Topic Course on Probabilistic Methods (Week 2)
Linearity of Expectation (2)

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Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)
Subtopics

Linearity of Expectation (2)

- Disjoint pairs
- $k$-sets
- Balancing vectors
- Unbalancing lights
- Brégman’s Theorem
- Hamliton paths
- Independence number
- Turán Theorem
Disjoint pairs

- \( \mathcal{F} \subseteq 2^{[n]} \).
- \( d(\mathcal{F}) := |\{(F, F') : F, F' \in \mathcal{F}, F \cap F' = \emptyset\}| \).
Disjoint pairs

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- $d(\mathcal{F}) := |\{(F, F'): F, F' \in \mathcal{F}, F \cap F' = \emptyset\}|$.

Daykin and Erdős conjectured if $|\mathcal{F}| = 2^{(1/2+\delta)n}$ then $d(\mathcal{F}) = o(|\mathcal{F}|^2)$. 
Disjoint pairs

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Daykin and Erdős conjectured if \( |\mathcal{F}| = 2^{(1/2 + \delta)n} \) then \( d(\mathcal{F}) = o(|\mathcal{F}|^2) \).

**Theorem [Alon-Frankl, 1985]:** If \( |\mathcal{F}| = 2^{(1/2 + \delta)n} \), then

\[
  d(\mathcal{F}) < |\mathcal{F}|^{2 - \delta^2/2}.
\]
Let $m := 2^{(1/2 + \delta)n}$. Suppose $d(\mathcal{F}) < m^{2 - \delta^2/2}$.
Proof

Let $m := 2^{(1/2 + \delta)n}$. Suppose $d(\mathcal{F}) < m^{2-\delta^2/2}$.

Pick independently $t$ members $A_1, A_2, \ldots, A_t$ of $\mathcal{F}$ with repetitions at random.
Proof

Let $m := 2^{(1/2+\delta)n}$. Suppose $d(F) < m^{2-\delta^2/2}$.

Pick independently $t$ members $A_1, A_2, \ldots, A_t$ of $F$ with repetitions at random.

\[
\Pr(|\bigcup_{i=1}^t A_i| \leq \frac{n}{2}) \leq \sum_{|S| = \frac{n}{2}} \Pr(\bigwedge_{i=1}^t (A_i \subset S)) \leq 2^n \left( \frac{2^{n/2}}{2^{(1/2+\delta)n}} \right)^t = 2^n(1-\delta t).
\]
Let $\nu(B) = |\{A \in \mathcal{F} : B \cap A = \emptyset\}|$. Then

$$\sum_{B} \nu(B) = 2d(\mathcal{F}) \geq 2m^{2-\delta^2/2}.$$
Let $v(B) = |\{A \in \mathcal{F} : B \cap A = \emptyset\}|$. Then

$$\sum_{B} v(B) = 2d(\mathcal{F}) \geq 2m^{2-\delta^2/2}.$$ 

Let $Y$ be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all $A_i$ $1 \leq i \leq t$. 
Let \( v(B) = |\{ A \in \mathcal{F} : B \cap A = \emptyset \}| \). Then

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\sum_{B} v(B) = 2d(\mathcal{F}) \geq 2m^{2-\delta^2/2}.
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Let \( Y \) be a random variable whose value is the number of members \( B \in \mathcal{F} \) that is disjoint to all \( A_i \ 1 \leq i \leq t \). Then

\[
E(|Y|) = \sum_{B \in \mathcal{F}} \left( \frac{v(B)}{m} \right)^t \\
\geq \frac{1}{m^{t-1}} \left( \frac{\sum_{B} v(B)}{m} \right)^t \\
\geq 2m^{1-t\delta^2/2}.
\]
Since $Y \leq m$, we get

$$\Pr(Y \geq m^{1-t\delta^2/2}) \geq m^{-t\delta^2/2}.$$
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Choose $t = \lceil 1 + \frac{1}{\delta} \rceil$. We have $m^{-t\delta^2/2} > 2^n(1-\delta t)$. 
Since $Y \leq m$, we get

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Choose $t = \lceil 1 + \frac{1}{\delta} \rceil$. We have $m^{-t\delta^2/2} > 2^{n(1-\delta)}$.

