

# Connected Balanced Subgraphs in Random Regular Multigraphs Under the Configuration Model\*

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## Abstract

Our previous paper [9] applied a lopsided version of the Lovász Local Lemma that allows negative dependency graphs [5] to the space of random matchings in  $K_{2n}$ , deriving new proofs to a number of results on the enumeration of regular graphs with excluded cycles through the configuration model [3]. Here we extend this from excluded cycles to some excluded balanced subgraphs, and derive asymptotic results on the probability that a random regular multigraph from the configuration model contains at least one from a family of balanced subgraphs in question.

## 1 The Tool

In [9] we proved the following theorem on extensions of (partial) matchings that allows (among other things) proving asymptotic enumeration results about regular graphs through the configuration model.

**Theorem 1** *Let  $\Omega$  be the uniform probability space of perfect matchings in the complete graph  $K_N$  ( $N$  even) or the complete bipartite graph  $K_{N,N'}$  (with  $N \leq N'$ ). Let  $r = r(N)$  be a positive integer and  $1/16 > \epsilon = \epsilon(N) > 0$*

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as  $N$  approaches infinity. Let  $\mathcal{M} = \mathcal{M}(N)$  be a collection of (partial) matchings in  $K_N$  or  $K_{N,N'}$ , respectively, such that none of these matchings is a subset of another. For any  $M \in \mathcal{M}$ , let  $A_M$  be the event consisting of perfect matchings extending  $M$ . Set  $\mu = \mu(N) = \sum_{M \in \mathcal{M}} \Pr(A_M)$ . Suppose that  $\mathcal{M}$  satisfies

1.  $|M| \leq r$ , for each  $M \in \mathcal{M}$ .
2.  $\Pr(A_M) < \epsilon$  for each  $M \in \mathcal{M}$ .
3.  $\sum_{M': A_{M'} \cap A_M = \emptyset} \Pr(A_{M'}) < \epsilon$  for each  $M \in \mathcal{M}$ .
4.  $\sum_{M: uv \in M} \Pr(A_M) < \epsilon$  for each single edge  $uv$ .
5.  $\sum_{H \in \mathcal{M}_F} \Pr_{N-2r}(A_H) < \epsilon$  for each  $F \in \mathcal{M}$ .

Then, if  $r\epsilon = o(1)$ , we have

$$\Pr(\bigwedge_{M \in \mathcal{M}} \overline{A_M}) = e^{-\mu + O(r\epsilon\mu)}, \quad (1)$$

and furthermore, if  $r\epsilon\mu = o(1)$ , then

$$\Pr(\bigwedge_{M \in \mathcal{M}} \overline{A_M}) = \left(1 + O(r\epsilon\mu)\right) e^{-\mu}. \quad (2)$$

In the theorem above  $\Pr(A_M)$  denotes the probability according to the counting measure, and  $\Pr_{N-2r}(A_H)$  indicates the probability of  $A_H$  in a setting, when  $2r$  of the  $N$  vertices (none of them is an endpoint of an edge in the partial matching  $H$ ) are eliminated, and the probability is considered in this smaller instance of the problem.

## 2 The Configuration Model and the Enumeration of $d$ -Regular Graphs

For a given sequence of positive integers with an even sum,  $(d_1, d_2, \dots, d_n) = \mathbf{d}$ , the *configuration model of random multigraphs with degree sequence  $\mathbf{d}$*  is defined as follows [3].

1. Let us be given a set  $U$  that contains  $N = \sum_{i=1}^n d_i$  distinct mini-vertices. Let  $U$  be partitioned into  $n$  classes such that the  $i$ th class consists of  $d_i$  mini-vertices. This  $i$ th class will be associated with vertex  $v_i$  after identifying its elements through a *projection*.
2. Choose a random matching  $M$  of the mini-vertices in  $U$  uniformly.

3. Define a random multigraph  $G$  associated with  $M$  as follows: For any two (not necessarily distinct) vertices  $v_i$  and  $v_j$ , the number of edges joining  $v_i$  and  $v_j$  in  $G$  is equal to the total number of edges in  $M$  between mini-vertices associated with  $v_i$  and mini-vertices associated with  $v_j$ .

