Diamond-free families

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Given a finite poset P, we consider the largest size $\mathrm{La}(n, P)$ of a family of subsets of $[n] := \{1, \ldots, n\}$ that contains no (weak) subposet P. This problem has been studied intensively in recent years, and it is conjectured that $\pi(P) := \lim_{n \to \infty} \mathrm{La}(n, P)/\binom{n}{\lfloor \frac{n}{2} \rfloor}$ exists for general posets P, and, moreover, it is an integer. For $k \geqslant 2$ let $\mathcal{D}_k$ denote the k-diamond poset $\{A < B_1, \ldots, B_k < C\}$. We study the average number of times a random full chain meets a P-free family, called the Lubell function, and use it for $P = \mathcal{D}_k$ to determine $\pi(\mathcal{D}_k)$ for infinitely many values k. A stubborn open problem is to show that $\pi(\mathcal{D}_2) = 2$; here we make progress by proving $\pi(\mathcal{D}_2) \leqslant 2\frac{3}{11}$ (if it exists).

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1. Introduction

We are interested in how large a family of subsets of the n-set $[n] := \{1, \ldots, n\}$ there is that avoids a given (weak) subposet P. The foundational result of this sort, Sperner's Theorem from 1928 [16], solves this problem for families that contain no two-element chain (that is, for antichains), determining that the maximum size is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. For other excluded subposets, it is interesting to compare the maximum size of a P-free family to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We give background and our new results for this study in the next section. One small forbidden poset that continues to stymie all interested researchers is the diamond poset on four elements. We present a better new upper bound on the size of diamond-free families. For k-diamond-free families for general k, we provide bounds that, surprisingly, turn out to be best-possible for infinitely many values of k.

In Section 3 we introduce our method for this subject, the Lubell function of a family, which gives the average number of times a random full chain meets the family. The Lubell function yields an upper

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bound on the size of a family. For diamond-free families, we observe that the maximum possible Lubell function decreases with n, and a calculation gives our bound. For excluding k-diamonds, our new idea is to partition the set of full chains, obtaining bounds on each block of the partition. The Lubell and full chain partition methods hold promise for other families of forbidden subposets.

Section 4 contains our detailed proofs, except for the long proof of Theorem 2.4, which is given in Section 5. The paper concludes with our ideas for advancing the project.

2. Background and main results

For posets $P = (P, \leqslant)$ and $P' = (P', \leqslant')$, we say P' is a *weak subposet of P* if there exists an injection $f : P' \to P$ that preserves the partial ordering, meaning that whenever $u \leqslant' v$ in P', we have $f(u) \leqslant f(v)$ in P [17]. Throughout the paper, when we say subposet, we mean weak subposet. We say the *height* $h(P)$ of poset P is the maximum size of any chain in P.

Let the Boolean lattice $\mathcal{B}_n$ denote the poset $(2^{[n]}, \subseteq)$. We consider collections $\mathcal{F} \subseteq 2^{[n]}$. Then $\mathcal{F}$ can be viewed as a subposet of $\mathcal{B}_n$. If $\mathcal{F}$ contains no subposet P, we say $\mathcal{F}$ is *P-free*. We are interested in determining the largest size of a P-free family of subsets of $[n]$, denoted $\mathrm{La}(n, P)$.

In this notation, Sperner's Theorem [16] gives that $\mathrm{La}(n, \mathcal{P}_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, where $\mathcal{P}_k$ denotes the path poset on k points, usually called a chain of size k. Moreover, Sperner determined that the largest antichains in $\mathcal{B}_n$ are the middle level, $\binom{[n]}{n/2}$ (for even n) and either of the two middle levels, $\binom{[n]}{(n-1)/2}$ or $\binom{[n]}{(n+1)/2}$ (for odd n), where for a set S, $\binom{S}{i}$ denotes the collection of i-subsets of S. More generally, Erdős solved the case of $\mathcal{P}_k$-free families. Let us denote by $\Sigma(n, k)$ the sum of the k middle binomial coefficients in n, and let $\mathcal{B}(n, k)$ denote the collection of subsets of $[n]$ of the k middle sizes, that is, the sizes $\lfloor (n-k+1)/2 \rfloor, \ldots, \lfloor (n+k-1)/2 \rfloor$ or else the sizes $\lceil (n-k+1)/2 \rceil, \ldots, \lceil (n+k-1)/2 \rceil$. So there are either one or two possible families $\mathcal{B}(n, k)$, depending on the parities of n and k, and regardless, $|\mathcal{B}(n, k)| = \Sigma(n, k)$. Then we have

Theorem 2.1. *(See [6].) For $n \geqslant k - 1 \geqslant 1$, $\mathrm{La}(n, \mathcal{P}_k) = \Sigma(n, k-1)$. Moreover, the $\mathcal{P}_k$-free families of maximum size in $\mathcal{B}_n$ are given by $\mathcal{B}(n, k)$.*

It follows that for fixed k, $\mathrm{La}(n, \mathcal{P}_k) \sim (k-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$ as $n \to \infty$. Katona and his collaborators promoted the problem of investigating $\mathrm{La}(n, P)$ for other posets P, especially its asymptotic behavior for large n. Consider the r-fork poset $\mathcal{V}_r$, which has elements $A < B_1, \ldots, B_r$, $r \geqslant 2$. In 1981 he and Tarján [13] obtained bounds on $\mathrm{La}(n, \mathcal{V}_2)$ that he and DeBonis [4] extended in 2007 to general $\mathcal{V}_r$, $r \geqslant 2$, proving that

$$\left(1 + \frac{r-1}{n} + \Omega\left(\frac{1}{n^2}\right) \right) \binom{n}{\lfloor \frac{n}{2} \rfloor} \leqslant \mathrm{La}(n, \mathcal{V}_r) \leqslant \left(1 + 2\frac{r-1}{n} + O\left(\frac{1}{n^2}\right) \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

While the lower bound is strictly greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, we see that $\mathrm{La}(n, \mathcal{V}_r) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Earlier, Thanh [18] had investigated the more general class of broom-like posets. Griggs and Lu [11] studied the even more general class of baton posets. These posets mentioned so far have Hasse diagrams that are trees. Carroll and Katona [3] considered forbidding induced V_2 subposets. Another survey of the subject is [12].

In [5] it is shown that for the butterfly poset $\mathcal{B}$, with elements A, B both less than C, D, one can give an exact answer, $\mathrm{La}(n, \mathcal{B}) = \Sigma(n, 2)$, for $n \geqslant 3$, which is asymptotic to $2\binom{n}{\lfloor \frac{n}{2} \rfloor}$. More generally, for any $s, t \geqslant 2$, the complete bipartite poset $\mathcal{K}_{s,t}$ with elements $A_1, \ldots, A_s$ all less than $B_1, \ldots, B_t$, satisfies $\mathrm{La}(n, \mathcal{K}_{s,t}) \sim 2\binom{n}{\lfloor \frac{n}{2} \rfloor}$ [4]. The $\mathcal{N}$-poset, with elements A, B, C, D such that $A < B$, $C < B$, $C < D$, is intermediate between $\mathcal{V}_2$ and $\mathcal{B}$. It is shown in [8] that $\mathrm{La}(n, \mathcal{N}) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Based on the examples for which $\mathrm{La}(n, P)$ was known, Griggs and Lu [11] proposed the conjecture that was certainly apparent to Katona et al.:

Conjecture 2.2. *For every finite poset P, the limit $\pi(P) := \lim_{n \to \infty} \frac{\mathrm{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists and is integer.*

All of the examples above agree with the conjecture, and Griggs and Lu verified it for additional examples, including tree posets of height 2. For the crown $\mathcal{O}_{2k}$, $k \geqslant 2$, which is the poset of height 2 that is a cycle of length $2k$ as an undirected graph, they extended the butterfly result above and proved that $\mathrm{La}(n, \mathcal{O}_{2k}) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all even $k \geqslant 4$. For odd $k \geqslant 3$, it remains a daunting problem to determine the asymptotic behavior of $\mathrm{La}(n, \mathcal{O}_{2k})$. At least, Griggs and Lu can show $\mathrm{La}(n, \mathcal{O}_{2k})/\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is asymptotically at most $1 + \frac{1}{\sqrt{2}}$, which is less than 2.

