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EXPLICIT CONSTRUCTION OF SMALL FOLKMAN GRAPHS*

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Abstract. A Folkman graph is a K_4-free graph G such that if the edges of G are 2-colored, then there exists a monochromatic triangle. Erdős offered a prize for proving the existence of a Folkman graph with at most 1 million vertices. In this paper, we construct several "small" Folkman graphs within this limit. In particular, there exists a Folkman graph on 9697 vertices.

Key words. Folkman graph, spectrum, K_4-free, monochromatic triangle, circulant graph

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1. Introduction.

For two graphs G and H, the Rado arrow notation $G \to (H)_p$ is the statement that if the edges of G are p-colored, then there exists a monochromatic subgraph of G isomorphic to H. In 1967 Erdős and Hajnal [2] (also see [3]) conjectured that for each p there exists a graph G, containing no K_4, which has the property that $G \to (K_3)_p$. This conjecture was proved by Folkman [4] for $p = 2$. A Folkman graph is a K_4-free graph G with $G \to (K_3)_2$. Nešetřil and Rödl [9] proved the conjecture for general p. In particular, for any $k_1 < k_2$ and any $p \geq 2$, one could ask what is the smallest integer n such that there is a K_{k_2}-free graph G on n vertices satisfying

$$G \to (K_{k_1})_p.$$

Let $f(p, k_1, k_2)$ denote this smallest integer n. Graham [6] proved that $f(2, 3, 6) = 8$ by showing

$$K_8 \setminus C_5 \to (K_3)_2.$$

Irving [7] proved that $f(2, 3, 5) \leq 18$, and it was further improved by Khadzhiivanov and Nenov [8] to $f(2, 3, 5) \leq 16$. Finally, Piwakowski, Radziszowski, and Urbański [13] and Nenov [12] proved $f(2, 3, 5) = 15$. However, both upper bounds of Folkman and of Nešetřil and Rödl for $f(2, 3, 4)$ are extremely large. Frankl and Rödl [5] first gave a reasonable bound

$$f(2, 3, 4) \leq 7 \times 10^{11}.$$

Erdős set a prize of \$100 for the challenge $f(2, 3, 4) \leq 10^{10}$. This reward was claimed by Spencer [10, 11], who proved that

$$f(2, 3, 4) < 3 \times 10^9.$$

Erdős then offered another \$100 prize (see [1, page 46]) for the new challenge

$$f(2, 3, 4) < 10^6.$$

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Here we claim the reward.

THEOREM 1.

$$f(2,3,4) \le 9697.$$

In fact, we construct several "small" Folkman graphs. This paper is organized as follows. In section 2, we use spectral analysis to establish a sufficient condition for $G \to (K_3)_2$. This allows us to test a set of graphs efficiently. In section 3, we examine a special class of graphs and find four "small" Folkman graphs.

2. Spectral analysis.

2.1. Localization. Our starting point is the following lemma from Spencer [10]. We will use the following notation.

For any graph H and a vertex-set partition $V(H) = X \cup Y$, let $e(X,Y)$ be the number of edges in H with one end in X and the other end in Y. Let $b(H)$ be the maximum of $e(X,Y)$ among all partition $V(H) = X \cup Y$.

Consider a random partition $V(H) = X \cup Y$ by putting each vertex independently into X or Y with equal probability. The expected number of $e(X,Y)$ is exactly $\frac{1}{2}|E(H)|$. Thus we have

$$b(H) \ge \frac{1}{2}|E(H)|.$$

DEFINITION 1. *For $0 < \delta < \frac{1}{2}$, a graph H is said to be* δ-fair *if $b(H) < (\frac{1}{2} + \delta)|E(H)|$.*

Supposing $G \not\to (K_3)_2$, we see that the edges of G can be colored in red and blue with no monochromatic triangle. For each triangle, there are two possible colorings (two red edges and a blue edge or vice versa). Each triangle has two vertices incident with a red edge and a blue edge. Thus

$$|\{xyz\colon xy \text{ is a red edge}, xz \text{ is a blue edge, and } yz \text{ is an edge}\}| = 2|\{\text{all triangles}\}|.$$

For any vertex $v \in V(G)$, let $\Gamma(v)$ be the set of neighbors of v in G. Let G_v be the induced subgraph on $\Gamma(v)$. The left-hand side of the above equation is at most $\sum_v b(G_v)$ while the right-hand side is exactly $\frac{2}{3}\sum_v |E(G_v)|$. This observation leads to the following lemma.

