E7) (a) For a positive n, show that the equation $\lambda(n)\log(\lambda(n)) = n$ has a unique real solution $\lambda(n)$.
(b) Show that for $n \to \infty$,

$$\log \lambda(n) = \log n - O(\log\log n).$$

(c) Try harder to show that

$$\log \lambda(n) = \log n - \log\log n + o(1).$$

Solution. We denote natural logarithm by log. The function $x \mapsto x\log x$ is increasing on $[1/e, \infty)$ (see the derivative), and turns positive for $x > 1$. Therefore for positive n, the equation $\lambda(n)\log(\lambda(n)) = n$ has a unique real solution.

(a) Rearranging the equation $\lambda(n)\log(\lambda(n)) = n$ we obtain

$$\lambda(n) = \frac{n}{\log(\lambda(n))}. \tag{0.1}$$

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Taking the logarithm of (0.1), we have

$$\log(\lambda(n)) = \log n - \log\log(\lambda(n)). \tag{0.2}$$

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As $\log\log n$ is defined and increasing for $n \geq 3$, and clearly $\lambda(n) < n$, we obtain (a) from (0.2).

(b) Insert into (0.2) for the $\log(\lambda(n))$ on the RHS what (0.2) says $\log(\lambda(n))$ is equal to. You obtain

$$\log(\lambda(n)) = \log n - \log[\log n - \log\log(\lambda(n))] = \log n - \log[(\log n)\cdot\Big(1 - \frac{\log\log\lambda(n)}{\log n}\Big)] = \tag{0.3}$$

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$$\log n - \log\log n - \log\Big(1 - \frac{\log\log\lambda(n)}{\log n}\Big) = \log n - \log\log n - o(1)$$

as $\log\log\lambda(n) < \log\log n$, $\frac{\log\log n}{\log n} \to 0$, and $\log x$ is continuous at $x = 1$.

E8) Observe that

$$e^{e^x - 1} \geq \frac{B_n}{n!}x^n \tag{0.4}$$

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for all $0 < x < \infty$.
(a) Show from this observation that $n!e^{\lambda(n) - 1 - n\log\log\lambda(n)} \geq B_n$.
(b) Show that this is the best possible bound from (0.4).
Solution. Recall that $e^{e^x-1} = \sum_{n\geq 0}\frac{B_n}{n!}x^n$, since by the Exponential Formula it is exponential generating function of the Bell numbers. As the Bell numbers are positive, (0.4) must hold for any x less than the convergence radius of the series. As the convergence radius is $R = \frac{1}{\limsup_n \left(\frac{B_n}{n!}\right)^{1/n}}$, to show $R = \infty$, we have to show that for all $\delta > 0$ there exists an n_0 such that for all $n > n_0$

$$\left(\frac{B_n}{n!}\right)^{1/n} < \delta \tag{0.5}$$

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holds. Set $\epsilon = \frac{\delta}{2e}$. Let C_n count the number of partitions of an n-element set into at least ϵn classes; and let D_n count the number of partitions of an n-element set into less than ϵn classes. Clearly $B_n = C_n + D_n \leq \max\big(2C_n, 2D_n\big)$, and therefore it suffices to show (0.5) with B_n changed to $2C_n$ (resp. $2D_n$).

First observe that $D_n \leq (\epsilon n)^n$. Indeed, put down $\lfloor \epsilon n \rfloor$ numbered boxes boxes. There are exactly $(\lfloor \epsilon n \rfloor)^n$ placements of the elements of the underlying n-element set into the boxes. Making partition classes from the contents of the boxes creates all partitions to be counted by D_n at least once, proving $D_n \leq (\epsilon n)^n$.

Using Stirling's Formula, $n! > \frac{1}{2}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ for sufficiently large n. We obtain

$$\left(\frac{2D_n}{n!}\right)^{1/n} \leq \left(\frac{2(\epsilon n)^n}{\frac{1}{2}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\right)^{1/n} = \epsilon e\left(\frac{4}{\sqrt{2\pi n}}\right)^{1/n} \to \frac{\delta}{2}.$$

Regarding C_n, we claim that

$$C_n \leq \sum_{k\geq\epsilon n}\frac{n!\binom{n-1}{k-1}}{k!}.$$

To verify this formula, do the following procedure to create partitions of an n-element set into k classes. Put your n elements into linear order, select $k-1$ gaps between elements from $n-1$ gaps in your linear order. Make partition classes from the elements not separated by gaps. Clearly every partition created is created by at least $k!$ ways, as classes with their fixed internal order come up exactly $k!$ ways. Using $\binom{n-1}{k-1} \leq 2^{n-1}$, we conclude

$$\frac{2C_n}{n!} \leq \sum_{k \geq \epsilon n} \frac{2^n}{k!} \leq \frac{n 2^n}{\lceil \epsilon n \rceil !},$$

as every k is at least $\lceil \epsilon n \rceil$, and there are at most n of them. We continue this estimation by pretending that ϵn is integer. It is not, and we should play with the ceilings, but it further complicates the formula without making a substantial difference. This should not be done, but it is often done... From Stirling's Formula, $(\epsilon n)! > \frac{1}{2}\left(\frac{\epsilon n}{e}\right)^{\epsilon n}\sqrt{2\pi\epsilon n}$ for n sufficiently large, and

$$\left(\frac{2C_n}{n!}\right)^{\frac{1}{n}} \leq \frac{2(n)^{\frac{1}{n}}}{\frac{1}{2}\left(\frac{\epsilon n}{e}\right)^{\epsilon}(2\pi\epsilon n)^{\frac{1}{2n}}} < 4\left(\frac{e}{\epsilon}\right)^{\epsilon}\frac{1}{n^{\epsilon}} < \delta$$

for n sufficiently large.