When Griggs lectured on this work on forbidden subposets in 2008, Mike Saks and Peter Winkler observed a pattern in all of the examples where $\pi(P)$ was determined, which we describe as follows. For poset P define $e(P)$ to be the maximum m such that for all n, the union of the m middle levels $\mathcal{B}(n, m)$ does not contain P as a subposet. Their observation was $\pi(P) = e(P)$. For instance, the middle two levels $\mathcal{B}(n, 2)$ contain no butterfly $\mathcal{B}$, since no two sets of the same size k contain the same two subsets of size $k - 1$. One gets that $e(\mathcal{B}) = 2$, which is $\pi(\mathcal{B})$. In general, it is clear that when it exists, $\pi(P)$ must be at least $e(P)$.

Impressive progress in the development of the theory is the result of Bukh [2] that for any tree poset T (meaning that the Hasse diagram is a tree), $\pi(T) = e(T)$, so that the conjecture (and observation) are satisfied. It is easily verified that $e(T) = h(T) - 1$.

Is there a connection for general P between $\pi(P)$ and the height $h(P)$? The result of DeBonis and Katona for complete bipartite posets $\mathcal{K}_{s,t}$ implies that for any poset P of height 2, $\pi(P) \leqslant 2$, when it exists. However, there is no such bound for taller posets, as observed by Jiang and Lu (see [11]). Let the k-diamond poset $\mathcal{D}_k$, $k \geqslant 2$, consist of $k + 2$ elements $A < B_1, \ldots, B_k < C$. Then $h(\mathcal{D}_k) = 3$ for general k, while for $k = 2^r - 1$, the middle $r + 1$ levels $\mathcal{B}(n, r+1)$ cannot contain $\mathcal{D}_k$, since an interval in $\mathcal{B}(n, r+1)$ with an element in the lowest level and an element in the highest level has at most 2^r elements (a subposet $\mathcal{B}_r$), and so at most $2^r - 2$ elements in the middle. Hence it is $\mathcal{D}_k$-free.

The diamond $\mathcal{D}_2$ is the most challenging poset on at most four elements in this theory. (It is also the Boolean lattice $\mathcal{B}_2$.) It is easily seen that $e(\mathcal{D}_2) = 2$. On the other hand, it is a subposet of the path $\mathcal{P}_4$. So, if $\pi(\mathcal{D}_2)$ exists (which has still not been shown), it would have to be in $[2, 3]$; Its conjectured value is 2.

As an illustration of the Lubell function method introduced in the next section, a short application is given that reduces the upper bound on $\pi(\mathcal{D}_2)$ from 3 to 2.5. A refinement of the Lubell function method, which involves partitioning the set of full chains in an appropriate way, gives our first improvement on the 2.5 bound:

Proposition 2.3. *For all sufficiently large n, $\mathrm{La}(n, \mathcal{D}_2)/\binom{n}{\lfloor \frac{n}{2} \rfloor} < 2.296$.*

We display this bound, not our best one, since its proof is simpler than our best bound, and since its proof gives us further insight into the Lubell function for $\mathcal{D}_2$. Some time after we had announced our bound above, Axenovich, Manske, and Martin [1] came up with a new approach which improves the upper bound to 2.283. Now using our methods with a much more careful analysis of diamond-free families for $n \leqslant 12$, we can provide a further slight improvement, which is the best-known upper bound:

Theorem 2.4. *For all sufficiently large n, $\mathrm{La}(n, \mathcal{D}_2)/\binom{n}{\lfloor \frac{n}{2} \rfloor} < 2\frac{3}{11} + o_n(1) < 2.273$. Consequently, if it exists, $\pi(\mathcal{D}_2) \in [2, 2\frac{3}{11}]$.*

Because this new bound requires considerably more care, its proof is given in its own section following the proofs of our other results. We shall see diamond-free families in the proof for which the Lubell function method cannot improve the upper bound on $\pi(\mathcal{D}_2)$ below 2.25. Therefore, new ideas are required to bring the upper bound down to the conjectured value of 2. Likewise, it appears that the methods of [1] cannot move below 2.25. See the final section of the paper for more discussion of how we can do better.

Given the great effort that has gone into improving the upper bound on $\pi(\mathcal{D}_2)$, it is then quite surprising that we can solve the π problem for many of the general diamonds $\mathcal{D}_k$ with $k > 2$. This can be regarded as our main result.

Theorem 2.5. *Let $k \geqslant 2$, and define $m := \lceil \log_2(k+2) \rceil$.*

(1) *If $k \in [2^{m-1} - 1, 2^m - \binom{m}{\lfloor \frac{m}{2} \rfloor} - 1]$, then*

$$\mathrm{La}(n, \mathcal{D}_k) = \Sigma(n, m).$$

Hence, $\pi(\mathcal{D}_k) = e(\mathcal{D}_k) = m$. Moreover, if $\mathcal{F}$ attains the bound $\mathrm{La}(n, \mathcal{D}_k)$, then $\mathcal{F} = \mathcal{B}(n, m)$.

(2) *If $k \in [2^m - \binom{m}{\lfloor \frac{m}{2} \rfloor}, 2^m - 2]$, then,*

$$\Sigma(n, m) \leqslant \mathrm{La}(n, \mathcal{D}_k) \leqslant \left(m + 1 - \frac{2^m - k - 1}{\binom{m}{\lfloor \frac{m}{2} \rfloor}} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Hence, if $\pi(\mathcal{D}_k)$ exists, then

$$m = e(\mathcal{D}_k) \leqslant \pi(\mathcal{D}_k) \leqslant \left(m + 1 - \frac{2^m - k - 1}{\binom{m}{\lfloor \frac{m}{2} \rfloor}} \right) < m + 1.$$

For $\pi(\mathcal{D}_2)$ this new theorem gives an upper bound of 2.5, not as good as the theorem before. However, this new result determines $\pi(\mathcal{D}_k)$ for "most" values of k, in that for the 2^{m-1} values of k in the range $[2^{m-1} - 1, 2^m - 2]$, case (1) applies to all but $\binom{m}{\lfloor \frac{m}{2} \rfloor} \sim C 2^m / m^{1/2}$ of them. Moreover, we are able to give $\mathrm{La}(n, \mathcal{D}_k)$ exactly, not just asymptotically for large n, for such values of k.

The poset $\mathcal{D}_k$ can be viewed as the "suspension" of an independent set of size k, where we mean that a maximum and a minimum element are added to it. We can consider a more general suspension of disjoint paths (chains). For $k \geqslant 1$ let $l_1 \geqslant \cdots \geqslant l_k \geqslant 3$, and define the *harp poset* $\mathcal{H}(l_1, \ldots, l_k)$ to consist of paths $\mathcal{P}_{l_1}, \ldots, \mathcal{P}_{l_k}$ with their top elements identified and their bottom elements identified. For instance, in this notation we have $\mathcal{D}_k$ is the harp $\mathcal{H}(3, \ldots, 3)$ where there are k 3's.

Theorem 2.6. *If $l_1 > \cdots > l_k \geqslant 3$, then*

$$\mathrm{La}\big(n, \mathcal{H}(l_1, \ldots, l_k)\big) = \Sigma(n, l_1 - 1).$$

Hence, for such harps, $\pi = e = l_1 - 1$. Moreover, for such harps, if $\mathcal{F}$ is a harp-free family of subsets of $[n]$ of maximum size, then $\mathcal{F}$ is $\mathcal{B}(n, l_1 - 1)$.

The theorem above only determines $\pi(\mathcal{H})$ for harps $\mathcal{H}$ that have strictly decreasing path lengths. However, for the general case in which path lengths can be equal there is no bound independent of k, since we have seen that for $\mathcal{D}_k$, which is a harp, $\pi(\mathcal{D}_k)$ is arbitrarily large as k grows. It is then remarkable that we can completely solve the problem of maximizing $\mathrm{La}(n, \mathcal{H})$ for harps with distinct path lengths. Another novel aspect of this result is that for $k \geqslant 2$ the harps it concerns are not ranked posets.