The configuration model of random  $d$ -regular multigraphs on  $n$  vertices is the instance  $d_1 = d_2 = \dots = d_n$ , where  $nd$  is even.

Bender and Canfield [2], and independently Wormald, showed in 1978 that for any fixed  $d$ , with  $nd$  even, the number of  $d$ -regular graphs is

$$(\sqrt{2} + o(1))e^{\frac{1-d^2}{4}} \left( \frac{d^d n^d}{e^d (d!)^2} \right)^{\frac{n}{2}}. \quad (3)$$

Bollobás [3] introduced probability to this enumeration problem by defining the configuration model, and brought the result (3) to the alternative form

$$(1 + o(1))e^{\frac{1-d^2}{4}} \frac{(dn - 1)!!}{(d!)^n}, \quad (4)$$

where the term  $(1 + o(1))e^{\frac{1-d^2}{4}}$  in (4) can be explained as the probability of obtaining a simple graph after the projection. (The semifactorial  $(dn - 1)!! = \frac{(dn)!}{(dn/2)! 2^{dn/2}}$  equals the number of perfect matchings on  $dn$  elements, and  $(d!)^n$  is just the number of ways matchings can yield the same simple graph after projection. Non-simple graphs, unlike simple graphs, can arise with different frequencies.) Bollobás also extended the range of the asymptotic formula to  $d < \sqrt{2 \log n}$ , which was further extended to  $d = o(n^{1/3})$  by McKay [10] in 1985. The strongest result is due to McKay and Wormald [11] in 1991, who refined the probability of obtaining a simple graph after the projection to  $(1 + o(1))e^{\frac{1-d^2}{4} - \frac{d^3}{12n} + O(\frac{d^2}{n})}$  and extended the range of the asymptotic formula to  $d = o(n^{1/2})$ . Wormald's Theorem 2.12 in [15] (originally published in [14]) asserts that for any fixed numbers  $d \geq 3$  and  $g \geq 3$ , the number of labelled  $d$ -regular graphs with girth at least  $g$ , is

$$(1 + o(1))e^{-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2^i}} \frac{(dn - 1)!!}{(d!)^n}. \quad (5)$$

[9] reproved the following theorem of McKay, Wormald and Wysocka [12] using Theorem 1, under a slightly stronger condition than  $(d-1)^{2g-3} = o(n)$  in [12]: (note that a power of  $g$  in (6) only restricts a second term in the asymptotic series of the bound on  $g$ ):

**Theorem 2** *In the configuration model, assume  $d \geq 3$  and*

$$g^6(d-1)^{2g-3} = o(n). \quad (6)$$

*Then the probability that the random  $d$ -regular multigraph has girth at least  $g \geq 1$  is  $(1 + o(1)) \exp\left(-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2^i}\right)$ , and hence the number of  $d$ -regular graphs on  $n$  vertices with girth at least  $g \geq 3$  is*

$$(1 + o(1)) e^{-\sum_{i=1}^{g-1} \frac{(d-1)^i}{2^i}} \frac{(dn-1)!!}{(d!)^n}.$$

*(The case  $g = 3$  means that the random  $d$ -regular multigraph is actually a simple graph.) Furthermore, the number of  $d$ -regular graphs not containing cycles whose length is in a set  $\mathcal{C} \subseteq \{3, 4, \dots, g-1\}$ , is*

$$(1 + o(1)) e^{-\frac{d-1}{2} - \frac{(d-1)^2}{4} - \sum_{i \in \mathcal{C}} \frac{(d-1)^i}{2^i}} \frac{(dn-1)!!}{(d!)^n}.$$

This is a special case of a more general result. The following definitions are used in random graph theory [1]. The *excess* of a graph  $G$ , denoted by  $\kappa(G)$ , is  $|E(G)| - |V(G)|$ . A graph  $G$  is *balanced*, if  $\kappa(H) < \kappa(G)$  for any proper subgraph  $H$  with at least one vertex. We first prove the following Lemma.

