being triangle-free, we have Y_i^j is an independent set for all $1 \leq i \leq k$ and $1 \leq j \leq 3$.

By Lemma 9, $G'(X_i)$ is 3-colorable for all $1 \leq i \leq k$. Let c_i be a 3-coloring of $G'(X_i)$ with the color set $\{s_i^1, s_i^2, s_i^3\}$. We use $\mathcal{P}(S)$ to denote the set of all subsets of S. We define $f_i : V(G) \to \mathcal{P}(\{s_i^1, s_i^2, s_i^3\})$ as

$$f_i(v) = \begin{cases} \{c_i(v)\} & \text{if } v \in V(G) - (\cup_{j=1}^3 X_i^j \cup_{j=1}^3 Y_i^j), \\ \{c_i(y_i^j)\} & \text{if } v \in Y_i^j, \\ \{s_i^1, s_i^2, s_i^3\} - c_i(y_i^j) & \text{if } v \in X_i^j. \end{cases}$$

Note that for a fixed $1 \leq i \leq k$, y_i^j denotes the vertex of $G'(X_i)$ obtained from contracting Y_i^j for $j = 1, 2, 3$. Observe that each vertex in X_i^j receives two colors from f_i and every other vertices receive one color. It is clear that any two adjacent vertices receive disjoint colors. Let $\sigma : V(G) \to \mathcal{P}(\cup_{i=1}^k \{s_i^1, s_i^2, s_i^3\})$ be a mapping defined as $\sigma(v) = \cup_{i=1}^k f_i(v)$. Now σ is a $k+1$-fold coloring of G such that each color are drawn from a palette of $3k$ colors and so $\chi_f(G) \le \frac{3k}{k+1} = 3 - \frac{3}{k+1}$.

We completed the proof of theorem 4. □

Before we prove Lemma 9, we first prove a lemma on coloring the graph obtained by splitting the hub of an odd wheel. Let $x_0, x_1, \ldots, x_{2k}$ be the set of vertices of an odd cycle C_{2k+1} in the circular order. Let $Y = \{y_1, y_2, y_3\}$. We construct a graph H as follows:

1. $V(H) = V(C) \cup Y$.
2. $E(C) \subset E(H)$.
3. Each x_i is adjacent to exactly one element of Y.
4. $y_1y_2, y_2y_3, y_1y_3 \in E(H)$.
5. H can have at most one vertex (of y_1, y_2, and y_3) with degree 2.

The graph H can be viewed as splitting the hub of the odd wheel into 3 new hubs where each spoke has to choose one new hub to connect and then connecting all new hubs. One special case it that one of the new hubs has no neighbor in C.

Lemma 10 *Let H be the graph as constructed above. Then H is 3-colorable.*

Proof: Without loss of generality, we assume $d_H(y_1) \ge 3$ and $d_H(y_2) \ge 3$. We construct a proper 3-coloring c of H as follows. First, let $c(y_1) = 1$, $c(y_2) = 2$, and $c(y_3) = 3$.

Again without loss of generality, we assume $(x_0, y_1) \in E(H)$. The neighbors of y_1 divide $V(C)$ into several intervals. The vertices in each interval are either connecting to y_2 or y_3 but not connected to y_1. Read the neighbors of y_2 and y_3 in Y counterclockwise, and omit repetitions of y_2y_2 and y_3y_3. There are 4 types of intervals:

Type I: $y_2, y_3, y_2, y_3, \ldots, y_3, y_2$.

Type II: $y_2, y_3, y_2, y_3, \ldots, y_2, y_3$.

Type III: $y_3, y_2, y_3, y_2, \ldots, y_3, y_2$.

Type IV: $y_3, y_2, y_3, y_2, \ldots, y_2, y_3$.

If y_3 is a vertex of degree two in H, then the interval I has only one type, which is degenerated into y_2.

Given an interval I, let $u(I)$ (or $v(I)$) be the common neighbor of y_1 and the left (or right) end of I respectively. We color $u(I)$ and $v(I)$ first and then try to extend it as a proper coloring of I. Sometimes we succeed while sometimes we fail. We ask the question whether we can always get a proper coloring. The yes/no result depends only on the type of I and the coloring combination of $u(I)$ and $v(I)$. In Table 1, the column is classified by the coloring combination of $u(I)$ and $v(I)$, while the row is classified by the types of I. Here "yes" means the coloring process always succeeds, while "no" means it sometimes fails.

