Topic Course on Probabilistic Methods
(Week 6)
Correlation Inequalities

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Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)
Correlation Inequalities

- Four Functions Theorem
- 4FT on distributive lattice
- FKG inequalities
Correlation Inequalities

- \((\Omega, \mathcal{F}, P)\): a probability space.
- \(A, B\): two events.
- \(A\) and \(B\) are independent if
  \[
  \Pr(AB) = \Pr(A)\Pr(B).
  \]
- \(A\) and \(B\) are **positively correlated** if
  \[
  \Pr(AB) \geq \Pr(A)\Pr(B).
  \]
- \(A\) and \(B\) are **negatively correlated** if
  \[
  \Pr(AB) \leq \Pr(A)\Pr(B).
  \]
Four Functions Theorem

- $N := \{1, 2, 3 \ldots, n\}$
- $P(N)$: the power set of $N$.
- $\alpha, \beta, \gamma, \delta: P(N) \rightarrow \mathbb{R}^+$
- For $A \subset P(N)$, and $\phi \in \{\alpha, \beta, \gamma, \delta\}$, let $\phi(A) = \sum_{A \in \mathcal{A}} \phi(A)$. 
Four Functions Theorem

- \( N := \{1, 2, 3 \ldots, n\} \)
- \( P(N) \): the power set of \( N \).
- \( \alpha, \beta, \gamma, \delta : P(N) \to \mathbb{R}^+ \)
- For \( \mathcal{A} \subset P(N) \), and \( \phi \in \{\alpha, \beta, \gamma, \delta\} \), let
  \[
  \phi(\mathcal{A}) = \sum_{A \in \mathcal{A}} \phi(A).
  \]

Theorem [Ahlswede, Daykin (1978)]: If for any \( A, B \subset N \),

\[
\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B),
\]

then for any \( \mathcal{A}, \mathcal{B} \subset P(N) \),

\[
\alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \cup \mathcal{B})\delta(\mathcal{A} \cap \mathcal{B}),
\]
Proof

Simplification:

■ Modifying $\alpha$ so that $\alpha(A) = 0$ for all $A \notin A$.
■ Modifying $\beta$ so that $\beta(B) = 0$ for all $B \notin B$.
■ Modifying $\gamma$ so that $\gamma(C) = 0$ for all $C \notin A \cup B$.
■ Modifying $\delta$ so that $\delta(D) = 0$ for all $D \notin A \cap B$. 
Proof

Simplification:

- Modifying $\alpha$ so that $\alpha(A) = 0$ for all $A \notin \mathcal{A}$.
- Modifying $\beta$ so that $\beta(B) = 0$ for all $B \notin \mathcal{B}$.
- Modifying $\gamma$ so that $\gamma(C) = 0$ for all $C \notin \mathcal{A} \cup \mathcal{B}$.
- Modifying $\delta$ so that $\delta(D) = 0$ for all $D \notin \mathcal{A} \cap \mathcal{B}$.

$$\alpha(A)\alpha(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

still holds. It is sufficient to prove for $\mathcal{A} = \mathcal{B} = P(N)$. 
Induction on $n$

Initial case $n = 1$: $P(N) = \{\emptyset, N\}$. Use index 0 for $\emptyset$ and 1 for $N$. We have

\[
\begin{align*}
\alpha_0\beta_0 & \leq \gamma_0\delta_0 \\
\alpha_0\beta_1 & \leq \gamma_1\delta_0 \\
\alpha_1\beta_0 & \leq \gamma_1\delta_0 \\
\alpha_1\beta_1 & \leq \gamma_1\delta_1.
\end{align*}
\]

We need prove

\[
(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \leq (\gamma_0 + \gamma_1)(\delta_0 + \delta_1).
\]

It can be directly verified.
**Inductive step**

Suppose it holds for $n - 1$ and let us prove it for $n \geq 2$. Let $N' = N \setminus \{n\}$ and for each $\phi \in \{\alpha, \beta, \gamma, \delta\}$ and $A \in N'$ define

$$\phi'(A) = \phi(A) + \phi(A \cup \{n\}).$$

Note that $\phi(P(N)) = \phi'(P(N'))$. Apply inductive hypothesis for functions $\alpha'$, $\beta'$, $\gamma'$, and $\delta'$. It suffices to check

$$\alpha'(A)\alpha'(B) \leq \gamma'(A \cup B)\delta'(A \cap B).$$

This is similar to the case $n = 1$. \qed
(L, ∨, ∧) is a lattice if it satisfies

- **Commutative laws:** \(a ∨ b = b ∨ a, a ∧ b = b ∧ a\).
\((L, \lor, \land)\) is a lattice if it satisfies

- **Commutative laws:** \(a \lor b = b \lor a, \ a \land b = b \land a\).
- **Associative laws:** \(a \lor (b \lor c) = (a \lor b) \lor c, \ a \land (b \land c) = (a \land b) \land c\)
Distributive lattice