3. The Lubell function

For now let us fix some family $\mathcal{F} \subseteq 2^{[n]}$. Let $\mathcal{C} := \mathcal{C}_n$ denote the collection of all $n!$ full (maximal) chains $\emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset [n]$ in the Boolean lattice $\mathcal{B}_n$. A method used by Katona et al. involves counting the number of full chains that meet $\mathcal{F}$. Here we collect information about the average

number of times chains $C \in \mathcal{C}$ meet $\mathcal{F}$, which can be used to give an upper bound on $|\mathcal{F}|$. Recall that the *height* of $\mathcal{F}$, viewed as a poset, is

$$h(\mathcal{F}) := \max_{C \in \mathcal{C}} |\mathcal{F} \cap C|.$$

We consider what we call the *Lubell function* of $\mathcal{F}$, which is

$$\bar{h}(\mathcal{F}) = \bar{h}_n(\mathcal{F}) := \underset{C \in \mathcal{C}}{\text{ave}} |\mathcal{F} \cap C|.$$

This is the expected value $\text{E}(|\mathcal{F} \cap C|)$ over a random full chain C in $\mathcal{B}_n$. Then $\bar{h}(\mathcal{F})$ is essentially the function of $\mathcal{F}$ at the heart of Lubell's elegant proof of Sperner's Theorem ([15], cf. [7]) with the observation.

Lemma 3.1. *Let $\mathcal{F}$ be a collection of subsets of $[n]$. Then $\bar{h}(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}$.*

Proof. We have that $\bar{h}(\mathcal{F}) = \text{E}(|\mathcal{F} \cap C|)$, where C is picked at random from $\mathcal{C}$. This expected value is, in turn, the sum over $F \in \mathcal{F}$ of the probability that a random C contains F. Since C meets the $\binom{n}{k}$ subsets of cardinality k with equal probability, it means that each set F contributes $1/\binom{n}{|F|}$ to the sum. $\quad\square$

Lubell's proof uses the simple facts that $|\mathcal{A} \cap C| \leqslant 1$ for any antichain $\mathcal{A}$ and that $\binom{n}{k}$ is maximized by taking $k = \lfloor \frac{n}{2} \rfloor$, to derive Sperner's Theorem that $|\mathcal{A}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}$. By similar reasoning for general families $\mathcal{F}$ we obtain a general upper bound.

Lemma 3.2. *Let $\mathcal{F}$ be a collection of subsets of $[n]$. If $\bar{h}(\mathcal{F}) \leqslant m$, for real number $m > 0$, then $|\mathcal{F}| \leqslant m\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Moreover, if m is an integer, then $|\mathcal{F}| \leqslant \Sigma(n, m)$, and equality holds if and only if $\mathcal{F} = \mathcal{B}(n, m)$ (when $n + m$ is odd), or if $\mathcal{F} = \mathcal{B}(n, m - 1)$ together with any $\binom{n}{(n-m)/2}$ subsets of sizes $(n - m)/2$ or $(n + m)/2$ (when $n + m$ is even).*

Proof. We use the symmetry and strict unimodality of the sequence of binomial coefficients $\binom{n}{k}$, $0 \leqslant k \leqslant n$. If $\bar{h}(\mathcal{F}) \leqslant m$, then $|\mathcal{F}| = \sum_{A \in \mathcal{F}} 1 \leqslant \sum_{A \in \mathcal{F}} \binom{n}{\lfloor \frac{n}{2} \rfloor}/\binom{n}{|A|} \leqslant m\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Now assume $m > 0$ is an integer. We construct a family $\mathcal{F}$ of maximum size, subject to $\bar{h}(\mathcal{F}) \leqslant m$, by selecting subsets A that contribute the least to $\bar{h}(\mathcal{F})$, which means that we minimize $1/\binom{n}{|A|}$. Essentially, we are solving the linear program of maximizing $\sum_i x_i \binom{n}{i}$ subject to $\sum_i x_i \leqslant m$, $0 \leqslant x_i \leqslant 1$ for all i. We maximize $|\mathcal{F}|$ by selecting $\mathcal{F}$ to be the m middle levels, $\mathcal{B}(n, m)$.

Further, if $|\mathcal{F}| = \Sigma(n, m)$, it must be that $\mathcal{F}$ is $\mathcal{B}(n, m)$ when $n + m$ is odd. If $n + m$ is even, the subsets of sizes $(n - m)/2$ and $(n + m)/2$ will tie for the m-th largest size, and we can freely choose any $\binom{n}{(n-m)/2}$ subsets of the two sizes so that $|\mathcal{F}| = \Sigma(n, m)$. $\quad\square$

We see that upper bounds on the average intersection size $|\mathcal{F} \cap C|$ lead to upper bounds on the ratio of particular interest in this paper, $|\mathcal{F}|/\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Hence, we get upper bounds on $\pi(P)$, when it exists, from upper bounds on $\bar{h}(\mathcal{F})$ for P-free families $\mathcal{F}$.

To illustrate how this can be useful, we now give a short proof that, if it exists, $\pi(\mathcal{D}_2) \leqslant 2.5$. Consider a diamond-free family $\mathcal{F} \subseteq 2^{[n]}$. No full chain $C \in \mathcal{C}_n$ meets $\mathcal{F}$ four times, or else $\mathcal{F}$ contains $\mathcal{P}_4$, which has $\mathcal{D}_2$ as a subposet. If no chain meets $\mathcal{F}$ three times, we immediately get $\bar{h}(\mathcal{F}) \leqslant h(\mathcal{F}) \leqslant 2$. Else, consider any three elements of F $X \subset Y \subset Z$, and let Y' be any set not equal to Y such that $X \subset Y' \subset Z$. Let σ be a permutation of $[n]$ that fixes X and Z and sends Y to Y'. Then σ sends full chains $C \in \mathcal{C}_n$ through X, Y, Z to full chains $C' \in \mathcal{C}$ through X, Y', Z. These chains C' meet $\mathcal{F}$ only twice, as $\mathcal{F}$ is diamond-free. These chains C' are distinct. We find then that $\bar{h}(\mathcal{F}) = \text{E}(|\mathcal{F} \cap C|) \leqslant 2.5$.

Unfortunately, the behavior of $\bar{h}(\mathcal{F})$ does not match that of $|\mathcal{F}|/\binom{n}{\lfloor \frac{n}{2} \rfloor}$ asymptotically – there can be a gap. We shall see examples of this for diamond-free families. Nonetheless, in many cases we

can obtain $\pi(P)$ from $\bar{h}(\mathcal{F})$. Besides that, it is interesting in its own right to maximize $\bar{h}(\mathcal{F})$ for P-free families $\mathcal{F}$, though obtaining a good bound on $\bar{h}(\mathcal{F})$ can be difficult. We have discovered that a "partition method" can be fruitful.

Specifically, we partition the set $\mathcal{C}$ of full chains into blocks $\mathcal{C}(i)$ and then, for each i separately, we bound the average size $|\mathcal{F} \cap C|$ over full chains $C \in \mathcal{C}(i)$. The principle is that the average size $|\mathcal{F} \cap C|$ over all full chains C is at most the maximum over i of the average over block $\mathcal{C}(i)$. An analogy to baseball is helpful for some readers: A hitter's average over a whole season is never more than his maximum monthly average over the months in the season.

We illustrate the partition method by sketching a derivation of $\mathrm{La}(n, \mathcal{B})$. Let $\mathcal{F}$ be a butterfly-free family of subsets of $[n]$, $n \geqslant 3$. One can check that if $\mathcal{F}$ contains $\emptyset$ or $[n]$, then $|\mathcal{F}| < \Sigma(n, 2)$ (although, one may have $\bar{h}(\mathcal{F}) > 2$). Else, suppose $\emptyset, [n] \notin \mathcal{F}$. We show $\bar{h}(\mathcal{F}) \leqslant 2$. Define the collection $\mathcal{M}$ of subsets $M \in \mathcal{F}$ of $[n]$ for which there exists a chain $C \in \mathcal{C}$ passing through $A, M, B \in \mathcal{F}$ with $A \subset M \subset B$. Notice that since $\mathcal{F}$ contains no butterfly $\mathcal{B}$, it contains no $\mathcal{P}_4$, and so the collection $\mathcal{M}$ is an antichain. Now partition the set of full chains $\mathcal{C}$ as follows: For $M \in \mathcal{M}$, $\mathcal{C}_M$ consists of all full chains meeting M, while $\mathcal{C}_\emptyset$ contains all full chains that do not meet $\mathcal{M}$.