(a) We concluded that for all $0 < x < \infty$, (0.4) holds. This gives an upper bound $f(x) \geq \log\left(\frac{B_n}{n!}\right)$, where $f(x) = e^x - 1 - n\log x$. $f(x)$ is minimized where its derivative is zero, $f'(x) = e^x - \frac{n}{x} = 0$. This happens when $x = \log\lambda(n)$. Therefore this method privides the upper bound

$$\frac{n!}{(\log\lambda(n))^n} e^{\lambda(n)-1}.$$

(b) This bound is the best possible with this method, as we optimized for x.

E9) Moser and Wyman proved the asymptotic formula, $B_n \sim \frac{1}{\sqrt{n}}\lambda(n)^{n+\frac{1}{2}}e^{\lambda(n)-1-n}$. Give a tight $O()$ estimation for the fraction of our upper bound over B_n in E8), and B_n as $n \to \infty$. (You will need Stirling's Formula.)
Solution.

$$\frac{\frac{n!}{(\log\lambda(n))^n}e^{\lambda(n)-1}}{\frac{1}{\sqrt{n}}\lambda(n)^{n+\frac{1}{2}}e^{\lambda(n)-1-n}} = \frac{n! e^n}{\frac{\sqrt{\lambda(n)}}{\sqrt{n}}[\lambda(n)\log\lambda(n)]^n} = \frac{\sqrt{n}}{\sqrt{\lambda(n)}} n! \left(\frac{e}{n}\right)^n,$$

which is by Stirling's Formula

$$< 2\frac{\sqrt{n}}{\sqrt{\lambda(n)}}\sqrt{2\pi n} = O\Big(\frac{n}{\sqrt{\lambda(n)}}\Big).$$

This upper bound can be made more explicit using E7(c):

$$\lambda(n) = e^{\log\lambda(n)} = e^{\log n - \log\log n + o(1)} = \frac{n}{\log n}e^{o(1)} = \frac{n}{\log n}(1 + o(1)),$$

and hence the fraction is $O(\sqrt{n\log n})$.

E10) Show that if $x > 1$ is a real number sufficiently close to one, then the sequence x, x^x, x^{x^x},... converges.
Solution. Recall $t(z)$, the exponential generating function of rooted labelled trees. From Cayley's Theorem, we know explicitly that

$$t(z) = \sum_{n\geq 1}\frac{n\cdot n^{n-2}}{n!}z^n = z + z^2 + ... \qquad (0.6)$$ tz

From the Exponential Formula we know the identity of formal power series

$$t(z) = ze^{t(z)}. \qquad (0.7)$$ azon1

It is easy to see by direct calculation that the convergence radius of (0.6) is $1/e$. Consider the power series $\mathcal{E}(z) = \frac{t(z)}{z} = \sum_{n\geq 1} \frac{n \cdot n^{n-2}}{n!} z^{n-1} = 1 + z + \ldots.$ It is easy to see that this series has the same convergence radius $1/e$, and the identity (0.7) translates into

$$\mathcal{E}(z) = e^{z\mathcal{E}(z)}. \tag{0.8}$$

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It follows that (0.7) and (0.8) are identities for numbers z: $|z| < 1/e$.

Consider now any $1 < z < e^{1/e}$. Note that $\mathcal{E}(\log z)$ is defined and > 1, as $\log z > 0$. Take any x with $1 < x < \min\big(z, \mathcal{E}(\log z)\big)$. We show that the sequence defined recursively as $x_1 = x$, $x_{n+1} = x^{x_n}$ is convergent. (This is exactly the sequence in the problem!)

We prove by induction that $x_n < \mathcal{E}(\log z)$. This holds for $n = 1$ by the choice of x. Note that

$$x_{n+1} = x^{x_n} \leq z^{\mathcal{E}(\log z)} = \mathcal{E}(\log z),$$

where the inequality holds by $x < z$ and the hypothesis $x_n < \mathcal{E}(\log z)$, and the last equation holds by (0.8).

We prove by induction that $x_n < x_{n+1}$. This holds for $n = 1$, as clearly $x_1 = x < x^x = x_2$ for any $x > 1$. Observe $x_{n+1} = x^{x_n} > x^{x_{n-1}} = x_n$.

The sequence (x_n) is bounded and increasing, hence it converges.