	$(2,2)$	$(2,3)$	$(3,2)$	$(3,3)$
Type I	Yes	Yes	Yes	No
Type II	Yes	Yes	No	Yes
Type III	Yes	No	Yes	Yes
Type IV	No	Yes	Yes	Yes

Table 1: Can a coloring be extended to I properly?

An ending vertex w of an interval I is called a *free* end if w's two neighbors outside I receiving the same color. Observe that in all *yes* entries, there exist at least one free end w. Note that each vertex on I has degree 3. We can always color the vertices of I greedily starting from the end not equaling w. Since w is a free end, there is no difficulty coloring w at the end.

Now we put them together. We color the neighbors of y_1 one by one counter-clockwise starting from x_0 according to the following rules:

1. When we meet an interval I of type II or III, we keep the colors of $u(I)$ and $v(I)$ the same.

2. When we meet an interval I of type I or IV, we keep the colors of $u(I)$ and $v(I)$ different.

There are two possibilities. If the last interval obeys the rules, then by Table 1, we can extend this partial coloring into a proper 3-coloring of H. If the last interval does not obey the rules, then we swap the colors 2 and 3 of the neighbors of y_1. By Table 1, this new partial coloring can be extended into a proper 3-coloring of H. We completed the proof. □

A maximal (by inclusion) 2-connected subgraph B of a graph is called a *block* of G. A *Gallai tree* is a connected graph in which all blocks are either complete graphs or odd cycles. A *Gallai forest* is a graph all of whose components are Gallai trees. A *k-Gallai tree (forest)* is a Gallai tree (forest) such that the degree of all vertices are at most $k-1$. A *k-critical graph* is a graph G whose chromatic number is k and the chromatic number of any subgraph is strictly less than k. Gallai showed the following Lemma.

Lemma 11 (Gallai [3]) *If G is a k-critical graph, then the subgraph of G induced on the vertices of degree $k-1$ is a k-Gallai forest.*

Now, we are ready to prove Lemma 9.

Proof of Lemma 9: Write $G' = G'(X)$ for short. Note that the only possible vertices of degree greater than 3 in G' are y_1, y_2, and y_3. We can color y_1, y_2, and y_3 by 1, 2, and 3, respectively. Since the remaining vertices have degree at most 3, we can color G' properly with 4 colors by greedy algorithm.

Suppose that G' is not 3-colorable. Let H be a 4-critical subgraph of G'. Then $d_H(v) = 3$ for all $v \in H$ except for possibly y_1, y_2, and y_3. Let T be the subgraph induced by all $v \in H$ such that $d_H(v) = 3$. Then T is not empty since $|T| \geq |H| - 3 \geq 1$. By Lemma 11 the induced subgraph on T is a 4-Gallai forest. T may contain one or more vertices in $\{y_1, y_2, y_3\}$. Let $T' = T \setminus \{y_1, y_2, y_3\} = V(H) \setminus \{y_1, y_2, y_3\}$. Observe that any induced subgraph of a 4-Gallai forest is still a 4-Gallai forest and so the induced subgraph on T' is also a 4-Gallai forest.

Recall the definition of an admissible set. If $u \in X_i$ and $v \in X_j$ for some i and j satisfying $1 \leq i \neq j \leq 3$, then $v \notin N_G^1(u) \cup N_G^2(u) \cup N_G^4(u)$. This implies that any vertex x in T' can have at most one neighbor in $\{y_1, y_2, y_3\}$. Note $d_H(x) = 3$. We have $d_{T'}(x) \geq 2$.

Let B be a leaf block in the Gallai-forest T'. Then B is a complete graph or odd cycles by the definition of the Gallai-forest. B can not be a single vertex or K_2 since every vertex in B has at least two neighbors in T'. Since G is triangle-free, then B must be an odd cycle C_{2r+1} with $r \geq 2$.

Case (a): $|N_H(B) \cap \{y_1, y_2, y_3\}| \geq 2$. Since H is 4-critical, $H \backslash B$ is 3-colorable. Let c be a proper 3-coloring of $H \backslash B$. Since $H \backslash B$ contains a triangle $y_1y_2y_3$, so y_1, y_2, and y_3 receive different colors. Since $|N_H(B) \cap \{y_1, y_2, y_3\}| \geq 2$ and Lemma 10, we can extend the coloring c to all vertices on B as well. Thus H is 3-colorable. Contradiction.