LEMMA 1 (see Spencer [10]). *If $\sum_v b(G_v) < \frac{2}{3}\sum_v |E(G_v)|$, then $G \to (K_3)_2$.*

COROLLARY 1. *Suppose for each vertex v the local graph G_v is $\frac{1}{6}$-fair. Then*

$$G \to (K_3)_2.$$

If in addition G_v is triangle-free for each v, then G is a Folkman graph.

2.2. δ-fair graphs. Suppose H is a graph on vertices $v_1, v_2, \ldots, v_n$. Let $A = (a_{ij})$ be the adjacency matrix of H so that

$$a_{ij} = \begin{cases} 1 & v_iv_j \text{ is an edge of } H; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{1}$ denote the n-dimensional column vector with all entries 1. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)'$ be the column vector of degrees. Here d_i is the degree of vertex v_i. By definition, we have

$$\mathbf{d} = A \cdot \mathbf{1}. \tag{1}$$

For any set $S \subset V(H)$, the volume of S is defined as

$$\mathrm{Vol}(S) = \sum_{v \in S} d_v.$$

We write $\mathrm{Vol}(H) = \mathrm{Vol}(V(H)) = \sum_v d_v = 2|E(H)|$. Let $\bar{d} = \frac{\mathrm{Vol}(H)}{n}$ be the average degree of H.

LEMMA 2. *If the smallest eigenvalue of $M = A - \frac{1}{\mathrm{Vol}(H)} \mathbf{d} \cdot \mathbf{d}'$ is greater than $-2\delta\bar{d}$, then H is δ-fair.*

Proof. For any partition of the vertex set $V(H) = X \cup Y$, let $\mathbf{1}_X$ be the n-dimensional column vector whose entries are 1 if the index is in X and 0 otherwise. The vector $\mathbf{1}_Y$ is defined similarly. By definition, we have

$$\mathbf{1}_X + \mathbf{1}_Y = \mathbf{1}. \tag{2}$$

From (1), we have

$$\begin{aligned} M \cdot \mathbf{1} &= \left(A - \frac{1}{\mathrm{Vol}(H)} \mathbf{d} \cdot \mathbf{d}' \right) \cdot \mathbf{1} \\ &= A \cdot \mathbf{1} - \frac{1}{\mathrm{Vol}(H)} \mathbf{d} \cdot \mathbf{d}' \cdot \mathbf{1} \\ &= \mathbf{d} - \frac{1}{\mathrm{Vol}(H)} \mathbf{d} \mathrm{Vol}(H) \\ &= 0. \end{aligned}$$

Thus, 0 is always an eigenvalue of M and $\mathbf{1}$ is the corresponding eigenvector.

Let $\alpha(t) = (1-t)\mathbf{1}_X - t\mathbf{1}_Y$. For any t, we claim

$$\alpha(t)' \cdot M \cdot \alpha(t) = -e(X,Y) + \frac{1}{\mathrm{Vol}(H)} \mathrm{Vol}(X)\mathrm{Vol}(Y).$$

From (2), we can rewrite

$$\alpha(t) = \mathbf{1}_X - t\mathbf{1} = -\mathbf{1}_Y + (1-t)\mathbf{1}.$$

We have

$$\begin{aligned} \alpha(t)' \cdot M \cdot \alpha(t) &= (\mathbf{1}_X - t\mathbf{1})' \cdot M \cdot (-\mathbf{1}_Y + (1-t)\mathbf{1}) \\ &= -\mathbf{1}_X' \cdot M \cdot \mathbf{1}_Y \\ &= -\mathbf{1}_X' \cdot A \cdot \mathbf{1}_Y + \frac{1}{\mathrm{Vol}(H)} \mathbf{1}_X' \cdot \mathbf{d} \cdot \mathbf{d}' \cdot \mathbf{1}_Y \\ &= -e(X,Y) + \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(H)}. \end{aligned}$$

Here we use the fact that $M \cdot \mathbf{1} = 0$.

