A FRACTIONAL ANALOGUE OF BROOKS' THEOREM

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Abstract. Let $\Delta(G)$ be the maximum degree of a graph G. Brooks' theorem states that the only connected graphs with chromatic number $\chi(G) = \Delta(G) + 1$ are complete graphs and odd cycles. We prove a fractional analogue of Brooks' theorem in this paper. Namely, we classify all connected graphs G such that the fractional chromatic number $\chi_f(G)$ is at least $\Delta(G)$. These graphs are complete graphs, odd cycles, C_8^2, $C_5 \boxtimes K_2$, and graphs whose clique number $\omega(G)$ equals the maximum degree $\Delta(G)$. Among the two sporadic graphs, the graph C_8^2 is the square graph of cycle C_8 while the other graph $C_5 \boxtimes K_2$ is the strong product of C_5 and K_2. In fact, we prove a stronger result; if a connected graph G with $\Delta(G) \geq 4$ is not one of the graphs listed above, then we have $\chi_f(G) \leq \Delta(G) - \frac{2}{67}$.

Key words. Fractional chromatic number, Brooks' theorem, chromatic number.

AMS subject classifications. 05C72, 05C15

1. Introduction. The chromatic number of graphs with bounded degrees has been studied for many years. Brooks' theorem perhaps is one of the most fundamental results; it is included by many textbooks on graph theory. Given a simple connected graph G, let $\Delta(G)$ be the maximum degree, $\omega(G)$ be the clique number, and $\chi(G)$ be the chromatic number. Brooks' theorem states that $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle. Reed [10] proved that $\chi(G) \leq \Delta(G) - 1$ if $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq \Delta_0$ for some large constant Δ_0. This excellent result was proved by probabilistic methods, and Δ_0 is at least hundreds. Before this result, Borodin and Kostochka [1] made the following conjecture.

Conjecture [1]: Suppose that G is a connected graph. If $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq 9$, then we have

$$\chi(G) \leq \Delta(G) - 1.$$

If the conjecture is true, then it is best possible since there is a K_8-free graph $G = C_5 \boxtimes K_3$ (actually K_7-free, see Figure 1) with $\Delta(G) = 8$ and $\chi(G) = 8$.

Here we use the following notation of the strong product. Given two graphs G and H, the *strong product* $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$, and (a, x) is connected to (b, y) if one of the following holds

- $a = b$ and $xy \in E(H)$,
- $ab \in E(G)$ and $x = y$,
- $ab \in E(G)$ and $xy \in E(H)$.

Reed's result [10] settled Borodin and Kostochka's conjecture for sufficiently large $\Delta(G)$, but the cases with small $\Delta(G)$ are hard to cover using the probabilistic method.

In this paper we consider a fractional analogue of this problem. The fractional chromatic number $\chi_f(G)$ can be defined as follows. A *b-fold coloring* of G assigns a set of b colors to each vertex such that any two adjacent vertices receive disjoint sets of colors. We say a graph G is *a:b-colorable* if there is a b-fold coloring of G in which each color is drawn from a palette of a colors. We refer to such a coloring as an a:b-coloring. The *b-fold coloring number*, denoted by $\chi_b(G)$, is the smallest integer a such that G has an a:b-coloring. Note

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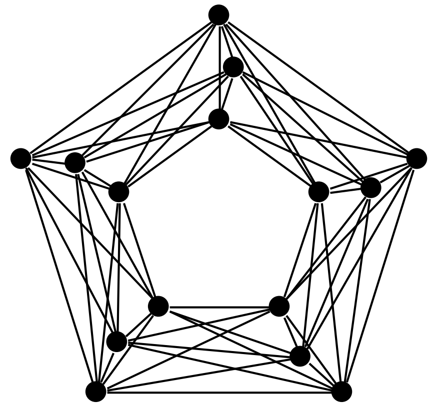


FIG. 1.1. *The graph $C_5 \boxtimes K_3$.*

that $\chi_1(G) = \chi(G)$. It was shown that $\chi_{a+b}(G) \le \chi_a(G) + \chi_b(G)$. The *fractional chromatic number* $\chi_f(G)$ is $\lim\limits_{b\to\infty} \frac{\chi_b(G)}{b}$.

By the definition, we have $\chi_f(G) \le \chi(G)$. The fractional chromatic number can be viewed as a relaxation of the chromatic number. Many problems involving the chromatic number can be asked again using the fractional chromatic number. The fractional analogue often has a simpler solution than the original problem. For example, the famous $\omega - \Delta - \chi$ conjecture of Reed [9] states that for any simple graph G, we have

$$\chi(G) \le \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

The fractional analogue of $\omega - \Delta - \chi$ conjecture was proved by Molloy and Reed [8]; they actually proved a stronger result with ceiling removed, i.e.,

$$\chi_f(G) \le \frac{\omega(G) + \Delta(G) + 1}{2}. \tag{1.1}$$

In this paper, we classify all connected graphs G with $\chi_f(G) \ge \Delta(G)$.

THEOREM 1.1. *A connected graph G satisfies $\chi_f(G) \ge \Delta(G)$ if and only if G is one of the following*

1. *a complete graph,*
2. *an odd cycle,*
3. *a graph with $\omega(G) = \Delta(G)$,*
4. *C_8^2,*
5. *$C_5 \boxtimes K_2$.*

For the complete graph K_n, we have $\chi_f(K_n) = n$ and $\Delta(K_n) = n-1$. For the odd cycle C_{2k+1}, we have $\chi_f(C_{2k+1}) = 2 + \frac{1}{k}$ and $\Delta(C_{2k+1}) = 2$. If G is neither a complete graph nor an odd cycle but contains a clique of size $\Delta(G)$, then we have

$$\Delta(G) \le \omega(G) \le \chi_f(G) \le \chi(G) \le \Delta(G). \tag{1.2}$$

The last inequality is from Brooks' theorem. The sequence of inequalities above implies $\chi_f(G) = \Delta(G)$.

If G is a vertex-transitive graph, then we have [11]

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)},$$

where $\alpha(G)$ is the independence number of G. Note that both graphs C_8^2 and $C_5 \boxtimes K_2$ are vertex-transitive and have the independence number 2. Thus we have

$$\chi_f(C_8^2) = 4 = \Delta(C_8^2) \quad \text{and} \quad \chi_f(C_5 \boxtimes K_2) = 5 = \Delta(C_5 \boxtimes K_2).$$

C_8^2 $\qquad$ $C_5 \boxtimes K_2$

FIG. 1.2. *The graph C_8^2 and $C_5 \boxtimes K_2$.*

Actually, Theorem 1.1 is a corollary of the following stronger result.

THEOREM 1.2. *Assume that a connected graph G is neither C_8^2 nor $C_5 \boxtimes K_2$. If $\Delta(G) \geq 4$ and $\omega(G) \leq \Delta(G) - 1$, then we have*

$$\chi_f(G) \leq \Delta(G) - \frac{2}{67}.$$

Remark: In the case $\Delta(G) = 3$, Heckman and Thomas [5] conjectured that $\chi_f(G) \leq 14/5$ if G is triangle-free. Hatami and Zhu [4] proved $\chi_f(G) \leq 3 - \frac{3}{64}$ for any triangle-free graph G with $\Delta(G) \leq 3$. The second and third authors showed an improved result $\chi_f(G) \leq 3 - \frac{3}{43}$ in the previous paper [7]. Thus we need only consider the cases $\Delta(G) \geq 4$. For any connected graph G with sufficiently large $\Delta(G)$ and $\omega(G) \leq \Delta(G) - 1$, Reed's result [10] $\chi(G) \leq \Delta(G) - 1$ implies $\chi_f(G) \leq \Delta(G) - 1$. The method introduced in [4] and strengthened in [7], has a strong influence on this paper. The readers are encouraged to read these two papers [4, 7].

Let $f(k) = \inf_G\{\Delta(G) - \chi_f(G)\}$, where the infimum is taken over all connected graphs G with $\Delta(G) = k$ and not one of the graphs listed in Theorem 1.1. Since $\chi_f(G) \geq \omega(G)$, by taking a graph with $\omega(G) = \Delta(G) - 1$, we have $f(k) \leq 1$. Theorem 1.2 says $f(k) \geq \frac{2}{67}$ for any $k \geq 4$. Reed's result [10] implies $f(k) = 1$ for sufficiently large k. Heckman and Thomas [5] conjectured $f(3) = 1/5$. It is an interesting problem to determine the value of $f(k)$ for small k. Here we conjecture $f(4) = f(5) = \frac{1}{3}$. If Borodin and Kostochka's conjecture is true, then $f(k) = 1$ for $k \geq 9$.

Theorem 2 is proved by induction on k. Because the proof is quite long, we split the proof into the following two lemmas.

LEMMA 1.3. *We have $f(4) \geq \frac{2}{67}$.*

LEMMA 1.4. *For each $k \geq 6$, we have $f(k) \geq \min\left\{f(k-1), \frac{1}{2}\right\}$. We also have $f(5) \geq \min\left\{f(4), \frac{1}{3}\right\}$.* It is easy to see the combination of Lemma 1.3 and Lemma 1.4 implies Theorem 1.2. The idea of reduction comes from the first author, who pointed out $f(k) \geq \min\left\{f(k-1), \frac{1}{2}\right\}$ for $k \geq 7$ based on his recent results [6]. The second and third authors orginally proved $f(k) \geq \frac{C}{k^5}$ (for some $C > 0$) using different method in the first version; they also prove the reductions at $k = 5, 6$, which are much harder than the case $k \geq 7$. We do not know whether a similar reduction exists for $k = 4$.