By definition of $\mathcal{M}$, no chain in $\mathcal{C}_\emptyset$ meets $\mathcal{F}$ three times, and so $\mathrm{ave}_{C \in \mathcal{C}_\emptyset} |\mathcal{F} \cap C| \leqslant 2$. For $M \in \mathcal{M}$, similar to the argument above for $\mathcal{D}_2$-free families, for any chain $C \in \mathcal{C}_M$ meeting $\mathcal{F}$ three times, it must meet $\mathcal{F}$ in A, M, B, and there is a corresponding chain in $\mathcal{C}_M$ meeting $\mathcal{F}$ only at M and avoiding A, B, so that $\mathrm{ave}_{C \in \mathcal{C}_M} |\mathcal{F} \cap C| \leqslant 2$. Hence, we have partitioned $\mathcal{C}$ into blocks such that $\mathcal{F}$ meets chains in each block at most twice, on average, and hence at most twice, on average, over all of $\mathcal{C}$. Thus, $\bar{h}(\mathcal{F}) \leqslant 2$, and it follows that $\mathrm{La}(n, \mathcal{B}) = \Sigma(n, 2)$, since $\mathcal{B}(n, 2)$ is butterfly-free.

Regarding the extremal butterfly-free families as far as achieving $\mathrm{La}(n, \mathcal{B})$, Lemma 3.2 above applies. In fact, it is known that $\mathcal{F}$ must be $\mathcal{B}(n, 2)$ for $n \geqslant 5$, though it is not true for $n = 4$: Consider $\mathcal{F}$ consisting of $\{\{1\}, \{2\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ and all six 2-subsets.

However, in some cases we can show that $\mathrm{La}(n, P)$ is attained only by $\mathcal{F} = \mathcal{B}(n, k)$:

Lemma 3.3. *Suppose that for poset P, $e(P) = m$, an integer. Suppose that for all n, all P-free families $\mathcal{F} \subseteq 2^{[n]}$ satisfy $\bar{h}(\mathcal{F}) \leqslant m$. Then for all n, $\mathrm{La}(n, P) = \Sigma(n, m)$, and if $\mathcal{F}$ is an extremal family, then $\mathcal{F} = \mathcal{B}(n, m)$.*

Proof. Let $\mathcal{F}$ be a P-free family with size $\mathrm{La}(n, P)$. According to Lemma 3.2, $\mathrm{La}(n, P) = \Sigma(n, m)$, since $\bar{h}(\mathcal{F}) \leqslant m$. Further, $\mathcal{F} = \mathcal{B}(n, m)$ when $n + m$ is odd. Hence, suppose $n + m$ is even, so that $\mathcal{F} = \mathcal{B}(n, m - 1)$ together with any $\binom{n}{(n-m)/2}$ subsets of sizes $(n - m)/2$ or $(n + m)/2$. Suppose for contradiction that $\mathcal{F}$ contains subsets of both sizes $(n - m)/2$ and $(n + m)/2$. By the natural generalization of Sperner's proof of Sperner's Theorem (or by using the normalized matching property on the ranks $\binom{[n]}{(n-m)/2}$ and $\binom{[n]}{(n+m)/2}$) [7], we can find subsets $A, B \in \mathcal{F}$ with $A \subset B$, $|A| = (n - m)/2$ and $|B| = (n + m)/2$. Then the interval $[A, B] \subset \mathcal{F}$. This interval is a Boolean lattice, $\mathcal{B}_m$. However, $\bar{h}_m(\mathcal{B}_m) = m + 1 > m$, so that by hypothesis, $\mathcal{B}_m$ must contain subposet P, which contradicts our assumption that $\mathcal{F}$ is P-free. Hence, $\mathcal{F}$ only contains one of the two sizes $(n - m)/2$ and $(n + m)/2$. $\square$

We saw that the conclusion of the lemma above fails for the butterfly $P = \mathcal{B}$ (but only for small n). The reason we could not apply this lemma to $P = \mathcal{B}$ is that the hypothesis fails for $n = 2$: The full Boolean lattice $\mathcal{B}_2$, which has one more element than $\mathcal{B}(2, 2)$, is butterfly-free.

4. Proofs of results 2.3, 2.5, 2.6

We now illustrate our partition method to bring the bound for $\mathcal{D}_2$-free families below 2.5. Our best bound is derived in the next section.

Proof of Proposition 2.3. Let $\mathcal{F}$ be a $\mathcal{D}_2$-free family of subsets of $[n]$ with maximum Lubell function value $\bar{h}(\mathcal{F})$, and let d_n denote this value. We claim that d_n is nonincreasing for $n \geqslant 2$. By easy direct case study we get that $d_2 = 2.5$ and $d_3 = d_4 = 7/3 \approx 2.33$.

For $n \geqslant 5$, if both $\emptyset$ and $[n]$ are in $\mathcal{F}$, then we have only one more subset in $\mathcal{F}$, and $\bar{h}(\mathcal{F}) \leqslant 2 + \frac{1}{n} \leqslant 2.2$. We will later give examples of families with Lubell function > 2.2, so $\mathcal{F}$ cannot satisfy

this condition. Then we may assume by symmetry that $[n] \notin \mathcal{F}$. We partition the set $\mathcal{C}$ of full chains into the blocks $\mathcal{C}_{n,i}$, where the chains $C \in \mathcal{C}_{n,i}$ pass through set $[n] \setminus \{i\}$. A random full chain in $\mathcal{C}$ is equally likely to belong to each $\mathcal{C}_{n,i}$, and $\bar{h}(\mathcal{F})$ is simply the average over i of the values $E(|\mathcal{F} \cap C|)$ taken over $C \in \mathcal{C}_{n,i}$, viewed as the Lubell function for the subsets of $[n] \setminus \{i\}$. That is, $\bar{h}(\mathcal{F})$ is the average of n terms, each of which is at most d_{n-1}. Hence, $d_n = \bar{h}(\mathcal{F}) \leqslant d_{n-1}$.

Returning to the calculations, for d_5 we note that since $\bar{h}(\mathcal{F})$ is a sum of terms, each 1 or 1/5 or 1/10, d_5 is a multiple of 1/10, and hence at most 2.3 since it is at most 7/3. Then d_7, which is similarly a multiple of 1/105, must be less than 2.3, and hence at most $241/105 < 2.2953$, and so by Lemma 3.2, for $n \geqslant 7$, $|\mathcal{F}| < 2.2953 \binom{n}{\lfloor \frac{n}{2} \rfloor}$, which implies the theorem. $\quad\square$

Next is the result for $\mathcal{D}_k$-free families for general k.

Proof of Theorem 2.5. Let $n, k \geqslant 2$, and define $m := \lceil \log_2(k+2) \rceil$.

For the lower bounds, consider $\mathcal{F} = \mathcal{B}(n, m)$. We have $|B - A| \leqslant m - 1$ for any two subsets $A \subseteq B$ in $\mathcal{F}$. There are at most $2^{m-1} - 2$ subsets S satisfying $A \subset S \subset B$. Hence $\mathcal{F}$ is $\mathcal{D}_k$-free. So $m \leqslant e(\mathcal{D}_k) \leqslant \pi(\mathcal{D}_k)$.

Now we derive the upper bounds. Let $\mathcal{F}$ be a largest $\mathcal{D}_k$-free family in $\mathcal{B}_n$. We take what we call the *min–max partition* of the set $\mathcal{C}$ of full chains in $\mathcal{B}_n$ according to $\mathcal{F}$: For subsets $A \subseteq B \subseteq [n]$ with $A, B \in \mathcal{F}$, the block $\mathcal{C}_{A,B}$ consists of the full chains C such that the smallest and the largest subsets in $\mathcal{F} \cap C$ are A and B, respectively. We denote by $\mathcal{C}_\emptyset$ the block of full chains that do not meet $\mathcal{F}$ at all. For $C \in \mathcal{C}_\emptyset$, we have $|\mathcal{F} \cap C| = 0$.