Case (b): $|N_H(B) \cap \{y_1, y_2, y_3\}| = 1$. Since B is a leaf block, at most one vertex, say v_0, can connect to another block in T'. List the vertices of B in the circular order as $v_0, v_1, v_2, \ldots, v_{2r}$. Then all $v_1, v_2, \ldots, v_{2r}$ connect to one y_i, say y_1. This implies that for $i = 1, 2, \ldots, 2r$, there exist a vertex $x_i \in X_1$ and a vertex $w_i \in Y_1$ so that v_i-w_i-x_i form a path of length 2. Since G is triangle free, we have $w_i \neq w_{i+1}$ for all $i \in \{1, \ldots, 2r-1\}$. Note that x_i-w_i-v_i-v_{i+1}-w_{i+1}-x_{i+1} forms a path of length 5 unless $x_i = x_{i+1}$.

Recall the admissible conditions: if $u, v \in X_i$ for some $i \in \{1, 2, 3\}$, then $v \notin N_G^1(u) \cup N_G^3(u) \cup N_G^5(u)$. We must have $x_1 = x_2 = \cdots = x_{2r}$. Denote this common vertex by x. Now have

$$\left| N_G^2(x) \cap B \right| \geq 2r.$$

Note $|N_G^2(x)| \leq 6$. We have $2r \leq 6$. The possible values for r are 2 or 3. If $r = 2$, B is a 5-cycle. This is a contradiction to Lemma 7. If $r = 3$, B is a 7-cycle. This is a contradiction to Lemma 8.

The proof of Lemma 9 is finished. □

4 Partition into 42 admissible sets

Theorem 5 *Let G be a triangle-free graph with maximum degree at most 3. Suppose that G is 2-connected and girth$(G) \leq 6$. Then G can be partitioned into at most 42 admissible sets.*

Proof of Theorem 5: We will define a coloring $c\colon V(G) \to [126]$ so that for $i = 1, \ldots, 42$, the i-th admissible set is $c^{-1}(\{3i-2, 3i-1, 3i\})$. We refer to

$\{3i-2, 3i-1, 3i\}$ as a color block for all $i \in \{1, \ldots, 42\}$. Since $4 \leq$ girth(G) ≤ 6, there is a cycle C of length 4, 5, or 6. Let v_{n-1} and v_n be a pair of adjacent vertices of C. Since G is 2-connected, then $G \setminus \{v_{n-1}, v_n\}$ is connected. We can find a vertex v_1 other than v_{n-1} and v_n such that $G \setminus v_1$ is connected. Inductively, for each $i \in \{2, \ldots, n-2\}$, we can find a vertex v_j other than v_{n-1} and v_n such that $G \setminus \{v_1, \ldots, v_{j-1}\}$ is connected. Therefore, we get an order of vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ such that for $j = 1, 2, \ldots, n-2$, the induced graph on $v_j, \ldots, v_n$ is connected.

We color the vertices greedily. Suppose that we have colored $v_1, v_2, \ldots, v_j$. For v_{j+1}, choose a color h satisfying the following:

1. For each $u \in N_G^1(v_{j+1}) \cap \{v_1, v_2, \ldots, v_j\}$, h is not in the same block of $c(u)$.
2. For each $u \in \left(N_G^3(v_{j+1}) \cup N_G^5(v_{j+1})\right) \cap \{v_1, v_2, \ldots, v_j\}$, $h \neq c(u)$.
3. For each $u \in \left(N_G^2(v_{j+1}) \cup N_G^4(v_{j+1})\right) \cap \{v_1, v_2, \ldots, v_j\}$, h could equal to $c(u)$ but not equal to the other two colors in the color block of $c(u)$.

For $j \leq n-2$, there is at least one vertex in $N_G^1(v_{j+1})$ and one vertex in $N_G^2(v_{j+1})$ still uncolored. Thus $\left|N_G^1(v_{j+1}) \cap \{v_1, v_2, \ldots, v_j\}\right| \leq 2$, $\left|N_G^2(v_{j+1}) \cap \{v_1, v_2, \ldots, v_j\}\right| \leq 5$, $|N_G^3(v_{j+1})| \leq 12$, $|N_G^4(v_{j+1})| \leq 24$, and $|N_G^5(v_{j+1})| \leq 48$. Since

$$3 \times 2 + 2 \times (5 + 24) + (12 + 48) = 124 < 126,$$

it is always possible to color the vertex v_{j+1} properly.