**Lemma 3** *Suppose that  $G$  is a connected balanced simple graph with  $\kappa(G) \geq 0$ . Then the number of subgraphs  $H$  with  $\kappa(H) = \kappa(G) - 1$  is at most  $2|V(G)|^2$ .*

**Proof:** First we claim that  $G$  has no leaf vertex. Otherwise, if  $v$  is a leaf vertex, then  $\kappa(G-v) = \kappa(G)$ , a contradiction.

Let  $H$  be a subgraph of  $G$  with  $\kappa(H) = \kappa(G) - 1$ . If  $V(H) = V(G)$ , then  $H$  is obtained by deleting one edge from  $G$ . The number of such  $H$ 's is  $|E(G)|$ . Now we assume  $V(H) \neq V(G)$ . For any vertex set  $S$ , let  $\Gamma(S)$  be the neighborhood of  $S$  in  $G$ . We define a sequence of graphs  $H_0, H_1, H_2, \dots$  as follows. Let  $H_0 = H$ . For  $i \geq 1$ , if  $V(H_{i-1}) \neq V(G)$ , we define the graph  $H_i$  as follows:  $V(H_i) = V(H_{i-1}) \cup \Gamma(V(H_{i-1}))$  and  $E(H_i) = E(H_{i-1}) \cup \{uv : u \in V(H_{i-1}), v \in \Gamma(V(H_{i-1})), \text{ and } uv \in E(G)\}$ . Let  $H_r$  be the last graph in the sequence. We have  $V(H_r) = V(G)$ . Observe

$$\kappa(H) = \kappa(H_0) \leq \kappa(H_1) \leq \kappa(H_2) \cdots \leq \kappa(H_r) \leq \kappa(G).$$

Since  $\kappa(H) = \kappa(G) - 1$ , equalities hold for all but at most one in the chain above. We have  $|\Gamma(V(H_i)) \setminus V(H_i)| \leq 2$  for all  $i \leq r-1$ , as  $G$  has no leaf.

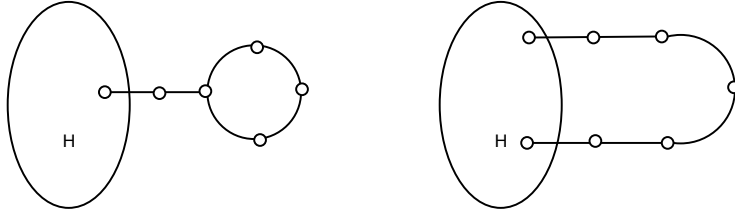


Figure 1:  $G - H$  is either a  $\rho$ -shape or a path when  $\kappa(H) = \kappa(G) - 1$ .

It is easy to check that the difference of  $G$  and  $H$  either forms a  $\rho$ -shape or is a path as shown in Figure 1. An  $H$  with a  $\rho$ -shape may occur at most  $2|E(G)|$  times, an  $H$  with a path may occur at most  $\binom{|V(G)|}{2}$  times.

Finally,  $|E(G)| + 2|E(G)| + \binom{|V(G)|}{2} \leq 2|V(G)|^2$ .  $\square$

Let  $\mathcal{G}$  be a family of connected balanced simple graphs with excess  $\kappa$ . We would like to estimate the probability that a random  $d$ -regular multigraph contains no graph in  $\mathcal{G}$ . Given a simple graph  $G$ , let  $|\text{Aut}(G)|$  be the number of automorphisms of  $G$ . For any  $k \geq 2$ , let  $a_k(G)$  be the number of vertices with degree at least  $k$ . We define a polynomial  $f_G(d) = \prod_{k=2}^{\infty} (d-k+1)^{a_k(G)} = \prod_{v \in V(G)} \binom{d-1}{d_v-1} (d_v-1)! = \frac{1}{d^{|V(G)|}} \prod_{v \in V(G)} \binom{d}{d_v} d_v! \leq (d-1)^{2|E(G)| - |V(G)|}$ . We have the following theorem.