We now bound the expected size of $|\mathcal{F} \cap C|$ for a random chain C in $\mathcal{C}_{A,B}$. If $|B - A| \leqslant m - 1$, then this is at most m immediately.

For the remainder, assume $|B - A| \geqslant m$. We use the Lubell function Lemma 3.1 to calculate $E(|\mathcal{F} \cap C|)$ by adding the contributions of each subset $S \in \mathcal{F} \cap [A, B]$, which is $1 / \binom{|B-A|}{|S-A|}$. Since $\mathcal{F}$ is $\mathcal{D}_k$-free and contains both A and B, there are at most $k - 1$ subsets $S \in \mathcal{F} \cap [A, B]$ besides A and B. Then $E(|\mathcal{F} \cap C|)$ is maximized if we take the $k - 1$ terms with largest contribution, *i.e.*, with minimum $\binom{|B-A|}{|S-A|}$, which means the sets S closest to the ends A or B, so with $|S - A|$ equal to 1 or $|B - A| - 1$, then 2 or $|B - A| - 2$, and so on. The contribution from each full level we include is then one.

For the case (1), where $k - 1 \leqslant 2^m - 2 - \binom{m}{\lfloor \frac{m}{2} \rfloor}$, we see that for $|B - A| = m$, the $k - 1$ terms are at most enough to account for all subsets $S \in [A, B]$ with $|S - A|$ not equal to $\lfloor m/2 \rfloor$, that is, we get Lubell function at most m (when we include the terms for A and B). For $|B - A| > m$, since the levels working up from A or down from B are larger, the $k - 1$ terms are no longer sufficient to cover as many full levels, and the Lubell function is strictly less than m. Since every block in our partition has expected value at most m, we conclude that $\bar{h}(\mathcal{F}) \leqslant m$. Lemma 3.2 gives us $|\mathcal{F}| \leqslant \Sigma(n, m)$. Furthermore, we also have $e(\mathcal{D}_k) \geqslant m$. Hence $m = e(\mathcal{D}_k) = \pi(\mathcal{D}_k) \leqslant \bar{h}(\mathcal{F}) \leqslant m$. By Lemma 3.3 the extremal family $\mathcal{F}$ must in fact be $\mathcal{B}(n, m)$.

For the case (2), where $k - 1 > 2^m - 2 - \binom{m}{\lfloor \frac{m}{2} \rfloor}$, we see that for $|B - A| = m$, the largest sum of $k - 1$ terms leads to Lubell function at most $m + 1 - (2^m - k - 1)/\binom{m}{\lfloor \frac{m}{2} \rfloor}$. As in case (1), if $|B - A| > m$, then since the levels working up from the bottom or down from the top in $[A, B]$ are larger, the Lubell function is strictly less than this bound. The bound holds for every block $\mathcal{C}_{A,B}$ of the min–max partition. Therefore, if $\pi(\mathcal{D}_k)$ exists, $\pi(\mathcal{D}_k) \leqslant \bar{h}(\mathcal{F}) \leqslant m + 1 - (2^m - k - 1)/\binom{m}{\lfloor \frac{m}{2} \rfloor}$. $\quad\square$

Now we use the min–max partition of the set of full chains to prove the Harp Theorem.

Proof of Theorem 2.6. We argue that $\bar{h}(\mathcal{F}) \leqslant l_1 - 1$ for any $\mathcal{H}(l_1, \ldots, l_k)$-free $\mathcal{F}$ using induction on k. The case $k = 1$ concerns a family $\mathcal{F}$ that contains no chain of height l_1, for which we get immediately that $\bar{h}(\mathcal{F}) = E(|\mathcal{F} \cap C|) \leqslant \max_C |\mathcal{F} \cap C| \leqslant l_1 - 1$ (which implies Erdős's Theorem 2.1).

Let $k \geqslant 2$, and assume the bound on $\bar{h}$ for harps with $k - 1$ paths. Let $\mathcal{F}$ be an $\mathcal{H}$-free family of subsets of $[n]$, where $\mathcal{H} = \mathcal{H}(l_1, \ldots, l_k)$, and consider a block $\mathcal{C}_{A,B}$ in the min–max partition of the set of full chains $\mathcal{C}$ induced by $\mathcal{F}$. Let t be the largest height of any chain in $\mathcal{F} \cap [A, B]$. If $t < l_1$ we get that for full chains C in this block, $E(|\mathcal{F} \cap C|) \leqslant l_1 - 1$.

Otherwise, $t \geqslant l_1$. Consider a largest chain Z in $\mathcal{F} \cap (A, B)$, say $S_1 \subset \cdots \subset S_{t-2}$, where $A \subset S_1$ and $S_{t-2} \subset B$. Let $\mathcal{F}'$ be $\mathcal{F} \cap [A, B]$ with the sets in Z removed.

Then Z and $\mathcal{F}'$ are disjoint and $E(\mathcal{F} \cap C)$ for random full chains C in this block is the sum of $E(Z \cap C)$ and $E(\mathcal{F}' \cap C)$. For the Z term, by Lemma 3.1 we get $\sum_i 1/\binom{|B-A|}{|S_i-A|} \leqslant (t-2)/|B-A| < 1$. For the other term, we observe that $\mathcal{F}'$ is $\mathcal{H}(l_2, \ldots, l_k)$-free in the Boolean lattice of subsets of $[A, B]$. By induction on k, $\bar{h}(\mathcal{F}') \leqslant l_2 - 1 \leqslant l_1 - 2$. So C meets $\mathcal{F}'$ on average at most $l_1 - 2$ times. Combining terms, we find that C meets $\mathcal{F}$ at most $l_1 - 1$ times on average for C in this block, and hence for all random full chains $C \in \mathcal{C}$. We have that $\bar{h}(\mathcal{F}) \leqslant l_1 - 1$. By Lemma 3.2 we get that $\mathrm{La}(n, \mathcal{H}(l_1, \ldots, l_k)) \leqslant \Sigma(n, l_1 - 1)$.

The family $\mathcal{B}(n, l_1 - 1)$ achieves the upper bound just given, since it does not contain an l_1-chain, and is thus $\mathcal{H}(l_1, \ldots, l_k)$-free. We see that

$$e\big(\mathcal{H}(l_1, \ldots, l_k)\big) = \pi\big(\mathcal{H}(l_1, \ldots, l_k)\big) = l_1 - 1.$$

Moreover, by Lemma 3.3, the only harp-free family of maximum size is $\mathcal{B}(n, l_1 - 1)$. $\quad\square$

5. Proof of $\mathcal{D}_2$ Theorem 2.4

We investigate the structure of $\mathcal{D}_2$-free families with maximum Lubell function, and use this information to improve our earlier bound. Before proving Theorem 2.4, we continue from the proof of Proposition 2.3 in the last section, assuming all notation and facts from that.

We adopt the notation that for any families $\mathcal{F}_1, \ldots, \mathcal{F}_m$ of sets, $\mathcal{F}_1 \vee \cdots \vee \mathcal{F}_m$ denotes the family $\{F_1 \cup \cdots \cup F_m \mid \forall i \; F_i \in \mathcal{F}_i\}$. Given disjoint sets S, T we define the following three constructions:

Construction. $C_1(S, T)$: $\mathcal{F} = \{\emptyset\} \cup \binom{S}{1} \cup (\binom{S}{1} \vee \binom{T}{1}) \cup \binom{T}{2}$.

Construction. $C_2(S, T)$: $\mathcal{F} = \{\emptyset\} \cup \binom{S}{2} \cup \binom{T}{2} \cup (\binom{S}{2} \vee \binom{T}{1}) \cup (\binom{S}{1} \vee \binom{T}{2})$.

Construction. $C_3(S, T)$: $\mathcal{F} = \binom{[n]}{1} \cup \binom{S}{2} \cup \binom{T}{2} \cup (\binom{S}{2} \vee \binom{T}{1}) \cup (\binom{S}{1} \vee \binom{T}{2})$.

We will typically partition $[n]$ into subsets S, T in using these constructions, and we write $C_i(s, n - s)$ for $C_i([s], [n] \setminus [s])$, for integers s, $0 < s < n$. The families above are $\mathcal{D}_2$-free and each $\bar{h}(C_i(s, n - s)) = 2 + \frac{s(n-s)}{n(n-1)}$. For $n \geqslant 2$, the maximum value over s is $2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} > 2.25$, achieved by $s = \lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$.