It remains to color v_{n-1} and v_n properly. Note both v_{n-1} and v_n are on the cycle C. Let us count color redundancy according to the type of the cycle C.

Case C_4**:** For any vertex v on C_4, there are two vertices in $N_G^1(v) \cap N_G^3(v)$. We also have $|N_G^2(v)| \leq 5$ and $|N_G^4(v)| \leq 23$. Thus the number of forbidden colors for v is at most

$$3 \times 3 + 2 \times (5 + 23) + (12 + 48) - 2 = 123 < 126.$$

Case C_5**:** For any vertex v on C_5, there are two vertices in $N_G^1(v) \cap N_G^4(v)$. We also have $|N_G^5(v)| \leq 47$. The number of forbidden colors for v is at most

$$3 \times 3 + 2 \times (6 + 24) + (12 + 47) - 2 \times 2 = 124 < 126.$$

Case C_6**:** For any vertex v on C_6, there are two vertices in $N_G^1(v) \cap N_G^5(v)$ and two vertices in $N_G^2(v) \cap N_G^4(v)$. We also have $|N_G^3(v)| \leq 11$. Thus the number of forbidden colors for v is at most

$$3 \times 3 + 2 \times (6 + 24) + (11 + 48) - 2 - 2 \times 2 = 122 < 126.$$

In each subcase, we can find a color for v_{n-1} and v_n. □

Proof of Theorem 1: Suppose that there exists a graph G which is triangle-free, $\Delta \leq 3$, and $\chi_f(G) > 3 - \frac{3}{43}$. Without loss of generality, we can assume G has the smallest number of vertices among all such graphs. Then G is 2-connected and fractionally-critical.

If girth$(G) \geq 7$, Hatami and Zhu, in [5], showed $\chi_f(G) \leq 2.78571 \leq 3 - \frac{3}{43}$. Contradiction!

If girth$(G) \leq 6$, Theorem 5 states that G can be partitioned into 42 admissible sets. By Theorem 4, we have $\chi_f(G) \leq 3 - \frac{3}{k+1} = 3 - \frac{3}{43}$. Contradiction! □

5 Concluding Remarks

The reader may notice that we developed a heavy mechanism to prove Lemma 7 and 8. This is because in general $G'(X)$ (in the proof of Lemma 9) could be 4-chromatic as shown in the figure 16.

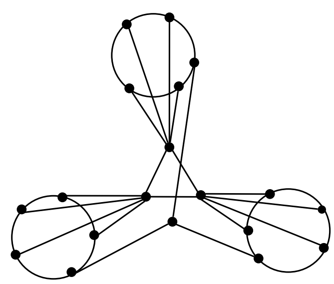


Figure 16: A difficult case: $\chi(G'(X)) = 4$.

However, if G is fractionally-critical, then these bad cases can be avoided by the admissible conditions together with Lemma 7 and 8. This is the motivation of section 2.

In this paper, we proved $\chi_f(G) \leq 3 - \frac{3}{43} < 2.9303$ for any triangle-free graphs with maximum degree at most 3. The gap to the conjectured value 2.8 is still substantial. Although there is a possibility to improve the upper bound on $\chi_f(G)$ slightly by modifying the definition of an admissible set, it certainly needs new ideas to settle Heckman and Thomas' conjecture.

In [5], Hatami and Zhu also considered the graph G with the girth $g \geq 5$ and the maximum degree at most 3. They proved $\chi_f(G) \leq c_g$, where the constant c_g goes to $3 - \frac{1}{3}$ as the girth g goes to infinity. Our method does not apply to these cases.

Although one can use our method to obtain an upper bound on $\chi_f(G)$ for general triangle-free graphs, the bound is worse than other known upper bounds. For example, Molloy and Reed, in [8], proved

$$\chi_f(G) \leq \frac{\Delta + \omega + 1}{2}, \tag{6}$$

which is the fractional version of Reed's famous conjecture,

$$\chi(G) \leq \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil . \tag{7}$$

Even for $\Delta = 4$, it already implies $\chi_f(G) \leq 3.5$. For large Δ, Johansson, in [7], proved the much stronger fact that there exists a fixed constant c so that $\chi(G) \leq \frac{c\Delta(G)}{\ln \Delta(G)}$ for every triangle-free graph G, provided Δ is large enough. So $\chi_f(G) \leq \frac{c\Delta(G)}{\ln \Delta(G)}$ for every triangle-free graph with sufficiently large maximum degree. Also, Reed, in [9], proved inequality (7) holds provided that Δ is sufficiently large and ω is sufficiently close to Δ.