Thus, with positive probability, $|\bigcup_{i=1}^{t} A_i| > \frac{n}{2}$ and $\bigcup_{i=1}^{t} A_i$ is disjoint to more than $2^{n/2}$ members of $\mathcal{F}$. Contradiction. $\square$
Let $X_1, X_2, \ldots, X_n$ be random variables and 
$X = \sum_{i=1}^{n} c_i X_i$. Then

$$E(X) = \sum_{i=1}^{n} c_i E(X_i).$$
Linearity of expectation

Let $X_1, X_2, \ldots, X_n$ be random variables and $X = \sum_{i=1}^{n} c_i X_i$. Then

$$E(X) = \sum_{i=1}^{n} c_i E(X_i).$$

**Philosophy:** There is a point in the probability space for which $X \geq E(X)$ and a point for $X \leq E(X)$. 
Theorem: Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. Then $G$ contains a bipartite subgraph with at least $m/2$ edges.
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$$E(X_{uv}) = \frac{1}{4}.$$
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$$E(X) = \sum_{uv \in E} E(X_{uv}) = \frac{m}{2}.$$
\( k \)-sets

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- $h: V^k \rightarrow \{-1, 1\}$. 
$\kappa$-sets

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A $k$-set $F$ is crossing if it contains precisely one point from each $V_i$.

**Theorem:** Suppose $h(F) = +1$ for all crossing $k$-sets $F$. Then there is an $S \subset V$ for which

$$|h(S)| \geq c_k n^k.$$  

Here $c_k > 0$, independent of $n$. 
Lemma: Let $P_k$ be the set of all homogeneous polynomials $f(p_1, \ldots, p_k)$ of degree $k$ with all coefficients have absolute value at most one and $p_1 p_2 \cdots p_k$ having coefficient one. Then for all $f \in P_k$ there exists $p_1, \ldots, p_k \in [0, 1]$ with

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**A Lemma**
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Here $c_k > 0$, independent of $n$.

Proof: Let $M(f) = \max_{p_1, \ldots, p_k} |f(p_1, \ldots, p_k)|$. Note $P_k$ is compact and $M$ is continuous. $M$ reaches its minimum value $c_k$ at some point $f_0$. We have

$$c_k = M(f_0) > 0.$$
Proof of theorem

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$$\Pr(x \in V) = p_i, \quad x \in V_i.$$
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$$X_F = \begin{cases} 
    h(F) & \text{if } F \subset S, \\
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Say $F$ has type $(a_1, \ldots, a_k)$ if $|F \cap V_i| = a_i$, $1 \leq i \leq k$. 
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Say $F$ has type $(a_1, \ldots, a_k)$ if $|F \cap V_i| = a_i$, $1 \leq i \leq k$. For these $F$,

$$E(X_F) = h(F)p_1^{a_1} \cdots p_k^{a_k}.$$
\[
E(X) = \sum_{\sum_{i=1}^{k} a_i = k} p_{1}^{a_1} \cdots p_{k}^{a_k} \sum_{F \text{ of type } (a_1, \ldots, a_k)} h(F).
\]
\[ E(X) = \sum_{\sum_{i=1}^{k} a_i = k} p_1^{a_1} \cdots p_k^{a_k} \sum h(F). \]

Let \( f(p_1, \ldots, p_k) = \frac{1}{n^k} E(X) \). Then \( f \in P_k \).
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Now select \( p_1, \ldots, p_k \in [0, 1] \) with \( |f(p_1, \ldots, p_k)| \geq c_k. \)
Then \( E(|X|) \geq |E(X)| \geq c_k n^k. \)
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Now select \( p_1, \ldots, p_k \in [0, 1] \) with \( |f(p_1, \ldots, p_k)| \geq c_k \). Then \( E(|X|) \geq |E(X)| \geq c_k n^k \).

There exists a \( S \) such that \( |h(S)| \geq c_k n^k \). \( \square \)
Theorem: Let $v_1, \ldots, v_n$ are $n$ unit vector in $\mathbb{R}^n$. Then there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n},$$

and also there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \geq \sqrt{n}.$$
Let $\epsilon_1, \ldots, \epsilon_n$ be selected uniformly and independently from \{+1, −1\}. Let $X = \|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|^2$. 

Proof
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$$E(X) = E\left( \sum_{i,j=1}^{n} \epsilon_i \epsilon_j v_i \cdot v_j \right)$$

$$= \sum_{i,j=1}^{n} E(\epsilon_i \epsilon_j) v_i \cdot v_j$$

$$= \sum_{i,j=1}^{n} \delta_i^j v_i \cdot v_j$$

$$= \sum_{i=1}^{n} \|v_i\|^2 = n.$$
**Theorem:** Let $v_1, \ldots, v_n \in \mathbb{R}^n$, all $\|v_i\| \leq 1$. Let $p_1, p_2, \ldots, p_n \in [0, 1]$ be arbitrary and set $w = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n$. Then there exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ so that setting $v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$,

$$\|w - v\| \leq \frac{\sqrt{n}}{2}.$$
An extension

**Theorem:** Let \( v_1, \ldots, v_n \in \mathbb{R}^n \), all \( \|v_i\| \leq 1 \). Let \( p_1, p_2, \ldots, p_n \in [0, 1] \) be arbitrary and set \( w = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n \). Then there exist \( \epsilon_1, \ldots, \epsilon_n \in \{0, 1\} \) so that setting \( v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n \),

\[
\|w - v\| \leq \frac{\sqrt{n}}{2}.
\]