In our approach the key to proving Theorem 2.4 is to focus on $\mathcal{D}_2$-free families $\mathcal{F}$ that contain $\emptyset$. Let δ_n be the maximum value of $\bar{h}(\mathcal{F})$ for all such families. Definitions give that $2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} \leqslant \delta_n \leqslant d_n$. Even though we do not obtain the values of d_n, we can obtain δ_n for n up to 12. This technical information (including the extremal families for δ_n) makes up the following lemma, which is the hard part in proving the theorem.

Lemma 5.1. *The sequence $\{\delta_n\}$ satisfies the following properties.*

(1) *It is nonincreasing for $n \geqslant 4$.*
(2) *For $4 \leqslant n \leqslant 12$, if $\mathcal{F}$ contains $\emptyset$ and $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$, then up to relabeling elements of $[n]$, $\mathcal{F}$ is $C_1(s, n - s)$ for $s = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$, or $C_2(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$. Hence, $\delta_n = 2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)}$.*

Proof. To show (1), let $\mathcal{F}$ be a $\mathcal{D}_2$-free family of subsets of $[n]$ such that $\bar{h}_n(\mathcal{F}) = \delta_n$. For $n \geqslant 5$, if both $\emptyset$ and $[n]$ are in $\mathcal{F}$, then we have only one more subset in $\mathcal{F}$, and $\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{1}{n} \leqslant 2.2$. Thus $[n] \notin \mathcal{F}$. Then similar to the proof of Proposition 2.3, we partition the set $\mathcal{C}$ of full chains into the blocks $\mathcal{C}_{n,i}$, where the chains $C \in \mathcal{C}_{n,i}$ pass through set $[n] \setminus \{i\}$. Again, $\bar{h}(\mathcal{F})$ is the average over i of the values $E(|\mathcal{F} \cap C|)$ taken over $C \in \mathcal{C}_{n,i}$, viewed as the Lubell function for the subsets of $[n] \setminus \{i\}$.

That is, $\bar{h}_n(\mathcal{F})$ is the average of n terms, each of which is at most δ_{n-1}. In other words, for $1 \leqslant i \leqslant n$, let $\mathcal{F}_i = \{F \in \mathcal{F} \mid i \notin F\}$. Then each $\mathcal{F}_i$ is a $\mathcal{D}_2$-free family in $2^{[n]\setminus\{i\}}$. We have

$$\bar{h}_n(\mathcal{F}) = \operatorname*{ave}_{1 \leqslant i \leqslant n} \left(\operatorname*{ave}_{C \in \mathcal{C}_{n,i}} |\mathcal{F} \cap C| \right) = \frac{1}{n} \sum_{i=1}^{n} \bar{h}_{n-1}(\mathcal{F}_i).$$

Hence, $\delta_n = \bar{h}_n(\mathcal{F}) \leqslant \delta_{n-1}$.

We claim the following two facts which are needed in showing (2).

Claim 1. The inequality $2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} - \frac{1}{\binom{n}{3}} < 2\frac{3}{11}$ holds for all $n \leqslant 12$.

This can be verified by a simple computation.

Claim 2. For $4 \leqslant n \leqslant 12$, suppose $\mathcal{F} \subset C_1(s, n-s)$ with $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$. Then $\mathcal{F} = C_1(s, n-s)$ with $s = \lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$. Similarly, if $\mathcal{F} \subset C_2(s, n-s)$ with $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$, then $\mathcal{F} = C_2(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ (the same as $C_2(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$) by relabeling the elements).

One can calculate that if $|s - (n-s)| > 1$, then $\bar{h}_n(C_i(s, n-s)) < 2\frac{3}{11}$. Furthermore, if $\mathcal{F} \subsetneq C_1(s, n-s)$ with $s = \lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$, then $\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} - \frac{1}{\binom{n}{2}} < 2\frac{3}{11}$. Similarly, if $\mathcal{F} \subsetneq C_2(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$, then $\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} - \frac{1}{\binom{n}{3}} < 2\frac{3}{11}$. So Claim 2 holds.

We show (2) by induction on n. When $n = 4$, it can be directly verified by enumeration. There are 17 classes (up to relabeling of elements of $[n]$) of $\mathcal{D}_2$-free families containing $\emptyset$. The classes $C_1(2, 2)$ and $C_2(2, 2)$ satisfy $\bar{h}(\mathcal{F}) = 2\frac{1}{3}$ while the rest of them have $\bar{h}(\mathcal{F})$ at most $2\frac{1}{4}$ which is less than $2\frac{3}{11}$.

Assume $n \geqslant 5$ and the statements are true for $n - 1$. Now we consider a $\mathcal{D}_2$-free family $\mathcal{F} \subset 2^{[n]}$ satisfying $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$ and $\emptyset \in \mathcal{F}$. Again, the full set $[n]$ is not in $\mathcal{F}$. Otherwise, $\mathcal{F}$ contains at most one more subset other than $\emptyset$ and $[n]$, and $\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{1}{n} < 2\frac{3}{11}$. Since $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$, there exists i so that $\bar{h}_{n-1}(\mathcal{F}_i) \geqslant 2\frac{3}{11}$. We may assume $\bar{h}_{n-1}(\mathcal{F}_n) \geqslant 2\frac{3}{11}$. By inductive hypothesis, $\mathcal{F}_n$ is $C_1(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil)$, $C_1(\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor)$, or $C_2(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil)$. We consider two cases.

Case 1. $\mathcal{F}_n = C_1(S, T)$ where $|S| = \lfloor \frac{n-1}{2} \rfloor$ or $\lceil \frac{n-1}{2} \rceil$.

It remains to decide the subsets in $\mathcal{F} \setminus \mathcal{F}_n$. Here are two subcases depending on whether $\{n\}$ is in $\mathcal{F}$.

Subcase 1a. $\{n\} \in \mathcal{F}$.

Since $\mathcal{F}$ is $\mathcal{D}_2$-free, it contains no subsets of forms $\{s_1, n\}$, $\{s_1, s_2, n, \ldots\}$, $\{s_1, t_1, n, \ldots\}$, and $\{t_1, t_2, n, \ldots\}$ for $s_i, \in S$ and $t_i \in T$. Thus, $\mathcal{F} \subset C_1(S \cup \{n\}, T)$. Since $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$, we conclude that $|S| + 1$ must be $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. Thus, by relabeling elements of $[n]$ we have that $\mathcal{F} = C_1(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$.

Subcase 1b. $\{n\} \notin \mathcal{F}$.

Let $S' = \{s \in S \mid \{s, n\} \in \mathcal{F}\}$ and $T' = \{t' \in T \mid \{t', n\} \in \mathcal{F}\}$. Since $\mathcal{F}$ is $\mathcal{D}_2$-free, $\mathcal{F}$ cannot have subsets of forms $\{s_1, s_2, n, \ldots\}$, $\{s_1, t_1, n, \ldots\}$, $\{t_1, t', n, \ldots\}$, and $\{t_1, t_2, t_3, \ldots\}$ for $s_i \in S$, $t_i \in T$, and $t' \in T'$. Equivalently,

$$\mathcal{F} \subset \mathcal{F}_n \cup \left(\binom{S'}{1} \vee \{\{n\}\} \right) \cup \left(\binom{T'}{1} \vee \{\{n\}\} \right) \cup \left(\binom{T \setminus T'}{2} \vee \{\{n\}\} \right).$$

Then

$$\bar{h}_n(\mathcal{F}) \leqslant \left(1 + \frac{|S|}{n} + \frac{|S||T| + \binom{|T|}{2}}{\binom{n}{2}}\right) + \frac{|S'| + |T'|}{\binom{n}{2}} + \frac{\binom{|T-T'|}{2}}{\binom{n}{3}}$$

$$\leqslant 1 + \frac{|S|}{n} + \frac{|S||T| + \binom{|T|}{2} + |S'|}{\binom{n}{2}} + f(|T'|).$$

Here $f(|T'|) = \frac{\binom{|T-T'|}{2}}{\binom{n}{3}} + \frac{|T'|}{\binom{n}{2}}$ is a quadratic function of $|T'|$ defined on the integer points of the interval $[0, |T|]$. Its maximum is reached at one of the two ends, namely $|T'| = 0$ or $|T'| = |T|$.

