Topic Course on Probabilistic Methods
(Week 3) Alterations

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Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)
Subtopics

Alteration

- Ramsey number $R(r, r)$
- Combinatorial geometry
- Ramsey number $R(k, r)$
- Property B problem revisited
Suppose that the “random” structure does not have all desired properties but many have a few “blemishes”. With a small alteration we remove the blemishes, giving the desired structures.
Ramsey number $R(r, r)$

**Theorem:** $R(r, r) > (1 + o(1)) \frac{1}{e} r 2^{r/2}$. 
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Proof: Color the edges of \( K_n \) in two colors with equal probability randomly and independently. Let \( X \) be the number of monochromatic \( K_r \). Then

\[
E(X) = \binom{n}{r} 2^{1-\binom{r}{2}}.
\]
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$$E(X) = \binom{n}{r}2^{1-(\frac{r}{2})}.$$

If $X < \frac{n}{2}$, then we can delete at most $\frac{n}{2}$ to destroy all monochromatic $K_r$. Thus, $R(r, r) > \frac{n}{2}$. 
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This gives $R(r, r) > (1 + o(1)) \frac{1}{e} r 2^{r/2}$. □
Combinatorial geometry

- $S$: a set of $n$ points in the unit square $[0, 1]^2$.
- $T(S)$: the minimum area of a triangle whose vertices are three distinct points of $S$.

Komlós, Pintz, Szemerédi (1982): There exists a set $S$ of $n$ points in the unit square such that $T(S) = \Omega\left(\frac{\log n}{n^2}\right)$. 
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Proof: Select $2n$ random points uniformly and independently from $[0, 1]^2$.

- $P, Q, R$: three random points.
- $\mu := \Delta PQR$: the area of $PQR$. 
Proof

\[ \Pr(x \leq |PQ| \leq x + \Delta x) \leq \pi(x + \Delta x)^2 - \pi x^2 \approx 2\pi x \Delta x. \]

If \( \mu \leq \epsilon \), then \( R \) is in the region of a rectangle of width \( \frac{4\epsilon}{x} \) and length at most \( \sqrt{2} \).
Proof

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\[ \Pr(\mu \leq \epsilon) \leq \int_0^{\sqrt{2}} (2\pi x) \left( \frac{4\sqrt{2}\epsilon}{x} \right) dx = 16\pi \epsilon. \]
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Let \( X \) be the number of triangles with areas < \( \frac{1}{100n^2} \).

\[ \mathbb{E}(X) \leq \binom{2n}{3} \frac{16\pi}{100n^2} < n. \]
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Let $X$ be the number of triangles with areas $< \frac{1}{100n^2}$.

$$E(X) \leq \left( \frac{2n}{3} \right) \frac{16\pi}{100n^2} < n.$$ 

Delete one vertex from each small triangle and leave at least $n$ vertices. Now no triangle has area less that $\frac{1}{100n^2}$. □
Theorem: For any $0 < p < 1$, we have

$$R(k, t) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{t} (1 - p)^{\binom{t}{2}}.$$
Ramsey number \( R(k, t) \)

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\]

**Proof:** Color each edge independently in red or blue; the probability of being red is \( p \) while the probability of being blue is \( 1 - p \). Let \( X \) be the number of red \( K_k \) and \( Y \) be the number of blue \( K_t \).

\[
E(X) = \binom{n}{k} p^{\binom{k}{2}}
\]

\[
E(Y) = \binom{n}{t} (1 - p)^{\binom{t}{2}}.
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Ramsey number \( R(3, t) \)

For \( k = 3 \), this alteration method gives \( R(3, t) \geq \left( \frac{t}{\ln t} \right)^{3/2} \).
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Before Shearer’s result, **Ajtai-Komlós and Szemerédi (1980)** proved $R(3, t) \leq \frac{c't^2}{\ln t}$.
Property B problem revisited:
Let $m(r)$ denote the minimum possible number of edges of an $r$-uniform hypergraph that does not have property $B$. 
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Theorem [Radhakrishnan-Srinivasan 2000]:

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**Theorem [Radhakrishnan-Srinivasan 2000]:**

$$m(r) \geq \Omega \left( \left( \frac{r}{\ln r} \right)^{1/2} 2^r \right).$$

**Proof:** For a fixed $r$-uniform hypergraph $H = (V, E)$ with $|E| = k2^{r-1}$. Let $p \in [0, 1]$ satisfying $k(1 - p)^r + k^2p < 1$. 
Here is a two-round coloring process.

- **First round**: Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected $k$ monochromatic edges. Let $U$ be the set of vertices in some monochromatic edges.
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- **First round:** Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected \( k \) monochromatic edges. Let \( U \) be the set of vertices in some monochromatic edges.

- **Second round:** Consider vertices in \( U \) sequentially in the (random) order of \( V \). A vertex \( u \in U \) is **still dangerous** if there is some monochromatic edge in the first coloring and for which no vertex has yet changed color.
  - If \( u \) is not dangerous, do nothing.
  - If \( u \) is still dangerous; with probability \( p \), flip the color of \( u \).
**Claim:** The algorithm fails with probability at most $k(1 - p)^r + k^2p$. 
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Bad events: An edge \( e \) is red in the final coloring if

- \( e \) was red in the first coloring and remained red through the final coloring; call this event \( A_e \).
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Bad events: An edge $e$ is red in the final coloring if

- $e$ was red in the first coloring and remained red through the final coloring; call this event $A_e$.
- $e$ was not red in the first coloring but was red in the final coloring; call this event $C_e$. 
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\]
\[
2 \sum_{e \in E(H)} \Pr(A_e) = k(1 - p)^r.
\]
For two edge $e, f$, we say $e$ *blames* $f$ if

- $e \cap f = \{v\}$ for some $v$.
- In the first coloring $f$ was blue and in the final coloring $e$ was red.
- $v$ was the last vertex of $e$ that changed color from blue to red.
- When $v$ changed its color $f$ was still entire blue.
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- When $v$ changed its color $f$ was still entire blue.

Call this event $B_{ef}$. Then

$$\sum_e \Pr(C_e) \leq \sum_{e \neq f} \Pr(B_{ef}).$$
Let $e, f$ with $e \cap f = \{v\}$ be fixed. The random ordering of $V$ induced a random ordering $\sigma$ on $e \cup f$. 
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- $i = i(\sigma)$: the number of $v' \in e$ coming before $v$.
- $j = j(\sigma)$: the number of $v' \in f$ coming before $v$. 
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\Pr(B_{ef} \mid \sigma) \leq \frac{p}{2} 2^{-r+1} (1 - p)^j 2^{-r+1+i} \left(\frac{1 + p}{2}\right)^i.
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We have

\[
\Pr(B_{ef}) \leq 2^{1-2r} p E[(1 + p)^{i} (1 - p)^{j}].
\]
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The failure probability is at most

\[ 2 \sum_{e \in E(H)} (\Pr(A_e) + \Pr(C_e)) \leq k(1 - p)^r + k^2 p < ke^{-pr} + k^2 p. \]
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The function \( f(p) = ke^{-pr} + k^2 p \) reaches its minimum at

\[ p = \frac{\ln(r/k)}{r}. \]

The minimum value is less than 1 if

\[ k < (1 + o(1)) \sqrt{\frac{2r}{\ln r}}. \]
Spencer modified the Radhakrishnan-Srinivasan’s proof slightly. To assign a random ordering of the vertex in $V$, it is sufficient to assign each vertex $v$ a birth time $x_v \in [0, 1]$. The birth time $x_v$ is assigned uniformly and independently.
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The rest of proof is the same.