**Hint:** Pick \( \epsilon_i \) independently with

\[
\Pr(\epsilon_i = 1) = p_i, \quad \Pr(\epsilon_i = 0) = 1 - p_i.
\]

The proof is similar.
Theorem: Let $a_{ij} = \pm 1$ for $1 \leq i, j \leq n$. Then there exist $x_i, y_j = \pm 1$, $1 \leq i, j \leq n$ so that

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq \left( \sqrt{\frac{2}{\pi}} + o(1) \right) n^{3/2}.$$
**Theorem:** Let $a_{ij} = \pm 1$ for $1 \leq i, j \leq n$. Then there exist $x_i, y_j = \pm 1, 1 \leq i, j \leq n$ so that

$$
\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq \left( \sqrt{\frac{2}{\pi}} + o(1) \right) n^{3/2}.
$$

**Proof:** Choose $y_j = 1$ or $-1$ randomly and independently. Let $R_i = \sum_{i=1}^{n} a_{ij} y_j$. Let $x_i$ be the sign of $R_i$. Then

$$
\sum_{i,j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} |R_i|.
$$
Each $R_i$ has the distribution $S_n = \sum_{i=1}^{n} X_i$, where $X_i$’s are independent uniform $\{-1, 1\}$ random variables.
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$$E(|S_n|) = n2^{1-n} \left( \frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right)$$

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Hence,

$$\sum_{i=1}^{n} E(|R_i|) = \left( \sqrt{\frac{2}{\pi}} + o(1) \right) n^{3/2}.$$
Brégman’s Theorem

- $A = (a_{i,j})$: an $n \times n$ matrix with all $a_{i,j} \in \{0, 1\}$. 
Brégman’s Theorem

- \( A = (a_{ij}) \): an \( n \times n \) matrix with all \( a_{i,j} \in \{0, 1\} \).
- \( S \): the set of permutations \( \sigma \in S_n \), with \( a_{i,\sigma(i)} = 1 \) for all \( i \).
Brégman’s Theorem

- $A = (a_{i,j})$: an $n \times n$ matrix with all $a_{i,j} \in \{0, 1\}$.
- $S$: the set of permutations $\sigma \in S_n$, with $a_{i,\sigma(i)} = 1$ for all $i$.
- $\text{per}(A) = |S|$: the permanent of $A$. 
Brégman’s Theorem

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- \( \text{per}(A) = |S| \): the permanent of \( A \).
- \( r_i \): the \( i \)-th row sum.

Brégman’s Theorem (1973): \( \text{per}(A) \leq \prod_{1 \leq i \leq n} (r_i!)^{1/r_i} \).
Proof [Schrijver 1978]

Pick \( \sigma \in S \) and \( \tau \in S_n \) independently and uniformly.
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Pick $\sigma \in S$ and $\tau \in S_n$ independently and uniformly.

- Let $A^{(1)} := A$; and $A^{(i)}$ is the submatrix obtained by deleting row $\tau(i - 1)$ and column $\sigma(\tau(i - 1))$ for $2 \leq i \leq n$. 
Proof [Schrijver 1978]

Pick $\sigma \in S$ and $\tau \in S_n$ independently and uniformly.

- Let $A^{(1)} := A$; and $A^{(i)}$ is the submatrix obtained by deleting row $\tau(i-1)$ and column $\sigma(\tau(i-1))$ for $2 \leq i \leq n$.

- $R_{\tau(i)}$: the $\tau(i)$’s row sum of $A^{(i)}$. 
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- \( L = L(\sigma, \tau) := \prod_{i=1}^{n} R_{\tau(i)} \).
Proof [Schrijver 1978]

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- $L = L(\sigma, \tau) := \prod_{i=1}^{n} R_{\tau(i)}$.

- $G(L) := e^{\mathbb{E} (\ln L)} = e^{\sum_{i=1}^{n} \mathbb{E} (\ln R_{\tau(i)})}$. 
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- $R_{\tau(i)}$: the $\tau(i)$’s row sum of $A^{(i)}$.

- $L = L(\sigma, \tau) := \prod_{i=1}^{n} R_{\tau(i)}$.