Let ρ be the largest eigenvalue of $-M$. By assumption, $\rho < 2\delta\bar{d}$. We have

$$\begin{aligned} e(X,Y) - \frac{1}{\mathrm{Vol}(H)} \mathrm{Vol}(X)\mathrm{Vol}(Y) &= \alpha(t)' \cdot (-M) \cdot \alpha(t) \\ &\le \rho \|\alpha(t)\|^2. \end{aligned}$$

Choose $t = \frac{|X|}{n}$ so that $\|\alpha(t)\|^2$ reaches its minimum $\frac{|X||Y|}{n}$. We have

$$e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(H)} \leq \rho\frac{|X||Y|}{n}.$$

Apply the Cauchy–Schwarz inequalities to $\mathrm{Vol}(X)\mathrm{Vol}(Y)$ and to $|X||Y|$. We have

$$\begin{aligned}
e(X,Y) &\leq \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(H)} + \rho\frac{|X||Y|}{n}.\\
&\leq \frac{(\mathrm{Vol}(X)+\mathrm{Vol}(Y))^2}{4\mathrm{Vol}(H)} + \rho\frac{(|X|+|Y|)^2}{4n}\\
&= \frac{\mathrm{Vol}(H)}{4} + \rho\frac{n}{4}\\
&< \frac{\mathrm{Vol}(H)}{4} + 2\delta\bar{d}\frac{n}{4}\\
&= (1+2\delta)\frac{\mathrm{Vol}(H)}{4}\\
&= (\frac{1}{2}+\delta)|E(H)|.
\end{aligned}$$

Since this holds for any partition $X \cup Y$, we have

$$b(H) \leq \left(\frac{1}{2}+\delta\right)|E(H)|.$$

H is δ-fair as claimed. ∎

COROLLARY 2. *Suppose H is a d-regular graph and that the smallest eigenvalue of its adjacency matrix A is greater than $-2\delta d$. Then H is δ-fair.*

Proof. Since H is d-regular, we have $\mathbf{d} = d\mathbf{1}$ and $\mathrm{Vol}(H) = nd$. Thus,

$$M = A - \frac{d}{n}\mathbf{1}\cdot\mathbf{1}'.$$

Note that $\mathbf{1}$ is the eigenvector of A with respect to the eigenvalue d. Suppose α is another eigenvector of A with respect to an eigenvalue λ ($\lambda \neq d$). The eigenvector α is orthogonal to $\mathbf{1}$. We have $M\alpha = A\alpha = \lambda\alpha$. Suppose A has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n = d$. Then M has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, and 0. In particular, the smallest eigenvalue of M equals the smallest eigenvalue of A. The conclusion follows from Lemma 2. ∎

Remark. The largest Laplacian eigenvalue of graph H can also be used to derive the δ-fairness of H. However, in practice, it is not as effective as the matrix M.

2.3. The spectrum of circulant graphs. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n. A circulant graph H generated by a subset $S \subset \mathbb{Z}_n$ is a graph with the vertex set $V(H) = \mathbb{Z}_n$ and the edge set $E(H) = \{xy \mid x - y \in S\}$. Here $S \subset \mathbb{Z}_n$ is a subset satisfying that

- if $s \in S$, then $-s \in S$;
- $0 \notin S$.

The following lemma determines the spectrum of circulant graphs.

LEMMA 3. *The eigenvalues of the adjacency matrix for the circulant graph generated by $S \subset \mathbb{Z}_n$ are*

$$\sum_{s\in S} \cos\frac{2\pi i s}{n}$$

for $i = 0, \ldots, n-1$.

Proof. Let $J = (J_{ij})$ be the adjacency matrix of the directed cycle on n vertices. Namely, $J_{ij} = 1$ if $j - i \equiv 1 \mod n$, and 0 otherwise. The adjacency matrix of the circulant graph generated by $(\mathbb{Z}_n, S)$ can be expressed as

$$A = \sum_{s \in S} J^s.$$

We identify elements $\mathbb{Z}_n$ with $0, 1, 2, \ldots, n-1$ and define a polynomial $f(x) = \sum_{s \in S} x^s$. Note that $A = f(J)$. The eigenvalues of A are completely determined by the eigenvalues of J and the polynomial $f(x)$.