**Theorem 4** *Let  $\mathcal{G}$  be a family of connected balanced simple graphs with non-negative excess  $\kappa$ . Set  $r = \max_{G \in \mathcal{G}} |E(G)|$ . In the configuration model, assume  $d \geq 3$  and*

$$\frac{r^3}{n} \sum_{G \in \mathcal{G}} (d-1)^{|E(G)|-1} = o(1) \text{ and } \ell = \frac{r^3 d}{n^{\kappa+1}} \left( \sum_{G \in \mathcal{G}} (d-1)^{|E(G)|-1} \right)^2 = o(1). \quad (7)$$

*Then the probability that the random  $d$ -regular multigraph arising from the configuration model contains no subgraph in  $\mathcal{G}$  is*

$$(1 + O(\ell)) \exp\left(- \sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}\right).$$

**Proof:** Let  $\epsilon = \frac{Kr^2}{n} \sum_{G \in \mathcal{G}} (d-1)^{|E(G)|-1}$  with a large constant  $K$ . The first condition makes sure  $r\epsilon = o(1)$ , the second condition makes sure  $r\epsilon\mu = O(\ell) = o(1)$  in Theorem 1.

For any  $G \in \mathcal{G}$ , let  $M_G$  be the family of (partial) matchings of  $U$  whose projection is a copy of  $G$ . Suppose that  $G$  has  $s$  vertices  $v_1, \dots, v_s$  and  $t$  edges  $e_1, \dots, e_t$ . For  $1 \leq i \leq s$ , let  $C_i$  be the class of  $d$  mini-vertices

associated to  $v_i$  and  $Q_i$  be an (ordered) queue of  $d_{v_i}$  mini-vertices in  $C_i$ . Let  $\mathcal{C}$  be the parameter space of all possible  $(C_1, \dots, C_s, Q_1, \dots, Q_s)$ . We define a mapping  $\psi: \mathcal{C} \rightarrow M_G$  as follows. For  $1 \leq j \leq t$ , suppose that the edge  $e_j$  has two end-vertices  $v_{j_1}$  and  $v_{j_2}$ . We pop a mini-vertex  $x_j$  from the queue  $Q_{j_1}$ , pop a mini-vertex  $y_j$  from the queue  $Q_{j_2}$ , and join  $x_j y_j$ . Denote by  $M$  the collection of edges  $\{x_j y_j\}_{1 \leq j \leq t}$ . Clearly  $M$  forms a partial matching whose projection is a copy of  $G$ . We define  $\psi(C_1, \dots, C_s, Q_1, \dots, Q_s) = M$ . Since every partial matching in  $M_G$  can be constructed in this way,  $\psi$  is surjective.

For any  $M \in M_G$  and any  $(C_1, \dots, C_s, Q_1, \dots, Q_s) \in \psi^{-1}(M)$ , it uniquely determines an ordering of edges in  $M$ . The number of such orderings that give the same projection  $G$  is exactly  $|\text{Aut}^*(G)|$ , the number of edge automorphisms of  $G$ . By Whitney's Theorem [7], for a connected  $G$ , which is not  $K_2$  or  $K_1$ ,  $|\text{Aut}^*(G)| = |\text{Aut}(G)|$ . We have  $|\psi^{-1}(M)| = |\text{Aut}(G)|$ .

There are  $\binom{n}{|V(G)|} |V(G)|!$  ways to choose  $(C_1, \dots, C_s)$ . For  $1 \leq i \leq s$ , there are  $\binom{d}{d_{v_i}} d_{v_i}!$  ways to choose the queue  $Q_i$ . We have

$$|\mathcal{C}| = \binom{n}{|V(G)|} |V(G)|! \prod_{v \in V(G)} \binom{d}{d_v} d_v!.$$

Thus,

$$\begin{aligned} |M_G| &= \frac{|\mathcal{C}|}{|\text{Aut}(G)|} \\ &= \frac{1}{|\text{Aut}(G)|} \binom{n}{|V(G)|} |V(G)|! \prod_{v \in V(G)} \binom{d}{d_v} d_v! \\ &= \frac{f_G(d)}{|\text{Aut}(G)|} \binom{n}{|V(G)|} |V(G)|! d^{|V(G)|}. \end{aligned} \quad (8)$$