- $G(L) := e^{E(\ln L)} = e^{\sum_{i=1}^{n} E(\ln R_{\tau(i)})}$.

Claim: $\per(A)) \leq G(L)$.
For any fixed $\tau$. Assume $\tau(1) = 1$. By re-ordering, assume the first row has ones in the first $r := r_1$ columns. For $1 \leq j \leq r$ let $t_j$ be the permanent of $A$ with the first row and $j$-th column removed (i.e., $\sigma(1) = j$). Let

$$t = \frac{t_1 + \cdots + t_r}{r} = \frac{\text{per}(A)}{r}.$$
For any fixed $\tau$. Assume $\tau(1) = 1$. By re-ordering, assume the first row has ones in the first $r := r_1$ columns. For $1 \leq j \leq r$ let $t_j$ be the permanent of $A$ with the first row and $j$-th column removed (i.e., $\sigma(1) = j$). Let

$$t = \frac{t_1 + \cdots + t_r}{r} = \frac{\text{per}(A)}{r}.$$

By induction,

$$G(R_2 \cdots R_n | \sigma(1) = j) \geq t_j.$$

$$G(L) \geq \prod_{j=1}^{r} \left( r t_j \right)^{t_j/\text{per}(A)} = r \prod_{j=1}^{r} \left( t_j \right)^{t_j/rt}.$$
Since \( \left( \prod_{j=1}^{r} t_{j}^{t_j} \right)^{1/r} \geq t^t \), we have

\[
G(L) \geq r \prod_{j=1}^{r} t_{j}^{t_j/rt} \geq r(t^t)^{1/t} = rt = \text{per}(A).
\]
Since \( \left( \prod_{j=1}^{r} t_{j}^{t_{j}} \right)^{1/r} \geq t^{t} \), we have

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\]

Now we calculate \( G[L] \) conditional on a fixed \( \sigma \). By reordering, assume \( \sigma(i) = i \) for all \( i \). Note

\[
G(R_{i}) = (r_{i}!)^{1/r_{i}}.
\]
Since \( \left( \prod_{j=1}^{r} t_{j}^{t_{j}} \right) \frac{1}{r} \geq t^{t} \), we have

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G(L) \geq r \prod_{j=1}^{r} t_{j}^{t_{j}/rt} \geq r(t^{t})^{1/t} = rt = \text{per}(A).
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G(R_{i}) = (r_{i}!)^{1/r_{i}}.
\]

\[
G(R) = G\left( \prod_{i=1}^{n} R_{i} \right) = \prod_{i=1}^{n} (r_{i}!)^{1/r_{i}}.
\]
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**Proof:** Let $X$ be the number of Hamiltonian paths in a random tournament. Write $X = \sum_{\sigma \in S_n} X_\sigma$. Here $X_\sigma$ is the indicator random variable for $\sigma$ giving a Hamilton path.

$$E(X_\sigma) = 2^{-(n-1)}.$$
**Theorem:** There is a tournament $T$ with $n$ players and at least $n!2^{-(n-1)}$ Hamiltonian paths.

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$$E(X_\sigma) = 2^{-(n-1)}.$$  

We have

$$E(X) = \sum_{\sigma \in S_n} E(X_\sigma) = n!2^{1-n}.$$  

Done! $\square$
Let $P(n)$ be the maximum possible number of Hamiltonian paths in a tournament on $n$ vertices.
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Szele [1943] proved

$$\frac{1}{2} \leq \lim_{n \to \infty} \left( \frac{P(n)}{n!} \right)^{1/n} \leq \frac{1}{2^{3/4}}.$$ 

He conjecture that $\lim_{n \to \infty} \left( \frac{P(n)}{n!} \right)^{1/n} = \frac{1}{2}$. 
Let $P(n)$ be the maximum possible number of Hamiltonian paths in a tournament on $n$ vertices.

**Szele [1943]** proved

\[
\frac{1}{2} \leq \lim_{n \to \infty} \left( \frac{P(n)}{n!} \right)^{1/n} \leq \frac{1}{2^{3/4}}.
\]

He conjectured that \( \lim_{n \to \infty} \left( \frac{P(n)}{n!} \right)^{1/n} = \frac{1}{2} \).

This conjecture was proved by Alon in 1990.
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**Theorem [Alon, 1990]:** $P(n) \leq cn^{3/2} \frac{n!}{2^{n-1}}$. 


Alon’s proof

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$$F(T) = \text{per}(A_T) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$  

Here $r_i$ is $i$-th row sum of $A_T$; $\sum_{i=1}^{n} r_i = \binom{n}{2}$.  