The rest of this paper is organized as follows. In section 2, we will introduce some notation and prove Lemma 1.4. In section 3 and section 4, we will prove $f(4) \geq \frac{2}{67}$.

2. Proof of Lemma 1.4. In this paper, we use the following notation. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex v in G, denoted by $\Gamma_G(v)$, is the set $\{u\colon uv \in E(G)\}$. The *degree* $d_G(v)$ of v is the value of $|\Gamma_G(v)|$. The *independent set* (or *stable set*) is a set S such that no edge with both ends in S. The *independence number* $\alpha(G)$ is the largest size of S among all the independent sets S in G. When $T \subset V(G)$, we use $\alpha_G(T)$ to denote the independence number of the induced subgraph of G on T. Let $\Delta(G)$ be the maximum degree of G. For any two vertex-sets S and T, we define $E_G(S,T)$ as $\{uv \in E(G) : u \in S \text{ and } v \in T\}$. Whenever G is clear under context, we will drop the subscript G for simplicity.

If S is a subset of vertices in G, then *contracting* S means replacing vertices in S by a single fat vertex, denoted by $\underline{S}$, whose incident edges are all edges that were incident to at least one vertex in S, except edges with both ends in S. The new graph obtained by contracting S is denoted by G/S. This operation is also known as *identifying vertices of* S in the literature. For completeness, we allow S to be a single vertex or even the empty set. If S only consists of a single vertex, then $G/S = G$; if $S = \emptyset$, then G/S is the union of G and an isolated vertex. When S consists of 2 or 3 vertices, for convenience, we write G/uv for $G/\{u,v\}$ and G/uvw for $G/\{u,v,w\}$; the fat vertex will be denoted by $\underline{uv}$ and $\underline{uvw}$, respectively. Given two disjoint subsets S_1 and S_2, we can contract S_1 and S_2 sequentially. The order of contractions does not matter; let $G/S_1/S_2$ be the resulted graph. We use $G-S$ to denote the subgraph of G induced by $V(G)-S$.

In order to prove Lemma 1.4, we need use the following theorems due to King [6].

THEOREM 2.1 (King [6]). *If a graph G satisfies $\omega(G) > \frac{2}{3}(\Delta(G)+1)$, then G contains a stable set S meeting every maximum clique.*

THEOREM 2.2 (King [6]). *For a positive integer k, let G be a graph with vertices partitioned into cliques $V_1, \ldots, V_r$. If for every i and every $v \in V_i$, v has at most $\min\{k, |V_i|-k\}$ neighbors outside V_i, then G contains a stable set of size r.*

LEMMA 2.3. *Suppose that G is a connected graph with $\Delta(G) \le 6$ and $\omega(G) \le 5$. Then there exists an independent set meeting all induced copies of K_5 and $C_5 \boxtimes K_2$.*

Proof: We first show that there exists an independent set meeting all copies of K_5. If G contains no K_5, then this is trivial. Otherwise, we can apply Theorem 2.1 to get the desired independent set since $\omega(G) > \frac{2}{3}(\Delta(G)+1)$ is satisfied.

Now we prove the Lemma by contradiction. Suppose the Lemma is false. Let G be a minimum counterexample (with the smallest number of vertices). For any independent set I, let $C(I)$ be the number of induced copies of $C_5 \boxtimes K_2$ in $G-I$. Among all independent sets which meet all copies of K_5, there exists one such independent set I such that $C(I)$ is minimized.

Since $C(I) > 0$, there is an induced copy of $C_5 \boxtimes K_2$ in $G-I$; we use H to denote it. In $C_5 \boxtimes K_2$, there is a unique perfect matching such that identifying the two ends of each edge in this matching results a C_5. An edge in this unique matching is called a *canonical* edge. We define a new graph G' as follows: First we contract all canonical edges in H to get a C_5, where its vertices are called *fat* vertices. Second we add five edges turning the C_5 into a K_5. Observe that each vertex in this C_5 can have at most two neighbors in $G-H$ and $\Delta(G') \le 6$. We will consider the following four cases.

Case 1: There is a K_6 in the new graph G'. Since the original graph G is K_6-free, the K_6 is formed by the following two possible ways.

Subcase 1a: This K_6 contains 5 fat vertices. By the symmetry of H, there is an induced C_5 in H such that the vertices in C_5 contain a common neighbor vertex v in $G \setminus V(H)$, see Figure 2.1. Since H is K_5-free, we can find x, y in this C_5 such that x, y is a non-edge. Let $I' := (I \setminus \{v\}) \cup \{x,y\}$; I' is also an independent set. Observe that v is not in any K_5 in $G-I'$. Thus the set I' is also an independent set and meets every K_5 in G. Since $C_5 \boxtimes K_2$ is a 5-regular graph, any copy of $C_5 \boxtimes K_2$ containing v must contain at least one of x and y.

Thus, $C(I') < C(I)$. Contradiction!

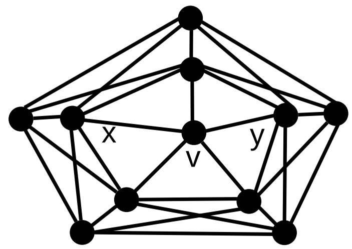


FIG. 2.1. *Subcase 1a.*

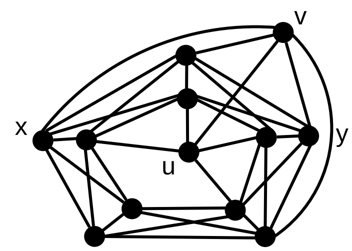


FIG. 2.2. *Subcase 1b.*

Subcase 1b: This K_6 contains 4 fat vertices. Let u, v be the other two vertices. By the symmetry of H, there is a unique way to connect u and v to H as shown by Figure 2.2. Since uv is an edge, one of u and v is not in I. We assume $u \notin I$. Let $\{x, y\} \subset \Gamma_G(v) \cap V(H)$ as shown in Figure 2.2 and $I' = I \setminus \{v\} \cup \{x, y\}$. Observe that I' is an independent set and v is not in a K_5 in $G - I'$. Thus I' is an independent set meeting each K_5 in G. Since each $C_5 \boxtimes K_2$ containing v must contain one of x and y. Thus $C(I') < C(I)$. Contradiction!

Case 2: There is a K_5 intersecting H with 4 vertices. Let v be the vertex of this K_5 but not in H, see Figure 2.3. We have two subcases.

Subcase 2a: The vertex v has another neighbor y in H but not in this K_5. Since H is K_5-free, we can select a vertex x in this K_5 such that xy is not an edge of G. Let $I' := I \setminus \{v\} \cup \{x, y\}$. Note that v is not in a K_5 in $G - I'$, and I' is an independent set. Thus I' is an independent set meeting each K_5 in G. Since any $C_5 \boxtimes K_2$ containing v must contain one of x and y, we have $C(I') < C(I)$. Contradiction!

Subcase 2b: All neighbors of v in H are in this K_5. Let x be any vertex in this K_5 other than v, and $I' := I \setminus \{v\} \cup \{x\}$. In this case, there is only one K_5 containing v. Thus, I' is also an independent set meeting every copy of K_5 in G. Observe that $\Gamma_G(v) \setminus \{x\}$ is disconnected. If $v \in H' = C_5 \boxtimes K_2$, then $\Gamma_G(v) \cap H'$ is connected. Thus v is not in a $C_5 \boxtimes K_2$ in $G - I'$ and $C(I') < C(I)$. Contradiction!

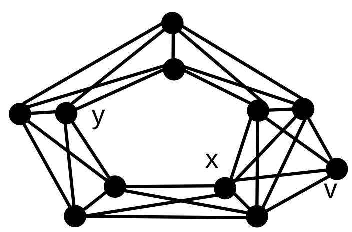


FIG. 2.3. *Case 2.*

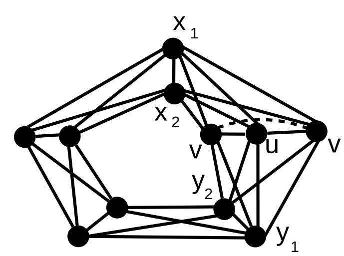


FIG. 2.4. *Case 3.*

Case 3: There is an induced subgraph H' isomorphic to $C_5 \boxtimes K_2$ such that H' and H are intersecting, see Figure 2.4. Since $V(H) \cap V(H') \neq \emptyset$ and $H \neq H'$, we can find a canonical edge uv of H and a canonical edge uv' of H' such that $v \notin V(H')$ and $v' \notin V(H)$. If vv' is a non-edge, then let $I' := I \setminus \{v'\} \cup \{u\}$. It is easy to check I' is still an independent set. We also observe that any possible K_5 containing v' must also contain u. Thus, I' meets every copy of K_5 in G. We have v' in no $C_5 \boxtimes K_2$ in $G - I'$ since vv' is not an edge. We therefore get $C(I') < C(I)$. Contradiction! If vv' is an edge, then locally there are two K_5 intersecting at u, v, and v'; say the other four vertices are x_1, x_2, y_1, y_2, where two cliques are $\{x_1, x_2, u, v, v'\}$ and $\{y_1, y_2, u, v, v'\}$, see Figure 2.4. Let $I' = I \cup \{x_1, y_1\} \setminus \{v'\}$. Note that I' is an independent set and v' is not in a K_5 in $G - I'$. Thus I' is an independent set meeting each K_5 in G. Observe that any copies of $C_5 \boxtimes K_2$ containing v' must contain one of x_1 and y_1; we have $C(I') < C(I)$. Contradiction!.