For  $i \geq 1$ , let  $\mathcal{G}_i$  be the set of graphs in  $\mathcal{G}$  with exactly  $i$  edges. Let  $\mathcal{M}_i$  be the set of matchings of  $U$  whose projection gives a graph  $G \in \mathcal{G}_i$ ; there are *exactly*  $|M_G|$  of them, and they are counted in (8). The bad events to be avoided are the projection of some matching from the union  $\mathcal{M} = \cup_{i=1}^r \mathcal{M}_i$ . For each  $M_i \in \mathcal{M}_i$  ( $i = 1, 2, \dots, r$ ), we have

$$\Pr(A_{M_i}) = \frac{1}{(nd-1)(nd-3) \cdots (nd-2i+1)}. \quad (9)$$

We have

$$\begin{aligned}
\mu &= \sum_{M \in \mathcal{M}} \Pr(A_M) \\
&= \sum_{i=1}^r \sum_{G \in \mathcal{G}_i} \frac{f_G(d)}{|\text{Aut}(G)|} \binom{n}{|V(G)|} |V(G)! d^{|V(G)|} \\
&\quad \frac{1}{(nd-1)(nd-3)\cdots(nd-2i+1)} \\
&= \sum_{i=1}^r \sum_{G \in \mathcal{G}_i} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{i-|V(G)|}} \left(1 + O\left(\frac{i^2}{n}\right)\right) \\
&= \left(1 + O\left(\frac{r^2|\mathcal{G}|}{n}\right)\right) \left(\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}\right). \tag{10}
\end{aligned}$$

Observe from (10) that  $\mu = O\left(\sum_{G \in \mathcal{G}} \frac{(d-1)^{|E(G)|+\kappa}}{(nd)^\kappa}\right) = O\left(\sum_{G \in \mathcal{G}} \frac{(d-1)^{|E(G)|}}{n^\kappa}\right)$ . Now we verify the conditions of Theorem 1. Item 1 and 2 are trivial by the definition of  $r$  and  $\epsilon$ . Item 3 can be verified as follows. For two matchings  $M$  and  $M'$ ,  $A_M \cap A_{M'} = \emptyset$  if and only if  $M$  and  $M'$  conflict. For  $M \in \mathcal{M}_j$ , we have

$$\begin{aligned}
\sum_{M': A_{M'} \cap A_M = \emptyset} \Pr(A_{M'}) &= \sum_{i=1}^r \sum_{M' \in \mathcal{M}_i: A_{M'} \cap A_M = \emptyset} \Pr(A_{M'}) \\
\text{(by symmetry argument)} &\leq \sum_{i=1}^r \sum_{M' \in \mathcal{M}_i} \frac{2j}{nd} \Pr(A_{M'}) \\
&\leq \frac{2r}{nd} \sum_{i=1}^r \sum_{M' \in \mathcal{M}_i} \Pr(A_{M'}) \\
&= \frac{2r}{nd} \mu \\
&< \epsilon. \tag{11}
\end{aligned}$$

Now we verify item 4. For any  $uv \in M \in \mathcal{M}$ , we have

$$\begin{aligned}
\sum_{M: uv \in M \in \mathcal{M}} \Pr(A_M) &\leq \sum_{i=2}^r \sum_{G \in \mathcal{G}_i} \frac{\binom{n}{|V(G)|-2} (|V(G)|-2)! d^{-2} \prod_{v \in V(G)} \binom{d}{d_v} d_v!}{(nd-1)(nd-3)\cdots(nd-2i+1)} \\
&< \sum_{i=2}^r \sum_{G \in \mathcal{G}_i} \frac{f_G(d)}{(nd)^{i-|V(G)|+2}} \left(1 + O\left(\frac{i^2}{n}\right)\right) \\
&< \epsilon. \tag{12}
\end{aligned}$$

(We omitted a  $\frac{1}{nd-1}$  additive term from the estimate, which was there in [9], as it was there to handle a loop.)