A convex inequality

**Lemma:** For every two integers $a, b$ satisfying $b \geq a + 2 > a \geq 1$, we have

$$\left(\frac{a!}{a}\right)^{1/a} \left(\frac{b!}{b}\right)^{1/b} < \left(\frac{(a + 1)!}{a+1}\right)^{1/(a+1)} \left(\frac{(b - 1)!}{b-1}\right)^{1/(b-1)}.$$
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**Lemma:** For every two integers $a, b$ satisfying $b \geq a + 2 > a \geq 1$, we have

$$(a!)^{1/a}(b!)^{1/b} < ((a + 1)!)^{1/(a+1)}((b - 1)!)^{1/(b-1)}.$$

**Proof:** Let $f(x) = \frac{(x!)^{1/x}}{((x+1)!)^{1/(1+x)}}$. We need to show $f(a) < f(b - 1)$. It suffices to show $f(x - 1) < f(x)$.

$$((x - 1)!)^{1/(x-1)}((x + 1)!)^{1/(1+x)} < (x!)^{2/x}.$$
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Simplifying it, we get \( \left( \frac{x^x}{x!} \right)^2 > \left( 1 + \frac{1}{x} \right)^{x(x-1)} \).

It can be proved using \( x! > \left( \frac{x+1}{2} \right)^x \) for \( x \geq 2 \). \( \square \)
Proof of theorem

Observe that $\sum_{i=1}^{n} (r_i!)^{1/r_i}$ achieves the maximum when all $r_i$'s are almost equal. We get

$$F(T) \leq (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2e}} n^{3/2} \frac{(n - 1)!}{2^n}.$$
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Construct a new tournament $T'$ for $T$ by adding a new vertex $v$, where the edges from $v$ to $T$ are oriented randomly and independently. Every Hamiltonian path in $T$ can be extended to a Hamiltonian cycle in $T'$ with probability $\frac{1}{4}$. We have

$$P(T) \leq \frac{1}{4} C(T') = O \left( n^{3/2} \frac{n!}{2^{n-1}} \right). \quad \square$$
Independence number

\( \alpha(G) \): the maximal size of an independent set of a graph \( G \).
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Theorem [Caro (1979), Wei(1981)] $\alpha(G) \geq \sum_{v \in V} \frac{1}{d_v + 1}$. 
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**Proof:** Pick a random permutation \( \sigma \) on \( V \). Define

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I = \{ v \in V : vw \in E \Rightarrow \sigma(v) < \sigma(w) \}.
\]

Then \( I \) is an independent set.
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Let \( X_v \) be the indicator random variable for \( v \in I \).

\[
E(X_v) = \Pr(v \in I) = \frac{1}{d_v + 1}.
\]

\[
\alpha(G) \geq E(|I|) = \sum_v \frac{1}{d_v + 1}.
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Turán number $t(n, H)$: the maximum integer $m$ such that there is a graph on $n$ vertices containing no subgraph $H$. 
**Turán number** \( t(n, H) \): the maximum integer \( m \) such that there is a graph on \( n \) vertices containing no subgraph \( H \).

**Turán Theorem:** For \( n = km + r \) \((0 \leq r < k)\),

\[
t(n, K_{k+1}) = m^2 \binom{k}{2} + rm(k - 1) + \binom{r}{2}.
\]

The equality holds if and only if \( G \) is the complete \( k \)-partite graph with equitable partitions, denoted by \( G_{n,k} \).
For any $k \leq n$, let $q, r$ satisfy $n = kq + r$, $0 \leq r < k$. Let $e = r \binom{q+1}{e} + (m - r) \binom{q}{2}$. 
For any $k \leq n$, let $q, r$ satisfy $n = kq + r$, $0 \leq r < k$. Let $e = r \left( \binom{q+1}{e} \right) + (m - r) \binom{q}{2}$.

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When the equality holds, $I$ is a constant. $G$ can not contain an induced $P_2$. Therefore $G = \overline{G}_{n,k}$. 

Dual version
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■ Turán (1941):
\[
t(n, K_k) = |E(G_{n,k-1})| = (1 - \frac{1}{k-1} + o(1)) \left(\frac{n}{2}\right).
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History

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Erdős-Bondy-Simonovits (1963,1974):
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t(n, C_{2k}) \leq ckn^{1+1/k}.
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Open conjectures

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- **Conjecture ($\$250$ for proof and $\$100$ for disproof:)** Suppose $H$ is a bipartite graph. Prove or disprove that $t(n, H) = O(n^{3/2})$ if and only if $H$ does not contain a subgraph each vertex of which has degree $> 2$. 