Case 4: This is the remaining case, G' is K_6-free. We have $\omega(G') \leq 5$ and $|V(G')| < |V(G)|$. By the minimality of G, there is an independent set I' of G' meeting every copy of K_5 and $C_5 \boxtimes K_2$. In I', there is a unique vertex x of the K_5 obtained from contracting canonical edges of H. Let uv be the canonical edge corresponding to x. Let $I'' = I' \setminus \{x\} \cup \{u\}$, we get an independent set I'' of G. Note that any $v \in H \setminus \{u\}$ is not in any K_5 of $G - I''$ by Case 2 as well as not in any $C_5 \boxtimes K_2$ of $G - I''$ by Case 3. Thus I'' hits each K_5 in G and $C(I'') = 0$. Contradiction! □

The following lemma extends Theorem 2.1 when $\omega(G) = 4$; a similar result was proved independently in [2].

LEMMA 2.4. *Let G be a connected graph with $\Delta(G) \leq 5$ and $\omega(G) \leq 4$. If $G \neq C_{2l+1} \boxtimes K_2$ for some $l \geq 2$, then there is an independent set I hitting all copies of K_4 in G.* **Proof:** We will prove it by contradiction. If the lemma is false, then let G be a minimum counterexample. If G is K_4-free, then there is nothing to prove. Otherwise, we consider the clique graph $\mathcal{C}(G)$, whose edge set is the set of all edges appearing in some copy of K_4. Because of $\Delta(G) = 5$, here are all possible connected component of $\mathcal{C}(G)$.

1. $C_t \boxtimes K_2$ for $t \geq 4$. If this type occurs, then every vertex in $C_t \boxtimes K_2$ has degree 5; thus, this is the entire graph G. If t is even, then we can find an independent set I meeting every K_4. If t is odd, then it is impossible to find such an independent set. However, this graph is excluded from the assumption of the Lemma.
2. $P_t \boxtimes K_2$ for $t \geq 3$. In this case, all internal vertices have degree 5 while the four end vertices have degree 4. Consider a new graph G' which is obtained by deleting all internal vertices and adding four edges to make the four end vertices as a K_4. It is easy to check $\Delta(G') \leq 5$ and $\omega(G') \leq 4$. Since $|G'| < |G|$, there is an independent set I of G' meeting every copy of K_4 in G'. Note that there is exactly one end vertex in I. Observe that any one end vertex can be extended into a maximal independent set meeting every copy of K_4 in $P_t \boxtimes K_2$. Thus, we can extend I to an independent set I' of G such that I' meets every copy of K_4 in G. Hence, this type of component does not occur in $\mathcal{C}(G)$.
3. There are four other types listed in Figure 2.5.

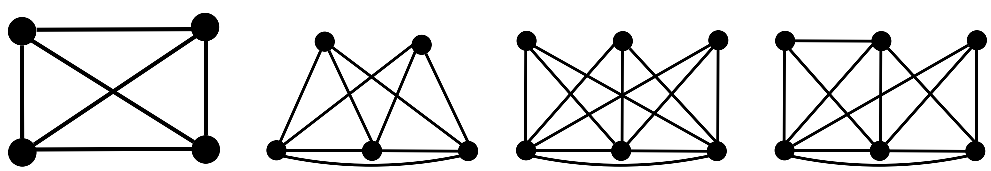


FIG. 2.5. *All types of components in the clique graph $C(G)$.*

For each component C_i in $\mathcal{C}(G)$, let V_i be the set of common vertices in all K_4's of C_i; for the leftmost figure in Figure 2.5, V_i is the set of all 4 vertices; for the middle two figures, V_i is the set of bottom three vertices; for the rightmost figure, V_i consists of the left-bottom vertex and the middle-bottom vertex. Note that all V_i's are pairwise disjoint. Let G' be the induce subgraph of G on $\cup_i V_i$. Note that G' does not contains any vertex in $C_i \setminus V_i$. By checking each type, we find out that for each i and each $v \in V_i$, v has at most $\min\{2, |V_i| - 2\}$ neighbors outside V_i in G' (not in G!). Applying Theorem 2.2 to G', we conclude that there exists an independent set I of G' meeting every V_i; thus I meets every K_4 in G. Contradiction!

□

Lemma 2.5. *Let G be a connected graph with $\Delta(G) \leq 5$ and $\omega(G) \leq 4$. If $G \neq C_{2l+1} \boxtimes K_2$ for some $l \geq 2$, then there exists an independent set meeting all induced copies of K_4 and C_8^2.*

Proof: We will use proof by contradiction. Suppose the Lemma is false. Let G be a minimum counterexample (with the smallest number of vertices). For any independent set I, let $C(I)$ be the number of induced copies of C_8^2 in $G - I$. Among all independent sets which meet all copies of K_4, there exists an independent set I such that $C(I)$ is minimized. Since $C(I) > 0$, let H be a copy of C_8^2 in $G - I$. The vertices of H are listed by u_i for $i \in \mathbb{Z}_8$ anticlockwise such that $u_i u_j$ is an edge of H if and only if $|i - j| \leq 2$. The vertex v_{i+4} is the *antipode* of v_i for any $i \in \mathbb{Z}_8$.

Case 1: There exists a vertex $v \notin V(H)$ such that v has five neighbors in H. By the Pigeonhole Principle, $\Gamma(v)$ contains a pair of antipodes. Without loss of generality, say $u_0, u_4 \in \Gamma(v)$. If the other three neighbors of v do not form a triangle, then we let $I' := I \setminus \{v\} \cup \{u_0, u_4\}$; note that v is not in any K_4 of $G - I'$. Thus I' is an independent set meeting every copy of K_4. Since every copy of C_8^2 containing v must contain one of u_0 and u_4, we have $C(I') < C(I)$. Contradiction! Hence, the other three neighbors of v must form a triangle. Without loss of generality, we can assume that the three neighbors are u_1, u_2, and u_3. Now we let $I'' := I \setminus \{v\} \cup \{u_0, u_3\}$; note that $v \notin K_4 \subset G - I'$. Thus I'' is also an independent set meeting every copy of K_4 of G. Since every copy of C_8^2 containing v must contain one of u_0 and u_3, we have $C(I'') < C(I)$. Contradiction!

Case 2: There exists a vertex $v \notin V(H)$ such that v has exactly four neighbors in H. Since H is K_4-free, we can find $u_i, u_j \in \Gamma(v) \cap V(H)$ such that $u_i u_j$ is a non-edge. Let $I' := I \setminus \{v\} \cup \{u_i, u_j\}$; I' is also an independent set. Note that $\Gamma(v) \setminus \{u_i, u_j\}$ can not be a triangle, v is not in any $K_4 \subset G - I'$. Thus I' meets every copy of K_4. Since every copy of C_8^2 containing v must contain one of u_i and u_j, we have $C(I') < C(I)$. Contradiction!

Case 3: There exists a vertex $v \notin V(H)$ such that v has exactly three neighbors in H. If the 3 neighbors do not form a triangle, then choose $u_i, u_j \in \Gamma(v) \cap V(H)$ such that $u_i u_j$ is a non-edge. Note that $\Gamma(v) \setminus \{u_i, u_j\}$ can not be a triangle; v is not in any $K_4 \subset G - I'$. Let $I' := I \setminus \{v\} \cup \{u_i, u_j\}$; I' is also an independent set meeting every copy of K_4. Since every copy of C_8^2 containing v must contain one of u_i and u_j, we have $C(I') < C(I)$. Contradiction! Else, the three neighbors form a triangle; let u_i be one of them and $I' := I \setminus \{v\} \cup \{u_i\}$; v is not in any $K_4 \subset G - I'$. Thus I' is an independent set meeting every copy of K_4. Note that $\Gamma(v) \setminus \{u_i\}$ has only two vertices in H. The induced graph on $\Gamma(v) \setminus \{u_i\}$ is disconnected. However, for any vertex v in $H' = C_8^2$, the subgraph induced by $\Gamma_G(v) \cap H'$ is a P_4. There is no C_8^2 in $G - I'$ containing v. Thus, $C(I') < C(I)$. Contradiction!

Case 4: Every vertex outside H can have at most 2 neighbors in H. We identify each pair of antipodes of H to get a new graph G' from G. After identifying, H is turned into a K_4; where the vertices of this K_4 are referred as fat vertices.