Finally, we have to verify item 5. For any  $F \in \mathcal{M}$ , we have to estimate  $\sum_{M \in \mathcal{M}_F} \Pr_{N-2r}(A_M)$ . By the inequality below, this boils down to estimating  $\sum_{M \in \mathcal{M}_F} \Pr(A_M) = \sum_{M \in \mathcal{M}_F} \Pr_N(A_M)$ , as with  $|M| = i$ ,

$$\begin{aligned} \frac{\Pr_{N-2r}(A_M)}{\Pr_N(A_M)} &\leq \prod_{j=1}^i \frac{nd - 2j - 1}{nd - r - 2j - 1} \\ &\leq \prod_{j=1}^i \left( 1 + \frac{2r}{n - 2r - 2j - 1} \right) \leq e^{\frac{2r^2}{nd-4r-1}}. \end{aligned}$$

Assume that  $M' \in \mathcal{M}$  intersects  $F$ ,  $M = M' \setminus F \neq \emptyset$ , and the projection of  $M'$  is a graph  $G' \in \mathcal{G}$ . Let  $H$  be the projection of  $F \cap M'$ . The graph  $H$  is a subgraph of  $G$  satisfying  $0 < |E(H)| < |E(G)|$ . (Otherwise,  $G' \subset G$  contradicts to the assumption that  $\mathcal{G}$  is balanced.)

We have

$$\begin{aligned} &\sum_{M \in \mathcal{M}_F} \Pr(A_M) \\ &\leq \sum_{i=2}^r \sum_{G' \in \mathcal{G}_i} \sum_{\substack{H \subset G' \\ E(H) \neq \emptyset}} \frac{(n - |V(G)|)^{|V(G')| - |V(H)|} (d(d-1))^{|E(G')| - |E(H)|}}{\prod_{j=1}^{|E(G')| - |E(H)|} (nd - 2j + 1)} \\ &= \sum_{i=2}^r \sum_{G' \in \mathcal{G}_i} \left( 1 + O\left(\frac{i^2}{n}\right) \right) \sum_{\substack{H \subset G' \\ E(H) \neq \emptyset}} \frac{(d-1)^{|E(G')| - |E(H)|}}{n^{\kappa(G') - \kappa(H)}} \\ &= \left( 1 + O\left(\frac{r^2 |\mathcal{G}|}{n}\right) \right) \sum_{G' \in \mathcal{G}} \sum_{\substack{H \subset G' \\ E(H) \neq \emptyset}} \frac{(d-1)^{|E(G')| - |E(H)|}}{n^{\kappa(G') - \kappa(H)}}. \end{aligned}$$

(For the  $d(d-1)$  base term in the second line, consider that we can build up  $G'$  sequentially by always adding an edge incident to a pre-existing component with at least one edge, starting with the components of  $H$  with at least one edge.)

Since  $G'$  is balanced, we have  $\kappa(G') - \kappa(H) \geq 1$  for any subgraph  $H$  with  $0 < |E(H)| < |E(G)|$ . The last summation can be partitioned into summations over two classes. The first class  $\mathcal{C}_1$  consists of  $H$  with  $\kappa(H) = \kappa(G') - 1$ . By Lemma 3, the number of such  $H$  is at most  $|V(G')|^2$ . The second class  $\mathcal{C}_2$  consists of  $H$  with  $\kappa(H) \leq \kappa(G') - 2$ ; there are most



$2^{|E(G')|}$  of them. We bound  $(d-1)^{|E(G')|-|E(H)|}$  by  $(d-1)^{|E(G')|-1}$ . We have

$$\begin{aligned} \sum_{M \in \mathcal{M}_F} \Pr(A_M) &\leq 2 \sum_{G' \in \mathcal{G}} \sum_{\substack{H \subset G' \\ E(H) \neq \emptyset}} \frac{(d-1)^{|E(G')|-|E(H)|}}{n^{\kappa(G')-\kappa(H)}} \\ &\leq 2 \sum_{G' \in \mathcal{G}} (d-1)^{|E(G')|-1} \left( \frac{2|V(G')|^2}{n} + \frac{2^{|E(G')|}}{n^2} \right) \\ &< \epsilon. \end{aligned}$$