Claim 3. If $\bar{h}(\mathcal{F}) \geqslant 2\frac{3}{11}$, then $|T'| = |T|$.

For $n = 5$, we have $|S| = |T| = 2$. If $|T'| < |T|$, then $f(|T'|) \leqslant \max\{f(0), f(|T| - 1)\} = \frac{1}{10}$. Thus, $\bar{h}(\mathcal{F}) \leqslant 1 + \frac{|S|}{n} + \frac{|S||T| + \binom{|T|}{2} + |S'|}{\binom{n}{2}} + f(|T'|) < 2\frac{3}{11}$, which contradicts our assumption, and so $|T'| = |T|$.

For $n = 6$, we have either $|S| = 2$ and $|T| = 3$, or else $|S| = 3$, $|T| = 2$. If $|T'| < |T|$, then $f(|T'|) \leqslant \max\{f(0), f(|T| - 1)\} = \frac{3}{20}$ for $(|S|, |T|) = (2, 3)$, and $f(|T'|) \leqslant \max\{f(0), f(|T| - 1)\} = \frac{1}{15}$ for $(|S|, |T|) = (3, 2)$. By direct computation, both cases give $\bar{h}(\mathcal{F}) \leqslant 1 + \frac{|S|}{n} + \frac{|S||T| + \binom{|T|}{2} + |S'|}{\binom{n}{2}} + f(|T'|) < 2\frac{3}{11}$, which is again a contradiction.

For $7 \leqslant n \leqslant 12$, both $f(0)$ and $f(|T| - 1)$ are at most $\frac{|T|}{\binom{n}{2}} - \frac{1}{\binom{n}{3}}$. Thus,

$$\bar{h}_n(\mathcal{F}) \leqslant 1 + \frac{|S|}{n} + \frac{|S||T| + \binom{|T|}{2} + |S'| + |T|}{\binom{n}{2}} - \frac{1}{\binom{n}{3}}$$

$$\leqslant \bar{h}_n\left(C_1(|S|, |T| + 1)\right) - \frac{1}{\binom{n}{3}}$$

$$\leqslant 2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} - \frac{1}{\binom{n}{3}} < 2\frac{3}{11}.$$

This contradiction again proves $|T'| = |T|$, and completes the proof of Claim 3.

Hence, $\binom{T \setminus T'}{2} \vee \{\{n\}\}$ is a null family. Namely, $\mathcal{F} \subset C_1(S, T \cup \{n\})$. By the condition $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$, we conclude by relabeling elements of $[n]$ that $\mathcal{F} = C_1(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$, which is one of the listed possibilities in (2).

Case 2. $\mathcal{F}_n = C_2(S, T)$ where $|S| = \lfloor \frac{n-1}{2} \rfloor$ and $|T| = \lceil \frac{n-1}{2} \rceil$.

We determine what are the possible subsets in $\mathcal{F} \setminus \mathcal{F}_n$. Consider the two subcases depending on whether $\{n\} \in \mathcal{F}$.

Subcase 2a. $\{n\} \in \mathcal{F}$.

Since $\mathcal{F}$ is $\mathcal{D}_2$-free, $\mathcal{F}$ cannot contain subsets of forms $\{s_1, s_2, n, \ldots\}$, $\{t_1, t_2, n, \ldots\}$, and $\{u, v, w, n, \ldots\}$ for $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $u, v, w \in [n]$. Let $S' = \{s \in S \mid \{s, n\} \in \mathcal{F}\}$ and $T' = \{t \in T \mid \{t, n\} \in \mathcal{F}\}$. Then

$$\mathcal{F} \subset \mathcal{F}_n \cup \{\{n\}\} \cup \left(\binom{S'}{1} \vee \{\{n\}\}\right) \cup \left(\binom{T'}{1} \vee \{\{n\}\}\right) \cup \left(\binom{S \setminus S'}{1} \vee \binom{T \setminus T'}{1} \vee \{\{n\}\}\right).$$

We have

$$\bar{h}_n(\mathcal{F}) \leqslant \left(1 + \frac{\binom{|S|}{2} + \binom{|T|}{2}}{\binom{n}{2}} + \frac{\binom{|T|}{2}|S| + \binom{|S|}{2}|T|}{\binom{n}{3}}\right) + \frac{1}{n} + \frac{|S'| + |T'|}{\binom{n}{2}} + \frac{(|S| - |S'|)(|T| - |T'|)}{\binom{n}{3}}$$

$$= \bar{h}_n\big(C_2\big(|S| + 1, |T|\big)\big) - b,$$

where we see that $b := -\frac{1}{n} + \frac{|S| - |S'| - |T'|}{\binom{n}{2}} + \frac{\binom{|T|}{2} + |S'||T| + |S||T'| - |S'||T'|}{\binom{n}{3}}$ is a bilinear function of $(|S'|, |T'|)$ defined on $[0, |S|] \times [0, |T|]$. To find the extremal values of b it suffices to check the four corner points $(0, 0)$, $(|S|, 0)$, $(0, |T|)$, $(|S|, |T|)$. We find the minimum value of b, at $(|S|, |T|)$, is $\frac{|S||T| - \binom{|S|}{2}}{3\binom{n}{3}} \geqslant \frac{1}{\binom{n}{3}}$. Hence,

$$\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} - \frac{1}{\binom{n}{3}} < 2\frac{3}{11},$$

which contradicts our assumption, so this subcase is impossible.

Subcase 2b. $\{n\} \notin \mathcal{F}$.

Similar to Subcase 2a, let $S' = \{s \in S \mid \{s, n\} \in \mathcal{F}\}$ and $T' = \{t \in T \mid \{t, n\} \in \mathcal{F}\}$. The family $\mathcal{F}$ cannot have subsets of forms $\{s, s', n, \ldots\}$, $\{s', t', n, \ldots\}$, $\{t, t', n, \ldots\}$, and $\{u, v, w, n, \ldots\}$ for $s \in S$, $t \in T$, $s' \in S'$, $t' \in T'$ and $u, v, w \in [n]$. Then

$$\mathcal{F} \subset \mathcal{F}_n \cup \left(\binom{S'}{1} \vee \{\{n\}\}\right) \cup \left(\binom{T'}{1} \vee \{\{n\}\}\right)$$

$$\cup \left(\left(\left(\binom{S}{1} \vee \binom{T}{1}\right) \setminus \left(\binom{S'}{1} \vee \binom{T'}{1}\right)\right) \vee \{\{n\}\}\right)$$

$$\cup \left(\binom{S \setminus S'}{2} \vee \{\{n\}\}\right) \cup \left(\binom{T \setminus T'}{2} \vee \{\{n\}\}\right).$$

We have

$$\bar{h}_n(\mathcal{F}) \leqslant \left(1 + \frac{\binom{|S|}{2} + \binom{|T|}{2}}{\binom{n}{2}} + \frac{\binom{|T|}{2}|S| + \binom{|S|}{2}|T|}{\binom{n}{3}}\right) + \frac{|S'| + |T'|}{\binom{n}{2}}$$

$$+ \frac{|S||T| - |S'||T'| + \binom{|S \setminus S'|}{2} + \binom{|T \setminus T'|}{2}}{\binom{n}{3}}$$

$$= \bar{h}_n\big(C_2(|S| + 1, |T|)\big) - \frac{g}{\binom{n}{3}},$$

where

$$g = \binom{n}{3}\left(\frac{|S| - |S'| - |T'|}{\binom{n}{2}}\right) + \binom{|T|}{2} + |S'||T'| - \binom{|S \setminus S'|}{2} - \binom{|T \setminus T'|}{2}$$

$$= \varepsilon_1 |T'| + \varepsilon_2 |S \setminus S'|,$$

with

$$\varepsilon_1 = \frac{2(n-2)}{3} - \frac{|T'| - 1}{2} - \frac{5(|S \setminus S'|)}{6} \quad \text{and} \quad \varepsilon_2 = \frac{n-2}{3} - \frac{|S| - |S'| - 1}{2} - \frac{|T'|}{6}.$$

If $(|S'|, |T'|) = (|S|, 0)$ or $(0, |T|)$, then $\mathcal{F} \subset C_2(S \cup \{n\}, T)$ or $\mathcal{F} \subset C_2(S, T \cup \{n\})$. Since $\bar{h}_n(\mathcal{F}) \geqslant 2\frac{3}{11}$, we have, by relabeling elements of $[n]$, $\mathcal{F} = C_2(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$, which is another of the alternatives listed in (2).