Subcase 4a: $G' \neq C_{2l+1} \boxtimes K_2$. Observe $\Delta(G') \leq 5$. We claim G' is K_5-free. Suppose not. Since every vertex in H has at most one neighbor outside H, then each fat vertex can have at most two neighbors outside H. Recall that the original graph G is K_5-free. If G' has some K_5, then this K_5 contains either 3 or 4 fat vertices. Let w be one of the other vertices in this K_5. We get w has at least three neighbors in H. However, this is covered by Case 1, Case 2, or Case 3. Thus, G' is K_5-free. Since $|G'| < |G|$, by the minimality of G, G' has an independent set I' meeting every copy of K_4 and C_8^2 in G'. There is exactly one fat vertex in I'. Now replacing this fat vertex by its corresponding pair of antipodal vertices, we get an independent set I''; we assume the pair of antipodal vertices are u_2 and u_6. It is easy to check that I'' is an independent set of G. Next we claim any $v \in V(H) \setminus \{u_2, u_6\}$ is neither in a $K_4 \subset V(G) - I''$ nor in a $C_8^2 \subset V(G) - I''$. Suppose there is some v such that $v \in K_4 \subset G - I''$. Recall each $v \in V(H)$ has at most one neighbor outside H and H

is K_4-free; there is some $w \notin V(H)$ such that w has at least three neighbors in H. This is already considered by Case 1, Case 2, or Case 3. We are left to show that $v \notin C_8^2 \subset G - I''$ for each $v \in V(H) \setminus \{u_2, u_6\}$. If not, there exists a copy H' of C_8^2 in $G - I''$ containing v. Note H' is 4-regular, any vertex in H' can have at most one neighbor in I''; in particular, $v \neq u_0, u_4$. Without loss of generality, we assume $v = u_3$. Then there is a vertex $w \notin V(H)$ such that u_3w is an edge, see Figure 2.6. Observe that the neighborhood of each vertex of an induced C_8^2 is is a P_4. Since u_1u_4 and u_1u_5 are two non-edges, we have wu_1 being an edge. Observe $\Gamma_G(u_1) = \{u_7, u_0, u_2, u_3, w\}$. Since $u_2 \notin H'$, we have $u_0 \in H'$; u_0 has two neighbors (u_2 and u_6) outside H', contradiction! Therefore, I'' meets every copy of K_4 and C_8^2 in G. Contradiction!

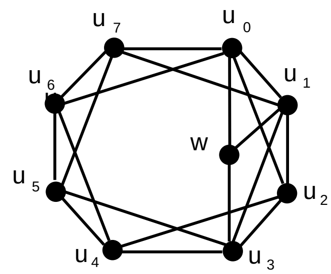


FIG. 2.6. *Subcase 4a.*

Subcase 4b: $G' = C_{2l+1} \boxtimes K_2$. The graph G can be recovered from G'. It consists of an induced subgraph $H = C_8^2$ and an induced subgraph $P_{2l-1} \boxtimes K_2$. For each vertex u in H, there is exactly one edge connecting it to one of the four end vertices of $P_{2l-1} \boxtimes K_2$; for each end vertex v of $P_{2l-1} \boxtimes K_2$, there are exactly two edges connecting v to the vertices in H. First, we take any maximum independent set I' of $P_{2l-1} \boxtimes K_2$. Observe that I' has exactly two end points of $P_{2l-1} \boxtimes K_2$; so I' has exactly four neighbors in H. In the remaining four vertices of H, there exists a non-edge u_iu_j since H is K_4-free. Let $I := I' \cup \{u_i, u_j\}$. Clearly I is an independent set of G meeting every copy of K_4 and C_8^2. Contradiction! □

We are ready to prove Lemma 1.4.

Proof of Lemma 1.4: We need prove for $k \geq 5$ and any connected graph G with $\Delta(G) = k$ and $\omega(G) \leq k - 1$ satisfies

$$\chi_f(G) \leq k - \min\left\{f(k-1), \frac{1}{2}\right\}. \tag{2.1}$$

If $\omega(G) \leq k - 2$, then by inequality (1.1), we have

$$\chi_f(G) \leq \frac{\Delta(G) + \omega(G) + 1}{2} \leq k - \frac{1}{2}.$$

Thus, inequality (2.1) is satisfied. From now on, we assume $\omega(G) = \Delta(G) - 1$.

For $\Delta(G) = k \geq 6$ and $\omega(G) = k - 1$, the condition $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ is satisfied. By Theorem 2.1, G contains an independent set meeting every maximum clique. Extend this independent set to a maximal independent set and denote it by I. Note that $\Delta(G - I) \leq k - 1$ and $\omega(G - I) \leq k - 2$.

Case 1: $k \geq 7$. From the definition of $f(k-1)$, we have $\chi_f(G - I) \leq \Delta(G - I) - f(k-1)$. Thus,

$$\chi_f(G) \leq \chi_f(G - I) + 1 \leq k - 1 - f(k-1) + 1 = k - f(k-1).$$

Thus, we have $f(k) \geq \min\{f(k-1), 1/2\}$.

Case 2: $k = 6$. By Lemma 2.3, we can find an independent set meets every copy of K_5 and $C_5 \boxtimes K_2$; we extend this independent set as a maximal independent set I. Note that $G - I$ contains no induced subgraph isomorphic $C_5 \boxtimes K_2$. We have $\chi_f(G-I) \le 5 - f(5)$; it implies $\chi_f(G) \le 6 - f(5)$. Thus, $f(6) \ge \min\{f(5), 1/2\}$ and we are done.
Case 3: $k = 5$. If $G = C_{2l+1} \boxtimes K_2$ for some $l \ge 3$; then G is vertex-transitive and $\alpha(G) = l$. It implies that

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)} = 4 + \frac{2}{l} \le 5 - \frac{1}{3}.$$

If $G \ne C_{2l+1} \boxtimes K_2$, then by Lemma 2.5, we can find an independent set meeting every copy of K_4 and C_8^2; we extend it as a maximal independent set I. Note that $G-I$ contains no induced subgraph isomorphic C_8^2. We have $\chi_f(G-I) \le 4-f(4)$; it implies $\chi_f(G) \le 5-f(4)$. Thus, $f(3) \ge \min\{f(4), 1/3\}$ and we are finished. □

3. The case $\Delta(G) = 4$. To prove $f(4) \ge \frac{2}{67}$, we will use an approach which is similar to those in [4, 7]. We will construct 133 4-colorable auxiliary graphs, and from these colorings we will construct a 134-fold coloring of G using 532 colors.

It suffices to prove that the minimum counterexample does not exist.

Let G be a graph with the smallest number of vertices and satisfying

1. $\Delta(G) = 4$ and $\omega(G) \le 3$;
2. $\chi_f(G) > 4 - \frac{2}{67}$;
3. $G \ne C_8^2$.

By the minimality of G, each vertex in G has degree either 4 or 3. To prove Lemma 1.3, we will show $\chi_f(G) \le 4 - \frac{2}{67}$, which gives us the desired contradiction.

For a given vertex x in $V(G)$, it is easy to color its neighborhood $\Gamma_G(x)$ using 2 colors. If $d_G(x) = 3$, then we pick a non-edge S from $\Gamma_G(x)$ and color the two vertices in S using color 1. If $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) \ge 3$, then we pick an independent set S in $\Gamma_G(x)$ of size 3 and assign the color 1 to each vertex in S. If $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, then we pick two disjoint non-edges S_1 and S_2 from $\Gamma_G(x)$; we assign color 1 to each vertex in S_1 and color 2 to each vertex in S_2.

The following Lemma shows that G has a key property, which eventually implies that this local coloring scheme works simultaneously for x in a large subset of $V(G)$.

Lemma 3.1. *For each $x \in V(G)$ with $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, there exist two vertex-disjoint non-edges $S_1(x), S_2(x) \subset \Gamma_G(x)$ satisfying the following property. If we contract $S_1(x)$ and $S_2(x)$, then the resulting graph $G/S_1(x)/S_2(x)$ contains neither K_5^- nor G_0. Here K_5^- is the graph obtained from K_5 by removing one edge and G_0 is the graph shown in Figure 3.1.* The proof of this lemma is quite long and we will present its proof in section 4.

For each vertex x in G, we associate a small set of vertices $S(x)$ selected from $\Gamma_G(x)$ as follows. If $d_G(x) = 3$, then let $S(x)$ be the endpoints of a non-edge in $\Gamma_G(x)$ and label the vertices in $S(x)$ as 1; if $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) \ge 3$, then let $S(x)$ be any independent set of size 3 in $\Gamma_G(x)$ and label all vertices in $S(x)$ as 1; if $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, then let $S(x) = S_1(x) \cup S_2(x)$, where $S_1(x)$ and $S_2(x)$ are guaranteed by Lemma 3.1; we label the vertices in $S_1(x)$ as 1 and the vertices in $S_2(x)$ as 2. For any $x \in V(G)$, we have $|S(x)| = 2$, 3, or 4.

The following definitions depend on the choice of $S(*)$, which is assumed to be fixed through this section. For $v \in G$ and $j \in \{1, 2, 3\}$, we define

$$N_G^j(v) = \{u | \text{ there is a path } vv_0 \dots v_{j-2}u \text{ in } G \text{ of length } j \text{ such that } v_0 \in S(v) \text{ and } v_{j-2} \in S(u)\}.$$

We now define $N_G^j(u)$ for $j \in \{4, 5, 7\}$; each $N_G^j(u)$ is a subset of the jth neighborhood of u. For $j = 4$, $v \in N_G^4(u)$ if $d_G(u) = 4$, $\alpha(\Gamma_G(u)) = 2$, u and v are connected as shown in

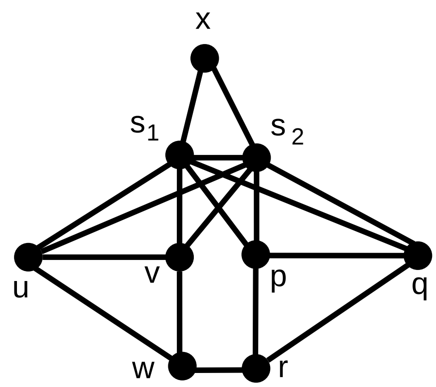


FIG. 3.1. *The graph G_0.*

Figure 3.2; otherwise $N^4_G(u) = \emptyset$. In Figure 3.2, w is connected to one of the two vertices in $S_2(u)$. Similarly, in Figure 3.3 and 3.3, a vertex is connected to a group of vertices means it is connected to any vertex in this group.