Borodin and Kostochka, in [1], conjectured if $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G)-1$ then

$$\chi(G) \leq \Delta(G) - 1. \tag{8}$$

Molloy and Reed proved this conjecture holds for sufficiently large Δ. It would be interesting to consider the fractional version of this conjecture for small Δ.

The frame-work of the convex structure of fractional colorings is very useful in studying the properties of fractionally-critical graphs. These results hold for the general graphs and are of independent interest. We have carefully separated them from the rest of the paper.

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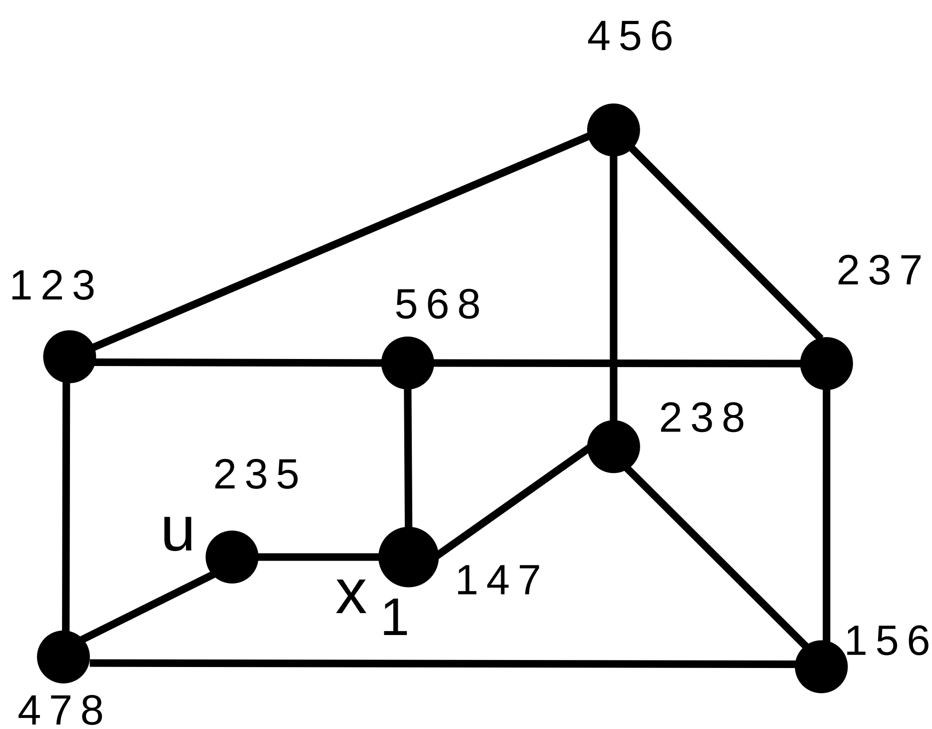
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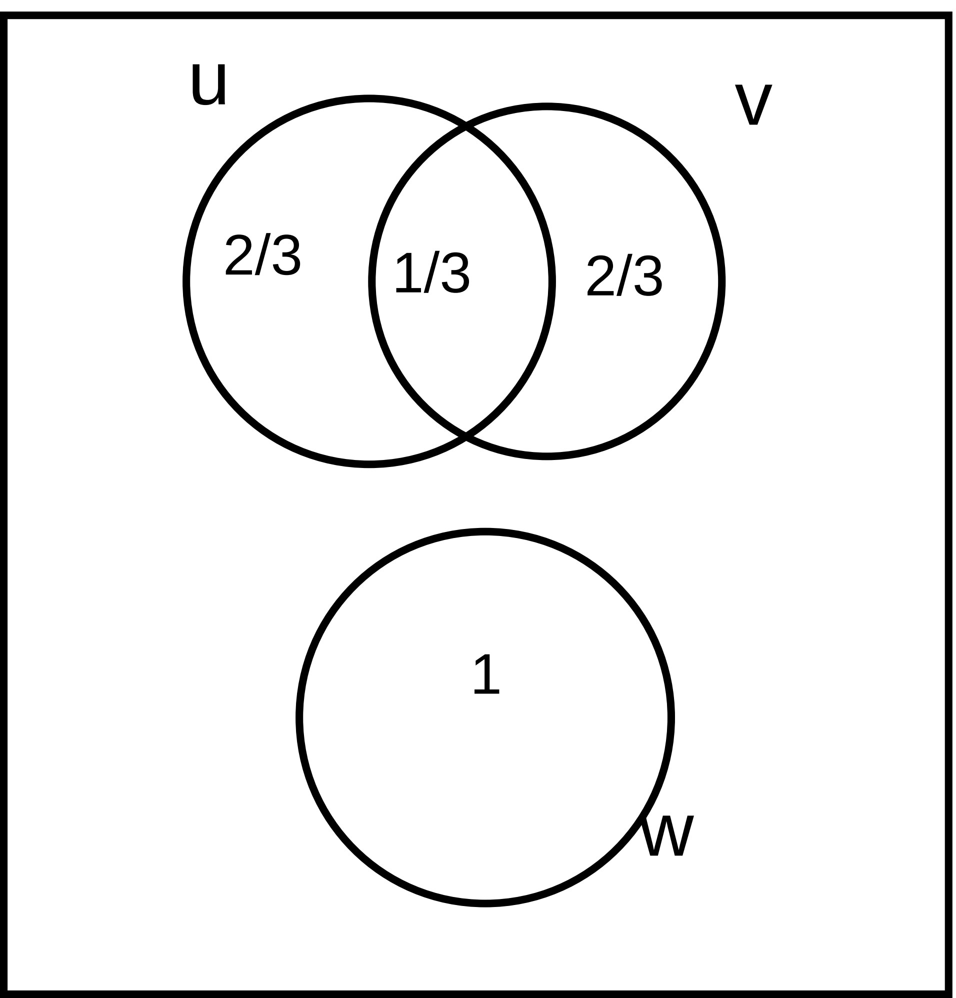
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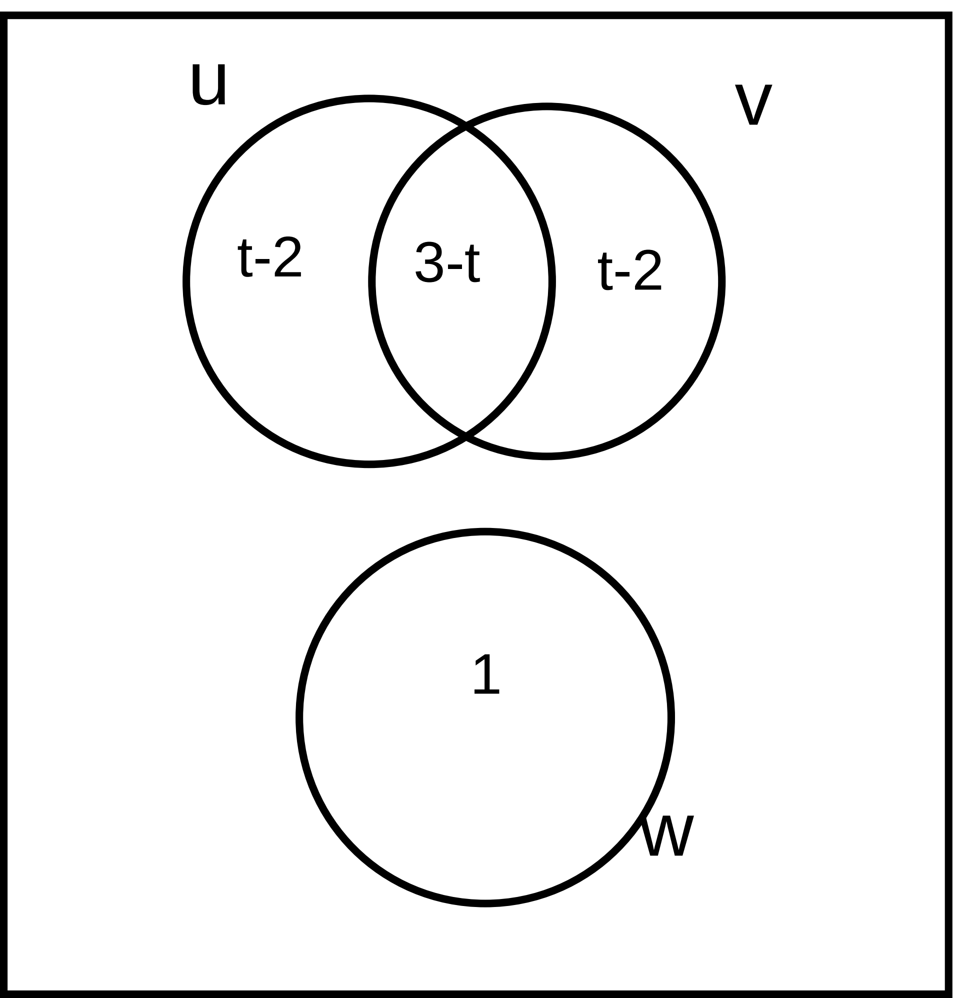