\((L, \lor, \land)\) is a lattice if it satisfies

- **Commutative laws:** \(a \lor b = b \lor a, a \land b = b \land a.\)
- **Associative laws:** \(a \lor (b \lor c) = (a \lor b) \lor c,\)
  \(a \land (b \land c) = (a \land b) \land c\)
- **Absorption laws:** \(a \lor (a \land b) = a, a \land (a \lor b) = a.\)

It is distributive if it further satisfies the distributive laws:

\[
a \land (b \lor c) = (a \land b) \lor (a \land c),
\]
\[
a \lor (b \land c) = (a \lor b) \land (a \lor c).
\]
Theorem [Ahlsweede, Daykin (1978)]: Let $L$ be a distributive lattice and $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}^+$. If for any $x, y \in L$,

$$\alpha(x)\alpha(y) \leq \gamma(x \lor y)\delta(x \land y),$$

then for any $X, Y \subset L$,

$$\alpha(X)\alpha(Y) \leq \gamma(X \lor Y)\delta(X \land Y),$$
Theorem [Ahlswede, Daykin (1978)]: Let $L$ be a distributive lattice and $\alpha, \beta, \gamma, \delta : L \rightarrow \mathbb{R}^+$. If for any $x, y \in L$,

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then for any $X, Y \subset L$,

$$\alpha(X)\alpha(Y) \leq \gamma(X \lor Y)\delta(X \land Y),$$

Note any distributive lattice can be embedded into $P([n])$. This is a corollary of the previous theorem.
A function $\mu: L \rightarrow \mathbb{R}^+$ is log-supermodular if
\[\mu(x)\mu(y) \leq \mu(x \lor y)\mu(x \land y)\]
for all $x, y$. 
A function $\mu : L \rightarrow \mathbb{R}^+$ is log-supermodular if
$\mu(x)\mu(y) \leq \mu(x \lor y)\mu(x \land y)$ for all $x, y$.

$f : L \rightarrow \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is decreasing if $f(x) \geq f(y)$ whenever $x \leq y$. 
FKG inequalities

- A function $\mu: L \rightarrow \mathbb{R}^+$ is log-supermodular if
  $$\mu(x)\mu(y) \leq \mu(x \lor y)\mu(x \land y)$$
  for all $x, y$.
- $f: L \rightarrow \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is decreasing if $f(x) \geq f(y)$ whenever $x \leq y$.

The FKG Inequality [Fortuin-Kasteleyn-Ginibre 1971]:
If $\mu$ is log-supermodular and $f, g$ are increasing, then
$$\sum_{x \in L} f(x)\mu(x) \sum_{x \in L} g(x)\mu(x) \leq \sum_{x \in L} f(x)g(x)\mu(x) \sum_{x \in L} \mu(x).$$
FKG inequalities

- A function $\mu: L \to \mathbb{R}^+$ is log-supermodular if $\mu(x)\mu(y) \leq \mu(x \lor y)\mu(x \land y)$ for all $x, y$.
- $f: L \to \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is decreasing if $f(x) \geq f(y)$ whenever $x \leq y$.

The FKG Inequality [Fortuin-Kasteleyn-Ginibre 1971]:
If $\mu$ is log-supermodular and $f, g$ are increasing, then

$$\sum_{x \in L} f(x)\mu(x) \sum_{x \in L} g(x)\mu(x) \leq \sum_{x \in L} f(x)g(x)\mu(x) \sum_{x \in L} \mu(x).$$

If one is increasing and the other is decreasing, then

$$\sum_{x \in L} f(x)\mu(x) \sum_{x \in L} g(x)\mu(x) \geq \sum_{x \in L} f(x)g(x)\mu(x) \sum_{x \in L} \mu(x).$$
A probabilistic view

- $(P(N), \mu)$: a probability space where $\mu$ is log-supermodular.
- An event $\mathcal{A}$ is monotone increasing if $A \in \mathcal{A}$ and $A \subset B$ implies $B \in \mathcal{A}$.

**Proposition:** If both $A$ and $B$ are monotone increasing or monotone decreasing, then

$$\Pr(AB) \geq \Pr(A)\Pr(B).$$

If one is monotone increasing and the other one is monotone decreasing, then

$$\Pr(AB) \leq \Pr(A)\Pr(B).$$
In $G(n, p)$, for any graph $H$,

$$
\mu(H) = \Pr(H) = p^{|E(H)|}(1 - p)^{|E(\bar{H})|}.
$$

Observe that this $\mu$ is log-supermodular. We get a lot of correlation inequalities on monotone events.
Applying to $G(n, p)$

In $G(n, p)$, for any graph $H$,

$$\mu(H) = \Pr(H) = p^{|E(H)|} (1 - p)^{|E(\bar{H})|}.$$ 

Observe that this $\mu$ is log-supermodular. We get a lot of correlation inequalities on monotone events.

Example of monotone events:
- Triangle-free.
- Being planar graph.
- $k$-connected.
- Having Hamiltonian cycle.
- $H$-free.
- Diameter less than $k$. 