Finally, the error in (10) does not hurt, as

$$\begin{aligned} e^{-\mu} &= e^{-\left(1+O\left(\frac{r^2|\mathcal{G}|}{n}\right)\right)} \left( \sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}} \right) \\ &= e^{-\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}} e^{-O\left(\frac{r^2|\mathcal{G}|}{n}\right)} \sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}} \\ &= e^{-\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}} \left( 1 - O\left(\frac{r^2|\mathcal{G}|}{n}\right) \sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}} \right) \\ &= (1 - O(\ell)) e^{-\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}}, \end{aligned}$$

and  $e^{-\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}} = (1 + O(\ell)) e^{-\mu}$ .  $\square$

**Corollary 4.1** *We obtain Theorem 2 from Theorem 4, with the following condition, which is slightly weaker than (6) in [9]:*

$$g^3(d-1)^{2g-3} = o(n) \tag{13}$$

**Proof:** Note that cycles are exactly the connected balanced graphs with  $\kappa = 0$ . Let  $C_1$  denote the graph of a one-vertex loop and  $C_2$  the graph of a pair of multiedges between two vertices. These are balanced multigraphs with  $\kappa = 0$ . Formally, we did not allow in Theorem 4 balanced multigraphs, however, minor changes in the arguments will allow the inclusion of these two graphs (see in [9] how to handle loops and parallel edges). The formulas extend for  $C_2$  and  $C_1$ , if we use as definition  $|\text{Aut}(C_2)| = 4$  and  $f_{C_2}(d) = (d-1)^2$ ;  $|\text{Aut}(C_1)| = 2$  and  $f_{C_1}(d) = d-1$ . Applying Theorem 4 to the family  $\mathcal{G} = \mathcal{C} \cup \{C_1, C_2\}$ , where  $\mathcal{C} \subseteq \{C_3, \dots, C_{g-1}\}$  for  $g \geq 3$  one obtains Theorem 2.  $\square$

**Corollary 4.2** *Under the conditions of Theorem 4, the probability of obtaining a balanced graph from  $\mathcal{G}$  after projection in the configuration model,*

(i) if  $\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}$  is separated from zero, is

$$1 - e^{-\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}} + O(\ell) e^{-\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}},$$

(ii) if  $\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}} = o(1)$ , and the first part of (7) is strengthened to  $\frac{r^3}{n} \sum_{G \in \mathcal{G}} (d-1)^{|E(G)|-1} = o(\frac{f_G(d)}{d^\kappa |\text{Aut}(G)|})$  uniformly, is

$$\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}} + O\left(\ell + \left(\sum_{G \in \mathcal{G}} \frac{f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}\right)^2\right),$$

where  $\ell$  is little-oh of the main term.

**Proof:** (i) is straightforward. To obtain (ii), use  $1 - (1 + O(\ell))e^x = x + O(\ell + x^2)$  for  $\ell = o(1), x = o(1)$ . The fact that  $\ell$  is little-oh of the main term follows from the extra assumption in (ii).  $\square$

In the *bipartite configuration model* we have two sets,  $U$  and  $V$ , each containing  $N$  mini-vertices, a fixed partition of  $U$  into  $d_1, \dots, d_n$  element classes, and a fixed partition of  $V$  into  $\delta_1, \dots, \delta_n$  element classes. Any perfect matching between  $U$  and  $V$  defines a bipartite multigraph with partite sets of size  $n$  after a projection contracts every class to single vertex. In the regular case,  $d_1 = \dots = d_n = \delta_1 = \dots = \delta_n = d$ . We have the following theorem