Let $\rho = e^{\frac{2\pi i}{n}}$ denote the primitive nth unit root. We observe that J has eigenvalues

$$1, \rho, \rho^2, \ldots, \rho^{n-1}.$$

Thus, the eigenvalues of A are

$$f(1), f(\rho), \ldots, f(\rho^{n-1}).$$

Since A is symmetric, the above eigenvalues are all real. For $i = 0, 1, 2, \ldots, n-1$, we have

$$f(\rho^i) = \Re(f(\rho^i)) = \sum_{s \in S} \cos \frac{2\pi i s}{n}. \qquad \square$$

3. Graph $L(m, s)$. The previous section allows us to test a special class of graphs efficiently.

Suppose m is an odd positive integer and $s < m$ is another positive integer relatively prime to m. Let $\phi(m)$ be the totient function of m, which is the number of positive integers not exceeding m and relatively prime to m. By Euler's theorem, we have $s^{\phi(m)} \equiv 1 \mod m$. Let n be the smallest positive integer satisfying $s^n \equiv 1 \mod m$. In particular, n is a factor of $\phi(m)$. Define a subset $S = S(s) \subset \mathbb{Z}_m$ as

$$S = \{s^i \mod m \mid i = 0, 1, 2, \ldots, n-1\}.$$

We observe that

- if $-1 \in S$, then for any $t \in S$, $-t \in S$;
- with inherited multiplication from $\mathbb{Z}_m$, S forms an abelian group isomorphic to $\mathbb{Z}_n$.

DEFINITION 2. *We define* **graph** $\mathbf{L(m, s)}$ *to be the circulant graph on m vertices generated by $S = S(s)$ provided $-1 \in S$.*

The graph $G = L(m, s)$ is a vertex-transitive graph on m vertices. All local graphs G_v are isomorphic to each other. The following lemma shows that G_v is also a circulant graph under isomorphism.

LEMMA 4. *The unique local graph of $L(m, s)$ is isomorphic to a circulant graph of order n.*

Proof. The local graph H of $L(m, s)$ can be described as follows.

1. $V(H) = S$.
2. $E(H) = \{xy \mid x \in S, \ y \in S, \text{ and } x - y \in S\}$.

We define a bijection $f : \mathbb{Z}_n \to S$ which maps i to $s^i \mod m$. This is a well-defined map since $s^n \equiv 1 \mod m$. The map f is a group isomorphism from $\mathbb{Z}_n$ to S:

$$f(i + j) = f(i) f(j).$$

We define $T \subset \mathbb{Z}_n$ as

$$T = \{i \mid f(i) - 1 \in S\}.$$

Let H' be the circulant graph generated by $(\mathbb{Z}_n, T)$. If suffices to show f is a graph homomorphism mapping H' to H.

On the one hand, for any edge $jk \in E(H')$, we have $j - k \in T$. Thus,

$$f(j-k) - 1 \in S.$$

Since $f(j)-f(k) = f(k)(f(j-k)-1)$ and S is a group, we conclude that $f(j)-f(k) \in S$. Equivalently, $f(j)f(k)$ is an edge of H.

On the other hand, for any edge $f(j)f(k) \in E(H)$, we have $f(j)-f(k) \in S$. Note that $f(-k)$ is the inverse of $f(k)$ in S. We conclude that

$$f(j-k) - 1 = f(-k)(f(j) - f(k)) \in S.$$

Thus, $j - k \in T$ and jk is an edge of H'. $\square$

3.1. Results from computation. For a fixed pair (m, s), let H be the local graph of $L(m, s)$ and A the adjacency matrix of H. Let $\sigma = \sigma(m, s)$ be the ratio of the smallest eigenvalue and the largest eigenvalue of A. If $\sigma > -\frac{1}{3}$, then H is $\frac{1}{6}$-fair from Corollary 2. Thus, from Corollary 1, $L(m, s) \to (K_3)_2$. Table 1 (except for the last row) shows graphs $L(m, s)$ satisfying that

1. $L(m, s)$ is K_4-free;
2. $\sigma = \sigma(m, s)$ is maximized in the sense that $\sigma(m, s) > \sigma(m', s')$, for all pairs (m', s') in the table and $m' < m$.