For $j = 5$, $v \in N^5_G(u)$ if $d_G(w) = 4$, $\alpha(\Gamma_G(w)) = 2$ for $w \in \{u, v\}$ and u and v are connected as shown in Figure 3.3; otherwise $N^5_G(u) = \emptyset$.

For $j = 7$, $v \in N^7_G(u)$ if $d_G(w) = 4$, $\alpha(\Gamma_G(w)) = 2$ for $w \in \{u, v\}$ and u and v are connected as shown in Figure 3.4; otherwise $N^7_G(u) = \emptyset$.

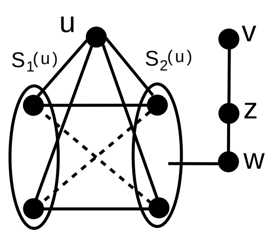


FIG. 3.2. *4-th neighborhood.*

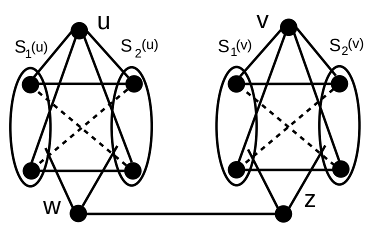


FIG. 3.3. *5-th neighborhood.*

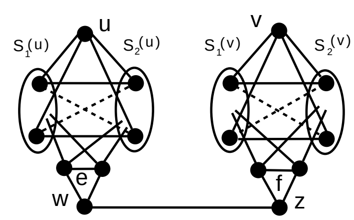


FIG. 3.4. *7-th neighborhood.*

Note that for $j \in \{1, 2, 3, 5, 7\}$, $v \in N^j_G(u)$ if and only if $u \in N^j_G(v)$; but this does not hold for $j = 4$. We have the following lemma.

LEMMA 3.2. *For $u \in V(G)$ such that $d_G(u) = 4$ and $\alpha(\Gamma_G(u)) = 2$, we have $|N^1_G(u) \cup N^2_G(u) \cup N^3_G(u) \cup N^4_G(u) \cup N^5_G(u) \cup N^7_G(u)| \leq 96$.* **Proof:** It is clear that $|N^1_G(u) \cup N^2_G(u) \cup N^3_G(u)| \leq 4 + 8 + 8 \times 3 = 36$. We next estimate $|N^4_G(u)|$. In Figure 3.2, observe that w is connected to one vertex of $S_2(u)$ and $w \notin \Gamma_G(u)$. For a fixed u, there are at most four choices for w, at most three choices for z, and at most three choices for v. Therefore, we have $|N^4_G(u)| \leq 4 \times 3 \times 3 = 36$.

Let us estimate $|N^5_G(u)|$. In Figure 3.3, for a fixed u, we have four choices for w and two choices for z. Fix a z. Assume $\Gamma_G(z) \setminus \{w\} = \{a, b, c\}$. Let $T_1 = \{a, b\}$, $T_2 = \{b, c\}$, and $T_3 = \{a, c\}$. We have the following claim.

Claim There are at most three $v \in N^5_G(u)$ such that for each v we have $\Gamma_G(z) \cap \Gamma_G(v) = T_i$ for some $1 \leq i \leq 3$ as shown in Figure 3.3.

Proof of the claim: For each $1 \leq i \leq 3$, there are at most three $v \in N^5_G(u)$ such that $\Gamma_G(z) \cap \Gamma_G(v) = T_i$ as shown in Figure 3.3 since each vertex in T_i has at most three neighbors other than z. If the claim is false, then there is $1 \le i \ne j \le 3$ such that $\Gamma_G(z) \cap \Gamma_G(v_i) = T_i$ and $\Gamma_G(z) \cap \Gamma_G(v_i') = T_i$ for some $v_i, v_i' \in N_G^5(u)$, and $\Gamma_G(z) \cap \Gamma_G(v_j) = T_j$ for some $v_j \in N_G^5(u)$, where v_i, v_i', v_j are distinct. Without loss of generality, we assume $\Gamma_G(z) \cap \Gamma_G(v_1) = \Gamma_G(z) \cap \Gamma_G(v_1') = T_1$ for $v_1, v_1' \in N_G^5(u)$, and $\Gamma_G(z) \cap \Gamma_G(v_2) = T_2$ for some $v_2 \in N_G^5(u)$, see Figure 3.5. Observe that $\Gamma_G(b) = \{v_1, v_1', v_2, z\}$. Since $\Gamma_G(z) \cap \Gamma_G(v_1) = T_1$ as shown in Figure 3.3, a and one of b's neighbors form $S_i(v_1)$ for some $i \in \{1, 2\}$; we assume it is $S_1(v_1)$. Note $\{z, v_1, v_1'\} \subset \Gamma_G(a)$. Thus $S_1(v_1) = \{a, v_2\}$ and $v_2 \in \Gamma_G(v_1)$. Similarly, we can show $S_1(v_1') = \{a, v_2\}$ and $v_2 \in \Gamma_G(v_1')$. Now, observe that $\Gamma_G(v_2) = \{v_1, v_1', b, c\}$. Since $\Gamma_G(z) \cap \Gamma_G(v_2) = T_2$ as shown in Figure 3.3, b and one of neighbors of v_2 form $S_i(v_2)$ for some $i \in \{1, 2\}$; we assume $i = 1$. Because $\{v_1, v_1'\} \subset \Gamma_G(b)$, then $S_1(v_2) = \{b, c\}$. However, b and c are not is in the same independent set in the definition of $N_G^5(u)$, see Figure 3.3. This is a contradiction and this case can not happen. The claim follows.

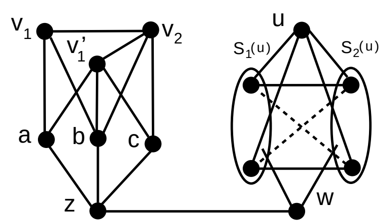


Fig. 3.5. *The picture for the claim.*

Therefore, $|N_G^5(u)| \le 4 \times 2 \times 3 = 24$.

In Figure 3.4, for a fixed u, we have two choices for the edge e, one choice for w, two choices for z, and three choices for the edge f. Fix a z. By considering the degrees of the endpoints of f, there is at most one f and at most one $v \in N_G^7(u)$ such that $|\Gamma_G(f) \cap \Gamma_G(v)| = 4$ as shown in Figure 3.4. Therefore, we have $|N_G^7(u)| \le 2 \times 2 \times 1 = 4$.

Last, we estimate $|N_G^5(u) \cup N_G^7(u)|$. If there is some $v \in N_G^7(u)$, then we observe that there are at most five z's (see Figure 3.3). We get the number of $v \in N_G^5(u)$ is at most $5 \times 3 = 15$. In this case, we have

$$|N_G^5(u) \cup N_G^7(u)| \le 4 + 15 < 24.$$

If $N_G^7(u) = \emptyset$, then also we have

$$|N_G^5(u) \cup N_G^7(u) \le 24.$$

Therefore

$$|N_G^1(u) \cup N_G^2(u) \cup N_G^3(u) \cup N_G^4(u) \cup N_G^5(u) \cup N_G^7(u)| \le 36 + 36 + 24 = 96.$$

$\square$

Based on the graph G, we define an auxiliary graph G^* on vertex set $V(G)$. The edge set is defined as follows: $uv \in E(G^*)$ precisely if either $u \in N_G^1(v) \cup N_G^2(v) \cup N_G^3(v) \cup N_G^4(v) \cup N_G^5(v) \cup N_G^7(v)$, or $v \in N_G^4(u)$. We have the following lemma.

Lemma 3.3. *The graph G^* is 133-colorable.*

Proof: Let σ be an increasing order of $V(G^*)$ satisfying the following conditions.

1: For u and v such that $d_G(u) = 3$ and $d_G(v) = 4$, we have $\sigma(u) < \sigma(v)$.

2: For u and v such that $d_G(u) = d_G(v) = 4$, $\alpha(\Gamma_G(u)) \ge 3$, and $\alpha(\Gamma_G(v)) = 2$, we have $\sigma(u) < \sigma(v)$.

We will color $V(G^*)$ according to the order σ. For each v, we have the following estimate on the number of colors forbidden to use for v.

1: For v such that $d_G(v) = 3$, the number of colors forbidden to use for v is at most $|N_G^1(v) \cup N_G^2(v) \cup N_G^3(v)| \leq 3 + 9 + 27 = 39$.
2: For v such that $d_G(v) = 4$ and $\alpha(\Gamma_G(v)) \geq 3$, the number of colors forbidden to use for v is at most $|N_G^1(v) \cup N_G^2(v) \cup N_G^3(v)| \leq 3 + 9 + 27 = 39$.
3: For v such that $d_G(v) = 4$ and $\alpha(\Gamma_G(v)) = 2$, the number of colors forbidden to use for v is at most $|N_G^1(v) \cup N_G^2(v) \cup N_G^3(v) \cup N_G^4(v) \cup N_G^5(v) \cup N_G^7(v)| + |N_G^4(v)| \leq 96 + 36 = 132$ by Lemma 3.2.

Therefore, the greedy algorithm shows G^* is 133-colorable. $\square$

Let X be a color class of G^*. We define a new graph $G(X)$ by the following process.