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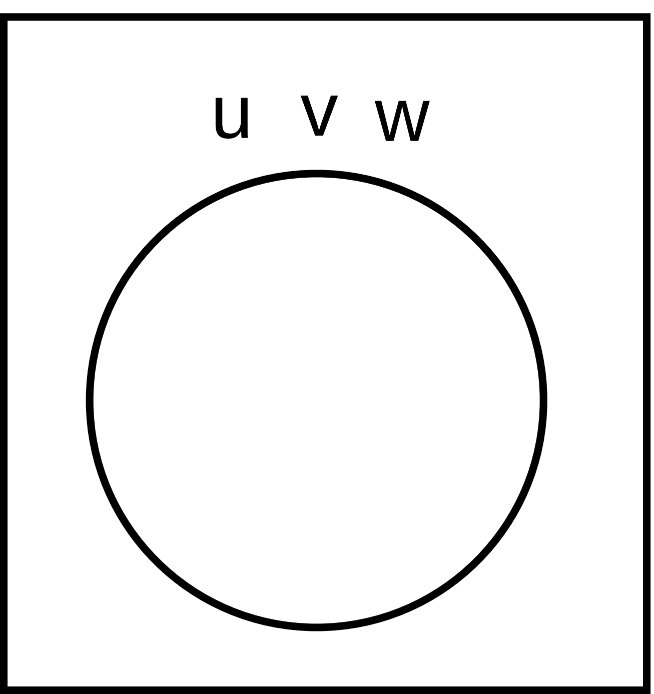
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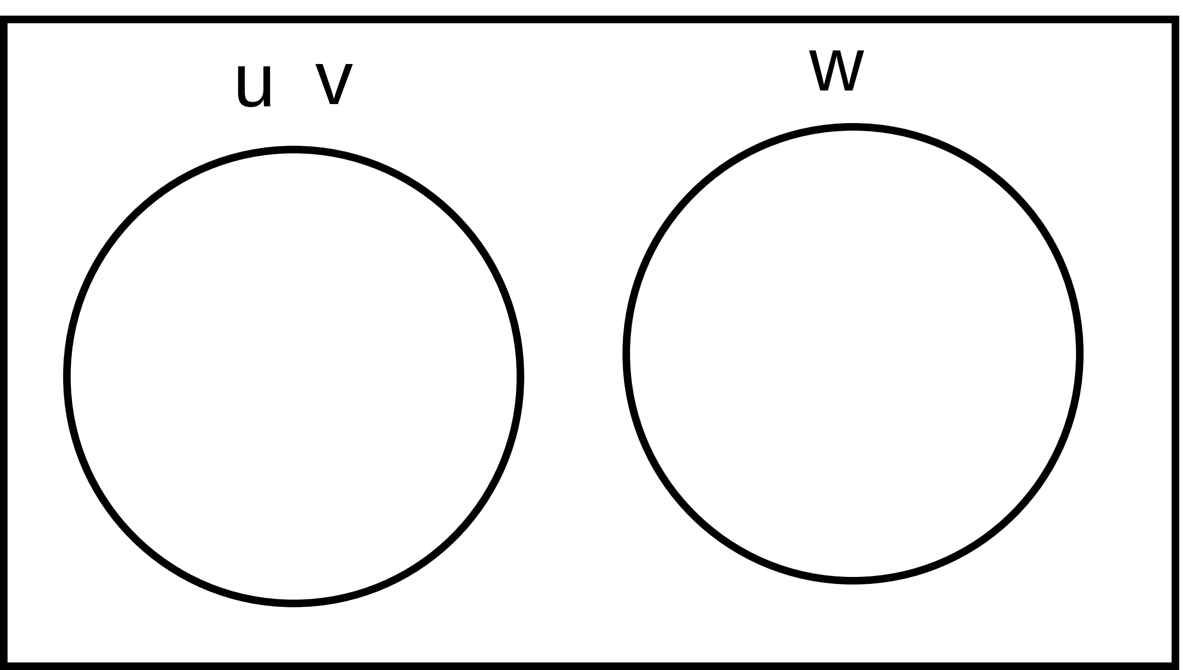
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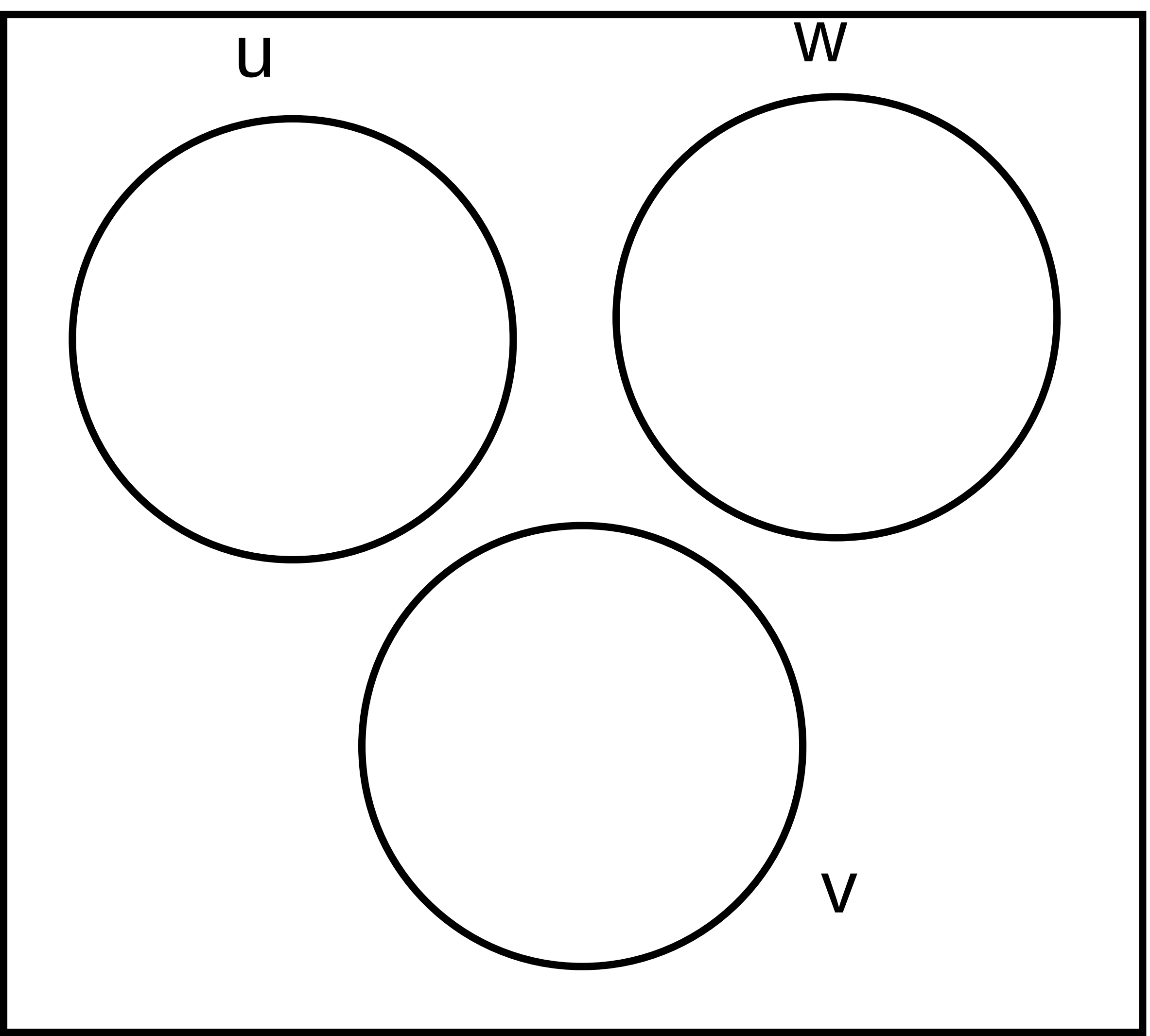
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