We note that $\sigma > -\frac{1}{3}$ in the last four rows of Table 1. Thus, $L(9697, 4)$, $L(30193, 53)$, $L(33121, 2)$, and $L(57401, 7)$ are Folkman graphs.

TABLE 1
A set of candidates for Folkman graphs.

$L(m, s)$	σ
$L(17, 2)$	$-0.8047\cdots$
$L(61, 8)$	$-0.7826\cdots$
$L(79, 12)$	$-0.7625\cdots$
$L(127, 5)$	$-0.6363\cdots$
$L(421, 7)$	$-0.6253\cdots$
$L(457, 6)$	-0.6
$L(631, 24)$	$-0.5749\cdots$
$L(761, 3)$	$-0.5613\cdots$
$L(785, 53)$	$-0.5404\cdots$
$L(941, 12)$	$-0.5376\cdots$
$L(1777, 53)$	$-0.5216\cdots$
$L(1801, 125)$	$-0.4912\cdots$
$L(2641, 2)$	$-0.4275\cdots$
$L(9697, 4)$	$-0.3307\cdots$
$L(30193, 53)$	$-0.3094\cdots$
$L(33121, 2)$	$-0.2665\cdots$
$L(57401, 7)$	$-0.3289\cdots$

Proof of Theorem 1. It suffices to show that $G = L(9697, 4)$ is a Folkman graph. The local graph of G is a circulant graph H generated by $T \subset \mathbb{Z}_n$. Here $n = 1212$

and

$$T = \{3, 9, 46, 57, 62, 70, 81, 91, 98, 115, 141, 166, 202, 204, 233, 271, 286, 301, 325, 342, 372, 376, 383, 396, 397, 403, 411, 428, 430, 436, 448, 450, 456, 471, 472, 479, 489, 516, 522, 532, 556, 564, 566, 588, 593, 595, 617, 619, 624, 646, 648, 656, 680, 690, 696, 723, 733, 740, 741, 756, 762, 764, 776, 782, 784, 801, 809, 815, 816, 829, 836, 840, 870, 887, 911, 926, 941, 979, 1008, 1010, 1046, 1071, 1097, 1114, 1121, 1131, 1142, 1150, 1155, 1166, 1203, 1209\}.$$

An easy calculation (by Maple) shows that H has the following properties:

1. H is a 92-regular and triangle-free graph.
2. The smallest eigenvalue of the adjacency matrix of H is

$$\sum_{t \in T} \cos \frac{2\pi \cdot 502t}{1212} \approx -30.43170597\ldots.$$

Since $30.43170597\ldots < \frac{92}{3}$, H is $\frac{1}{6}$-fair. Thus, $L(9697, 4)$ is a Folkman graph on 9697 vertices. ∎

Remark 1. We say G is a *strong* Folkman graph if G is K_4-free and $G \to (K_4 - e)_2$. Here $K_4 - e$ is the graph obtained by removing one edge from K_4. We can show that both $L(30193, 53)$ and $L(33121, 2)$ are strong Folkman graphs.

Remark 2. Graphs with relatively large σ (as shown in Table 1) are good candidates for Folkman graphs. Recently Exoo showed that $L(17, 2)$, $L(61, 8)$, $L(79, 12)$, $L(421, 7)$, and $L(631, 24)$ are not Folkman graphs. Little is known for other graphs. For example, is $L(2641, 2)$ a Folkman graph?

Remark 3. Exoo (see [14]) conjectured that $L(127, 5)$ is a Folkman graph. The set $S \subset \mathbb{Z}_{127}$ generated by 5 is precisely all nonzero cubes in $\mathbb{Z}_{127}$. Exoo did extensive computation on this graph. If his conjecture is true, then it implies $f(2, 3, 4) \le 127$.

Remark 4. Recently, Dudek and Rödl independently proved $f(2, 3, 4) < 130000$.

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