1. For each $x \in X$, if $|S(x)| = 2$ or $|S(x)| = 3$, then we contract $S(x)$ as a single vertex, delete the vertices in $\Gamma_G(v) \setminus S(v)$, and keep label 1 on the new vertex; if $|S(x)| = 4$, i.e., $S(x) = S_1(x) \cup S_2(x)$, then we contract $S_1(x)$ and $S_2(x)$ as single vertices and keep their labels. After that, we delete X. Let H be the resulting graph.
2. Note that $\Gamma_H(x) \cap \Gamma_H(y) = \emptyset$ and there is no edge from $\Gamma_H(x)$ to $\Gamma_H(y)$ for any $x, y \in X$ as X is a color class.
3. We identify all vertices with label i as a single vertex w_i for $i \in \{1, 2\}$. Let $G(X)$ be the resulted graph.

We have the following lemma on the chromatic number of $G(X)$.

Lemma 3.4. *The graph $G(X)$ is 4-colorable for each color class.* We postpone the proof of this lemma until the end of this section and prove Lemma 1.3 first.

Proof of Lemma 1.3: By Lemma 3.3, there is a proper 133-coloring of G^*. We assume $V(G^*) = V(G) = \cup_{i=1}^{133} X_i$, where X_i is the i-th color class.

For each $i \in \{1, \ldots, 133\}$, Lemma 3.4 shows $G(X_i)$ is 4-colorable; let $c_i \colon V(G(X_i)) \to T_i$ be a proper 4-coloring of the graph $G(X_i)$. Here $T_1, T_2, \ldots, T_{133}$ are pairwise disjoint; each of them consists of 4 colors. For $i \in \{1, \ldots, 133\}$, the 4-coloring c_i can be viewed as a 4-coloring of $G \setminus X_i$ since each vertex with label j receives the color $c_i(w_i)$ for $j = 1, 2$ and each removed vertex has at most three neighbors in $G \setminus X_i$.

Now we reuse the notation c_i to denote this 4-coloring of $G \setminus X_i$. For each $v \in X_i$, we have $|\cup_{u \in \Gamma_G(v)} c_i(u)| \leq 2$. We can assign two unused colors, denoted by the set $Y(v)$, to v. We define $f_i : V(G) \to \mathcal{P}(T_i)$ (the power set of T_i) satisfying

$$f_i(v) = \begin{cases} \{c_i(v)\} & \text{if } v \in V(G) \setminus X_i, \\ Y(v) & \text{if } v \in X_i. \end{cases}$$

Observe that each vertex in X_i receives two colors from f_i and every other vertex receives one color. Let $\sigma : V(G) \to \mathcal{P}(\cup_{i=1}^{k} T_i)$ be a mapping such that $\sigma(v) = \cup_{i=1}^{m} f_i(v)$. It is easy to verify σ is a 134-fold coloring of G such that each color is drawn from a palette of 532 colors; namely we have

$$\chi_f(G) \leq \frac{532}{134} = 4 - \frac{2}{67}.$$

The proof of Lemma 1.3 is finished. $\square$

Before we prove Lemma 3.4, we need the following definitions.

A *block* of a graph is a maximal 2-connected induced subgraph. A *Gallai tree* is a connected graph in which all blocks are either complete graphs or odd cycles. A *Gallai forest* is a graph all of whose components are Gallai trees. A *k-Gallai tree (forest)* is a Gallai tree (forest) such that the degree of all vertices are at most $k - 1$. A k-critical graph is a graph G whose chromatic number is k and deleting any vertex can decrease the chromatic number. Gallai showed the following Lemma.

Lemma 3.5. *[3] If G is a k-critical graph, then the subgraph of G induced on the vertices of degree $k - 1$ is a k-Gallai forest.*

Proof of Lemma 3.4: We use proof by contradiction. Suppose that $G(X)$ is not 4-colorable. The only possible vertices in $G(X)$ with degree greater than 4 are the vertices w_1 and w_2, which are obtained by contracting the vertices with label 1 and 2 in the intermediate graph H. The simple greedy algorithm shows that $G(X)$ is always 5-colorable. Let $G'(X)$ be a 5-critical subgraph of $G(X)$. Applying Lemma 3.5 to $G'(X)$, the subgraph of $G'(X)$ induced on the vertices of degree 4 is a 5-Gallai forest F. The vertex set of F may contain w_1 or w_2. Delete w_1 and w_1 from F if F contains one of them. Let F' be the resulting Gallai forest. (Any induced subgraph of a Gallai forest is still a Gallai forest.) The Gallai forest F' is not empty. Let T be a connected component of F' and B be a leaf block of T. The block B is either a clique or an odd cycle from the definition of a Gallai tree.

Let v be a vertex in B. As v has at most two neighbors (w_1 and w_2) outside F' in $G(X)$, we have $d_{F'}(v) \geq 2$. If v is not in other blocks of F', then we have $d_B(v) \geq 2$. It follows that $|B| \geq 3$. Since B is a subgraph of G and G is K_4-free, the block B is an odd cycle.

Let v_1v_2 be an edge in B such that v_1 and v_2 are not in other blocks. The degree requirement implies v_iw_j are edges in $G(X)$ for all $i, j \in \{1, 2\}$. For $i = 1, 2$, there are vertices $x_i, y_i \in X$ satisfying $S(x_i) \cap \Gamma_G(v_i) \neq \emptyset$ and $S(y_i) \cap \Gamma_G(v_i) \neq \emptyset$; moreover either $|S(x_i)| = 4$ or $|S(y_i)| = 4$ since one of its neighborhood has label 2. Without loss of generality, we assume $|S(x_i)| = 4$ for $i \in \{1, 2\}$. If $x_i \neq y_i$, then $y_i \in N_G^4(x_i)$, i.e., $y_i \in \Gamma_{G^*}(x_i)$; this contradicts X being a color class. Thus we have $x_i = y_i$ and $|S(x_i)| = 4$ for $i \in \{1, 2\}$. For $\{i, j\} = \{1, 2\}$, if $x_i \neq y_j$, then $y_i \in N_G^5(x_i)$, i.e., $y_i \in \Gamma_{G^*}(x_i)$; this is a contradiction of X being a color class. Thus we have

$$x_1 = x_2 = y_1 = y_2.$$

Let x denote this common vertex above. Then $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$.

Let v_0 be the only vertex in B shared by other blocks. Since $B - v_0$ is connected, the argument above shows there is a common x for all edges in $B - v_0$. If $\Gamma_{G(X)}(v_0) \cap \{w_1, w_2\} \neq \emptyset$, the there is some vertex $x_0 \in X$ such that $S(x_0) \cup \Gamma_G(v_0) \neq \emptyset$. By the similar argument, we also have $x_0 = x$.

Therefore, x depends only on B. In the sense that for any $y \in X$ and any $v \in B$, if $S(y) \cap \Gamma_G(v) \neq \emptyset$, then $y = x$.

The block B is an odd cycle as we mentioned above. Suppose $|B| = 2r + 1$. Let $v_0, v_1, \ldots, v_{2r}$ be the vertices of B in cyclic order and v_0 be the only vertex which may be shared by other block.

Let $x \in X$ be the vertex determined by B. Recall $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$. Each vertex in $\Gamma(x)$ can have at most 2 edges to B. We get

$$4r \leq |E(B, \Gamma(x))| \leq 8. \tag{3.1}$$

We have $r \leq 2$. The block B is either a C_5 or a K_3. We claim both v_0w_1 and v_0w_2 are non-edges of $G(X)$.

If $B = C_5$, then inequality (3.1) implies that v_0 has no neighbor in $\Gamma(x)$ and the claim holds. If $B = K_3$, then the claim also holds; otherwise $B \cup \{\underline{S_1(x)}, \underline{S_2(x)}\}$ forms a K_5^- in $G/S_1(x)/S_2(x)$, which is a contradiction to Lemma 3.1.

Let u_1 and u_2 be the two neighbors of v_0 in other blocks of F'. If u_1 and u_2 are in the same block, then this block is an odd cycle; otherwise, v_0u_1 and v_0u_2 are in two different blocks.

The union of non-leaf blocks of T is a Gallai-tree, denoted by T'. The argument above shows every leaf block of T' must be an odd cycle. Let C be such a leaf block of T'. Now C is an odd cycle, and C is connected to $|C| - 1$ leaf blocks of T. Let B and B' be two leaf blocks of T such that $B \cap C$ is adjacent to $B' \cap C$. Without loss of generality, we may assume B is the one we considered before. By the same argument, B' is an odd cycle of size

$2r'+1$ with $r' \in \{1,2\}$. Let $v'_0, v'_1, \ldots, v'_{2r'}$ be the vertices of B' and v'_0 be the only vertex in $B' \cap C$. For i in $\{1, 2, \ldots, 2r'\}$ and j in $\{1,2\}$, $v'_i w_j$ are edges in $G'(X)$. Similarly, there exists a vertex $x' \in X$ with $d_G(x') = 4$ and $\alpha(\Gamma_G(x')) = 2$ such that $|E(v_i, S_1(x'))| \geq 1$ and $|E(v_i, S_2(x'))| \geq 1$. We must have $x = x'$; otherwise $x' \in N^7_G(x)$, i.e., $x' \in \Gamma_{G^*}(x)$, and this contradicts the fact that X is a color class in D. Now we have $|E(\Gamma(x), B)| \geq 4r$ and $E(\Gamma(x), B')| \geq 4r'$. By counting the degrees of vertices in $\Gamma(x)$ in G, we have

$$4r + 4r' + 4 + 4 \leq 16.$$

We get $r = r' = 1$. Both B and B' are K_3's. In this case, $G/S_1(x)/S_2(x)$ contains the graph G_0, see figure 3.1. This contradicts Lemma 3.1.