**Theorem 5** *Let  $\mathcal{G}$  be a family of connected balanced simple bipartite graphs with non-negative excess  $\kappa$  and  $r = \max_{G \in \mathcal{G}} |E(G)|$ . In the regular case of the bipartite configuration model, assume  $d \geq 3$  and condition (7). Then the probability that the random  $d$ -regular multigraph contains no graph in  $\mathcal{G}$  is*

$$(1 + o(1)) e^{-\sum_{G \in \mathcal{G}} \frac{2f_G(d)}{|\text{Aut}(G)|(nd)^{\kappa(G)}}}.$$

**Proof:** We outline the proof. For  $i = 2, \dots, r$ , let  $\mathcal{M}_i$  be the set of matchings of  $U$  and  $V$ , whose projection gives a graph  $G \in \mathcal{G}$  with  $i$  edges. For a fixed  $G \in \mathcal{G}$ , since  $G$  is bipartite, let  $n_1(G)$  and  $n_2(G)$  be the size of vertex partition classes. The number of matchings, whose projection is  $G$ , is exactly

$$\frac{2}{|\text{Aut}(G)|} \binom{n}{n_1(G)} n_1(G)! \binom{n}{n_2(G)} n_2(G)! \prod_{v \in V(G)} \binom{d}{d_v} d_v!.$$

This formula is similar to Equation (8). If  $G$  has no automorphism switching its two colorclasses (in particular when  $n_1 \neq n_2$ ), then we can select

$n_1(G)$  classes from the  $n$  classes of  $U$  and select  $n_2(G)$  classes from the  $n$  classes of  $V$ , or vice versa. This explains the constant factor 2. If  $G$  has an automorphism switching its two colorclasses, then selecting  $n_1 = n_2$  classes from  $U$  and  $V$ , we obtain each copy of  $G$   $|Aut^*(G)|/2$  times from matchings. The bad events correspond to a matching from the union  $\mathcal{M} = \cup_{i=1}^r \mathcal{M}_i$ . For each  $M_i \in \mathcal{M}_i$  ( $i = 1, 2, \dots, r$ ), we have

$$\Pr(A_{M_i}) = \frac{(dn - 2i)!}{(dn)!}. \quad (14)$$

We have

$$\begin{aligned} & \sum_{M \in \mathcal{M}} \Pr(A_M) \\ &= \sum_{i=2}^r \sum_{G \in \mathcal{G}_i} \frac{2}{|Aut(G)|} \binom{n}{n_1(G)} \binom{n}{n_2(G)} \prod_{v \in V(G)} \binom{d}{d_v} d_v! \frac{(dn - 2i)!}{(dn)!} \\ &= \left(1 + O\left(\frac{r^2 |\mathcal{G}|}{n}\right)\right) \sum_{G \in \mathcal{G}} \frac{2f_G(d)}{|Aut(G)|(nd)^{\kappa(G)}}. \end{aligned} \quad (15)$$

All the estimates go through as in the proof of Theorem 2.  $\square$

Applying Theorem 5 to a family

$$\mathcal{G} = \{C_2\} \cup \mathcal{C} \quad \text{with} \quad \mathcal{C} \subseteq \{C_4, C_6, \dots, C_{2g-2}\}$$

(a slight extension to include  $C_2$ , like in [9]), we get another theorem of McKay, Wormald and Wysocka [12], who actually had it without  $g^3$  in (16). [9] reproved this theorem with  $g^6$  in the condition using Theorem 1.

**Theorem 6** *In the regular case of the bipartite configuration model, assume that  $g$  is even,  $d \geq 3$ , and*

$$g^3(d-1)^{2g-3} = o(n). \quad (16)$$

*Then the probability that the random bipartite  $d$ -regular multigraph does not contain a cycle of length  $C \subseteq \{2, 4, 6, \dots, g-2\}$ , is*

$$(1 + o(1))e^{-\sum_{i \in C} \frac{(d-1)^i}{i}}.$$

**Corollary 6.1** *Corollary 4.2 applies to the bipartite regular configuration model, changing  $f_G(d)$  to  $2f_G(d)$ .*

We are left with an open problem of finding asymptotics for the occurrence of an element of  $\mathcal{G}$  and obtaining a *simple* multigraph simultaneously.

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