Claim 4. When $(|S'|, |T'|) \neq (|S|, 0)$ or $(0, |T|)$, then $g \geqslant 1$.

Recall that $0 \leqslant |S'| \leqslant |S| = \lfloor \frac{n-1}{2} \rfloor$ and $0 \leqslant |T'| \leqslant |T| = \lceil \frac{n-1}{2} \rceil$.
Suppose $n = 2k$, so we have $|S| = k - 1$ and $|T| = k$, $k \geqslant 2$. Rewrite

$$\varepsilon_1 = \big(3\big|T \setminus T'\big| + 5\big|S'\big|\big)/6 \quad \text{and} \quad \varepsilon_2 = \big(\big|T \setminus T'\big| + 3\big|S'\big| + 2\big)/6.$$

Note that $|S'|$ and $|T \setminus T'|$ are not both zero, nor are $|T'|$ and $|S \setminus S'|$ both zero. One can see that ε_1 and ε_2 are each at least $1/2$, and so $g \geqslant 1$ unless $|T'| = 1$ and $|S \setminus S'| = 0$, or $|T'| = 0$ and $|S \setminus S'| = 1$. But either pair of conditions increases the ε's and still leads to $g \geqslant 1$.
Else, suppose $n = 2k + 1$, $|S| = |T| = k$ and $k \geqslant 2$. This time rewrite

$$\varepsilon_1 = \big(3\big|T \setminus T'\big| + 5\big|S'\big| - 1\big)/6 \quad \text{and} \quad \varepsilon_2 = \big(\big|T \setminus T'\big| + 3\big|S'\big| + 1\big)/6.$$

Again it is simple to check that $g \geqslant 1$, and Claim 4 holds.
From Claim 4, we have if $(|S'|, |T'|) \neq (|S|, 0)$ or $(0, |T|)$, then once again we get the contradiction

$$\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{n(n-1)} - \frac{1}{\binom{n}{3}} < 2\frac{3}{11}.$$

This completes the Case 2 and the proof of the lemma. $\quad\square$

Now we are ready to prove our improved bound.

Proof of Theorem 2.4. Let $\mathcal{F}$ be a $\mathcal{D}_2$-free family of subsets of $[n]$. Partition $\mathcal{F}$ into $\mathcal{F}_k$ and $\mathcal{F}'$ such that $\mathcal{F}_k$ contains subsets of sizes in $[k, n-k]$ where $k = \frac{n}{2} - 2\sqrt{n \ln n}$, and $\mathcal{F}' = \mathcal{F} \setminus \mathcal{F}_k$. We know that $|\mathcal{F}'|/\binom{n}{\lfloor \frac{n}{2} \rfloor} < \frac{2}{n^2}$ for large n (see [11], Lemma 1).

Now concentrate on the family of sets near the middle, $\mathcal{F}_k$. We take what we call the *min partition* of the set $\mathcal{C}$ of full chains in $\mathcal{B}_n$: Let $\mathcal{C}_\emptyset$ be the block containing the full chains that do not meet $\mathcal{F}_k$ at all. For each subset $A \in \mathcal{F}_k$, let $\mathcal{C}_A$ be the block containing all full chains C having A as the minimal element in $\mathcal{F}_k \cap C$. We see that the average number of times a chain in $\mathcal{C}_A$ meets $\mathcal{F}_k$ is obtained by considering only the subsets in $\mathcal{F}_k$ that contain A and viewing them (after removing A from each) as a diamond-free family of subsets of $[n] \setminus A$ containing $\emptyset$. We deduce that $\mathrm{ave}_{C \in \mathcal{C}_A}(|\mathcal{F}_k \cap C|) \leqslant \delta_{n-|A|}$. For n large enough, we have $n - |A| > \frac{n}{2} - 2\sqrt{n \ln n} \geqslant 12$. Since the δ_n are nonincreasing, we have for large n that $\bar{h}(\mathcal{F}_k) \leqslant \delta_{12} = 2\frac{3}{11}$. It follows that for sufficiently large n, all $\mathcal{D}_2$-free families $\mathcal{F}$ in $\mathcal{B}_n$ satisfy

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{|\mathcal{F}_k| + |\mathcal{F}'|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leqslant 2\frac{3}{11} + \frac{2}{n^2}.$$

Consequently, if it exists, the limit $\pi(\mathcal{D}_2) \leqslant 2\frac{3}{11}$. $\quad\square$

6. Further research

Beyond diamonds $\mathcal{D}_k$, we continue to investigate why the limit $\pi(P)$ exists for general posets P. The methods introduced in this paper have proven to be useful for determining $\pi(P)$ for several other small posets P, which we are collecting separately [9,10,14]. One example is the subposet $\mathcal{J}$ of $\mathcal{D}_2$ consisting of four elements A, B, C, D with $B < A$ and $B < C < D$. Forbidding $\mathcal{J}$ is more restrictive than forbidding $\mathcal{D}_2$. We show that for $n \geqslant 1$, $\mathrm{La}(n, \mathcal{J}) = \Sigma(n, 2)$, and hence $\pi(\mathcal{J}) = 2$. All known values of $\pi(P)$ satisfy Conjecture 2.2.

In order to resolve the asymptotics for diamond-free posets, and show that $\pi(\mathcal{D}_2) = 2$ as expected, it is not enough to work with the Lubell function due to families such as those in the constructions C_i described above. These examples show that the terms in the sequence δ_n, which was shown to be nonincreasing for $n \geqslant 2$, are at least 2.25 for all $n \geqslant 2$. We suspect that the limit $\lim_{n \to \infty} d_n$, which is known to exist, is 2.25. This would follow from the conjecture below. Here, the *conjugate* of $\mathcal{F}$ is the family $\bar{\mathcal{F}} = \{\bar{F} \mid F \in \mathcal{F}\}$ where $\bar{F} = [n] \setminus F$ is the complement of F.

Conjecture 6.1. *For every $n \geqslant 4$, the value $\bar{h}(\mathcal{F})$ of any $\mathcal{D}_2$-free family $\mathcal{F} \subset 2^{[n]}$ satisfies $\bar{h}_n(\mathcal{F}) \leqslant 2 + \frac{\lfloor \frac{n^2}{4} \rfloor}{n(n-1)}$ and equality holds if and only if, up to relabeling elements of $[n]$, $\mathcal{F}$ or $\bar{\mathcal{F}}$ is $C_i(S,T)$ $(i = 1, 2, 3)$ with $||S| - |T|| \leqslant 1$.*

Then how might we reduce our upper bound on $\pi(\mathcal{D}_2)$ (if it exists) to below 2.25? The examples above are nowhere near as large as $2\binom{n}{\lfloor \frac{n}{2} \rfloor}$, yet have very small sets that make large contributions to the Lubell function. To build large diamond-free families, we should restrict our attention to families with no small nor large sets, say $\mathcal{F} \subseteq \mathcal{B}(n,k)$ with $k = n - f(n) < n$. If we could show that $|\mathcal{F}|$ is at most $(2 + o_n(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$, for suitable $f(n)$, we would have $\pi(\mathcal{D}_2) = 2$ as we expect, since most of the 2^n subsets are concentrated near the middle rank.

Another indication of the challenge facing us for $\mathcal{D}_2$ is that we have constructed three $\mathcal{D}_2$-free families for $n = 6$ of size 36, which is one more than $\Sigma(6, 2)$. This is in contrast to the values k for which Theorem 2.5 determines $\mathrm{La}(n, \mathcal{D}_k)$ completely, and its value is exactly $\Sigma(n, m)$. Thus, the solution for $\mathcal{D}_2$, and probably also $\mathcal{D}_k$ for the unsettled values k, is likely going to be more complicated.

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