We can find the desired contradiction, so the lemma follows. □

4. Proof of Lemma 3.1. In this section, we will prove Lemma 3.1. We first review a Lemma from [7].

LEMMA 4.1. *Let G be a graph. Suppose that G_1 and G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{u, v\}$.*

1. *If uv is an edge of G, then we have*

$$\chi_f(G) = \max\{\chi_f(G_1), \chi_f(G_2)\}.$$

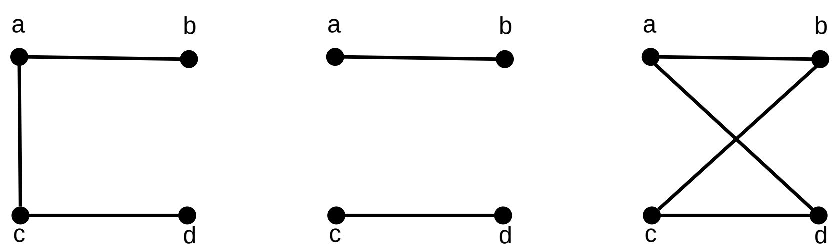
2. *If uv is not an edge of G, then we have*

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\},$$

where $G_2 + uv$ is the graph obtained from G_2 by adding edge uv and G_2/uv is the graph obtained from G_2 by contracting $\{u, v\}$.

Proof of Lemma 3.1: Recall that G is a connected K_4-free graph with minimum number of vertices such that $G \neq C_8^2$ and $\chi_f(G) > 4 - \frac{2}{67}$. Note that G is 2-connected. We will prove it by contradiction.

Suppose Lemma 3.1 fails for some vertex x in G. Observe $\Gamma_G(x)$ is one of the graphs in Figure 4.1. Here we assume $\Gamma_G(x) = \{a, b, c, d\}$. Through the proof of the lemma, let S_1 and S_2 be two vertex-disjoint independent sets in $\Gamma_G(x)$, H be a triangle in $V(G) \setminus (\{x\} \cup \Gamma_G(x))$, then say (S_1, S_2, H) is a bad triple if $\{\underline{S_1}, \underline{S_2}, H\}$ contains a K_5^- in $G/S_1/S_2$.



$\Gamma_G(x) = P_4$ $\Gamma_G(x) = 2\ e$ $\Gamma_G(x) = C_4$

FIG. 4.1. *Three possible cases of* $\Gamma_G(x)$.

If $\Gamma_G(x) = P_4$, then $\{a, d\}$ and $\{b, c\}$ is the only pair of disjoint non-edges. There is a triangle H with $V(H) = \{y, z, w\}$ such that $(\{a, d\}, \{b, c\}, H)$ is a bad triple. Note that $|E(\{a, b, c, d\}, \{y, z, w\})| = 5$ or 6. By an exhaustive search, the induced subgraph of G on $\{x, a, b, c, d, y, z, w\}$ is one of the following six graphs (see Figure 4.2).

If $\Gamma_G(x) = 2\ e$, then $(\{a, c\}, \{b, d\})$ and $(\{a, d\}, \{b, c\})$ are two pairs of disjoint non-edges. By considering the degrees of vertices in $\Gamma_G(x)$, there is only one triangle H with

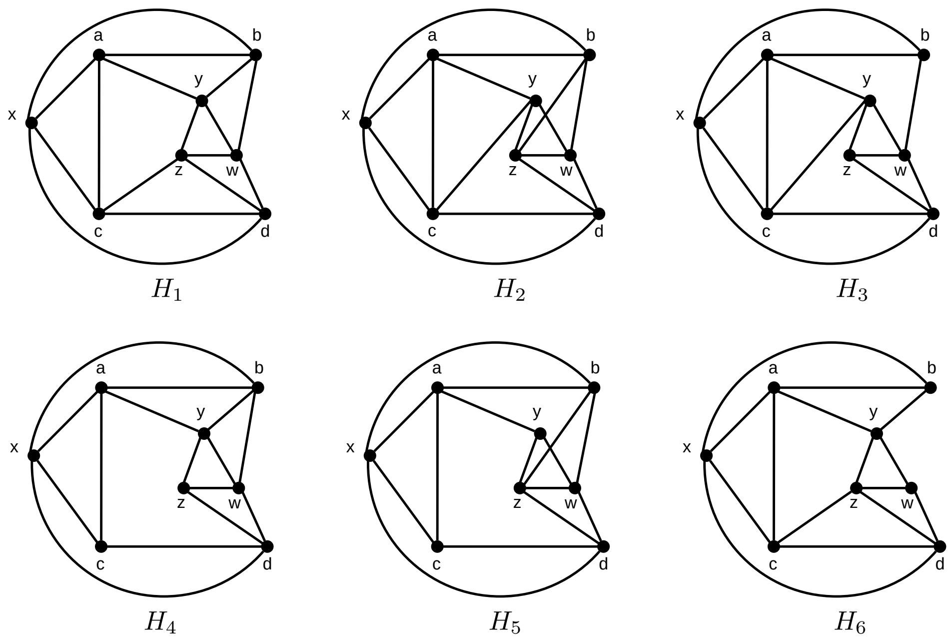


FIG. 4.2. *If $\Gamma_G(x) = P_4$, then there are six possible induced subgraphs.*

$V(H) = \{y, z, w\}$ such that $(\{a, c\}, \{b, d\}, H)$ and $(\{a, d\}, \{b, c\}, H)$ are two bad triples. By an exhaustive search, the induced subgraph of G on $\{x, a, b, c, d, y, z, w\}$ is one of the following three graphs (see Figure 4.3).

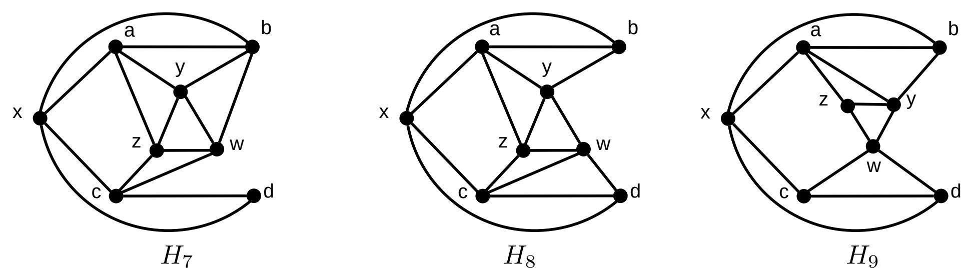


FIG. 4.3. *If $\Gamma_G(x) = 2\ e$, then there are three possible induced subgraphs.*

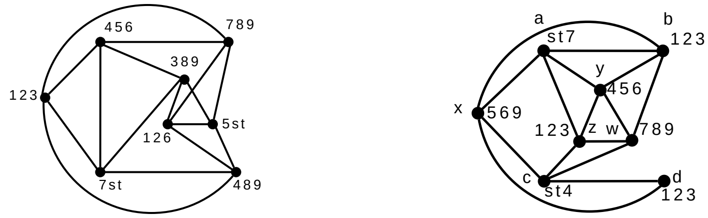
It suffices to show that G cannot contain H_i for $1 \leq i \leq 9$. Since all vertices in H_1 (and H_2) have degree 4, H_1 (and H_2) is the entire graph G. Observe that H_1 is isomorphic to C_8^2 and H_2 is 11:3-colorable (see Figure 4.4). Contradiction!

In H_7, the vertex d is the only vertex with degree less than 4. If H_7 is not the entire graph G, then d is a cut vertex of G. This contradicts the fact that G is 2-connected. Thus $G = H_7$. The graph H_7 is 11:3-colorable as shown by Figure 4.4. Contradiction!

Now we consider the case H_3. Note $H_3 + bz$ is the graph H_2. We have $\chi_f(H_3) \leq \chi_f(H_2) \leq 11/3$. The graph H_3 must be a proper induced subgraph of G, and the pair $\{b, z\}$ is a vertex cut of G. Let G' be the induced subgraph of G by deleting all vertices in H_3 but b, z. We apply Lemma 4.1 to $G + bz$ with $G_1 = H_3 + bz = H_2$ and $G_2 = G' + bz$. We have

$$\chi_f(G + bz) \leq \max\{\chi_f(H_2), \chi_f(G' + bz)\}.$$

Note $\chi_f(H_2) \leq 11/3$ and $11/3 < \chi_f(G) \leq \chi_f(G + bz)$. We have $\chi_f(G) \leq \chi_f(G' + bz)$. Both



H_2 H_7

FIG. 4.4. *The graph H_2 and H_7 are 11:3-colorable.*

b and z have at most 2 neighbors in $G' + bz$. Thus $G' + bz$ is K_4-free; $G' + bz \neq C_8^2$ and has fewer vertices than G. This contradicts to the minimality of G.

Note $H_5 + cy = H_2$. The case H_5 is similar to the case H_3.

Note that H_4, H_6, and H_8 are isomorphic to each other. It suffices to show G does not contain H_4. Suppose that H_4 is a proper induced subgraph of G. Let G_1 be the induced subgraph of G by deleting all vertices in H_4. Note C_8^2 is not a proper subgraph of any graph in $\mathcal{G}_4$. We have $G_1 \neq C_8^2$. Note that c and z have degree 3 while other vertices in H_4 have degree 4. Since G is 2-connected, c has a unique neighbor, denoted by u, in $V(G_1)$. Similarly, z has a unique neighbor, denoted by v, in $V(G_1)$. Observe that the pair $\{u, v\}$ forms a vertex cut of G. Let G_2 be the induced graph of G on $V(H_4) \cup \{u, v\}$. Applying Lemma 4.1 to G with G_1 and G_2, we have

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\}.$$

Figure 4.5 shows $\chi_f(G_2 + uv)$ and $\chi_f(G_2/uv)$ are at most 11/3.

$G_2 + uv$ G_2/uv

FIG. 4.5. *Case H_4: both graph $G_2 + uv$ and G_2/uv are 11:3-colorable.*

Since $\chi_f(G) > 11/3$, we have $\chi_f(G) \leq \chi_f(G_1)$. Now G_1 is K_4-free and has maximum degree at most 4; G_1 has fewer vertices than G. This contradicts the minimality of G.

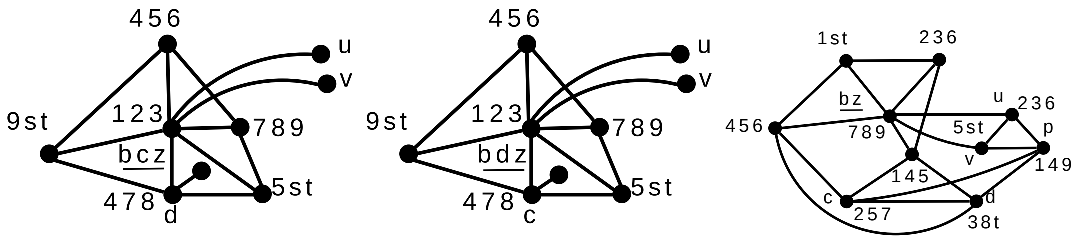
If $G = H_4$, then $\chi_f(H_4) \leq 11/3$, since H_4 is a subgraph of $G_2 + uv$ in Figure 4.5.

Now we consider the last case H_9. First, we contract b, c, z into a fat vertex denoted by $\underline{bcz}$. We write G/bcz for the graph after this contraction. Observe that $\{\underline{bcz}, d\}$ is a vertex-cut of G/bcz. Let G_4 and G_4' be two connected subgraphs of G/bcz such that $G_4 \cup G_4' = G/bcz$, $G_4 \cap G_4' = \{\underline{bcz}, d\}$, and $\{u, v\} \subset G_4'$. Note that G_4 is 11:3 colorable, see Figure 4.6. Now by Lemma 4.1, we have

$$\chi_f(G/bcz) \leq \max\{\chi_f(G_4), \chi_f(G_4')\}.$$

As $\{b, c, z\}$ is an independent set, each a:b-coloring of G/bcz gives an a:b-coloring of G, that is $\chi_f(G/bcz) \geq \chi_f(G) > 11/3$. The graph G_4 is 11:3-colorable; see Figure 4.6. Thus we have $\chi_f(G_4') \geq \chi_f(G/bcz) \geq \chi_f(G)$. It is easy to check that G_4' has maximum degree 4, K_4-free, and it is not C_8^2. Hence G'_4 must contain a K_4. Otherwise, it contradicts the minimality of G.

Second, we contract $\{b, d, z\}$ into a fat vertex $\underline{bdz}$ and denote the graph by G/bcz. Let G_5 and G'_5 be two connected subgraphs of G/bdz such that $G_5 \cup G'_5 = G/bzd$, $G_5 \cap G'_5 = \{\underline{bzd}, c\}$, and $\{u, v\} \subset G'_5$. Note that G_5 is 11:3-colorable; see Figure 4.6. By a similar argument, G'_5 must contain a K_4.



G_4 G_5 G_6

FIG. 4.6. *Case H_9: the graphs G_4, G_5, and G_6 are 11:3-colorable.*

The remaining case is that both G'_4 and G'_5 have a K_4 when we contract b and z. Since the original graph G is K_4-free, the K_4 in G'_4 (and in G'_5) must contain the fat vertex $\underline{bcz}$ (or $\underline{bdz}$), respectively. Note that each of the four vertices b, c, d, z has at most one edge leaving H_9. There must be a triangle uvp in G and these four outward edges are connected to some element of $\{u, v, p\}$. The graph G/bz must contain the subgraph G_6 as drawn in Figure 4.6.

Note that $\{u, v\}$ is a vertex-cut in G/bz. Let G_6 and G'_6 be two connected subgraphs of G/bz, which satisfy $G_6 \cup G'_6 = G$, $G_6 \cap G'_6 = \{u, v\}$, and $\underline{bz} \in G_6$. By Lemma 4.1, we have

$$\chi_f(G/bz) \leq \max\{\chi_f(G_6), \chi_f(G'_6)\}.$$

Note that G_6 is 11:3-colorable; see Figure 4.6. We also have $\chi_f(G/bz) \geq \chi_f(G) > \frac{11}{3}$. We obtain $\chi_f(G'_6) \geq \chi_f(G/bz) \geq \chi_f(G)$. Observe that G'_6 is a subgraph of G. We arrive at a contradiction of the minimality of G.

If $\Gamma_G(x) = C_4$, then the only possible choice for the two independent sets are $\{a, c\}$ and $\{b, d\}$. If there is some triangle H such that $(\{a, c\}, \{b, d\}, H)$ is a bad triple, then we have

$$|E(\Gamma_G(x), H)| \geq 5.$$

However, $|E(\Gamma_G(x), H)| \leq 4$. This is a contradiction. Thus the lemma follows in this case.

We can select two vertex disjoint non-edges S_1 and S_2 such that the graph $G/S_1/S_2$ contains no K_5^-. For these particular S_1 and S_2, if $G/S_1/S_2$ contains no G_0, then Lemma 3.1 holds.

Without loss of generality, we assume that $G/S_1/S_2$ does contain G_0. Let $s_i = \underline{S_i}$ for $i = 1, 2$. Observe that both s_1 and s_2 have four neighbors u, v, p, q other than x in G_0. It follows that

$$|E(S_1 \cup S_2, \{u, v, p, q\})| \geq 8.$$

On the one hand, we have

$$\begin{aligned} |E(G\,|_{S_1 \cup S_2})| &= \frac{1}{2}\left(\sum_{v \in S_1 \cup S_2} d(v) - |E(S_1 \cup S_2, \{u, v, p, q\})| - 4\right) \\ &\leq \frac{1}{2}(16 - 8 - 4) \\ &= 2. \end{aligned}$$

On the other hand, $\alpha(\Gamma(x)) = 2$ implies $G\mid_{S_1 \cup S_2}$ contains at least two edges. Thus, we have $\Gamma_G(x) = 2\ e$. Label the vertices in $\Gamma_G(x)$ by a, b, c, d as in Figure 4.1. We assume ab and cd are edges while ac, bd, ad, bc are non-edges. Observe that each vertex in $\{u, v, p, q\}$ has exactly two neighbors in $\{a, b, c, d\}$.

If one vertex, say u, has two neighbors forming a non-edge, say ac, then we can choose $S_1' = \{a, c\}$ and $S_2' = \{b, d\}$. It is easy to check that $G/S_1'/S_2'$ contains neither G_0 nor K_5^-. We are done in this case.

In the remaining case, we can assume that for each vertex y in $\{u, v, p, q\}$, the neighbors of y in $\{a, b, c, d\}$ always form an edge. Up to relabeling vertices, there is only one arrangement for edges between $\{u, v, p, q\}$ and $\{a, b, c, d\}$; see the graph H_{10} defined in Figure 4.7. The

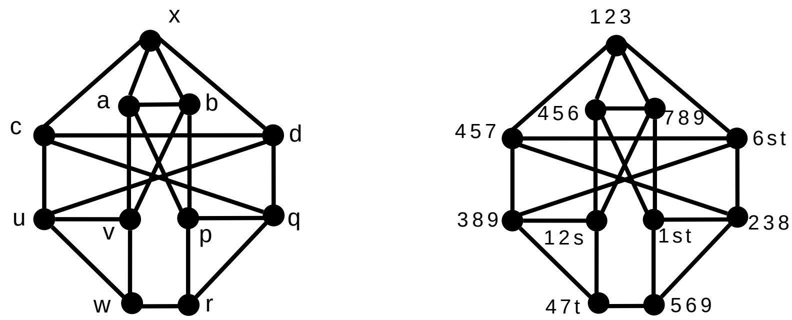


Fig. 4.7. *H_{10} and an 11:3-coloring of H_{10}.*

graph H_{10} is 11:3-colorable as shown in Figure 4.7. Since $\chi_f(G) > 11/3$, H_{10} is a proper subgraph of G. Note in H_{10}, every vertices except w and r has degree 4; both w and r have degree 3. Thus, $\{w, r\}$ is a vertex cut of G. Let $G_1 = H_{10}$ and G_2 be the subgraph of G by deleting vertices in $\{x, a, b, c, d, p, q, u, v\}$. Applying Lemma 4.1 with G_1 and G_2 defined above, we have

$$\chi_f(G) \le \max\{\chi_f(G_1), \chi_f(G_2)\}.$$

Since $\chi_f(G) > 11/3$ and $\chi_f(G_1) \le 11/3$ (see Figure 4.7), we must have $\chi_f(G_2) \ge \chi_f(G)$. Note that G_2 has fewer number of vertices than G. This contradicts the minimality of G. Therefore, the lemma follows. □

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