1 General Hints

It is worth submitting partial results: like some construction is required for all n, and you have it for all odd n, or a map has to be shown bijective, and you show it is injective. Partial credit will be given for them. Make yourself familiar with solutions of those problems that you did not solve, as these problems carry a substantial amount of material relevant for the course. Many problems will be discussed in class.

It is impossible and even counterproductive to forbid students from discussion of homework problems with each other. However, if a solution is "joint solution" of two or more students, the participants should indicate this on the submission, and the credit will be divided among them.

For problems requring a single number as answer, leave your results as a sum of powers, binomial coefficients, Stirling or Fibonacci numbers, etc., unless a *numerical answer* is asked for. Let me know about the typoes so that I can correct them.

2 Bijections and advanced differentiation rules.

B1) Show with a bijection that for $n \geq 1$, an n-element set has the same number of even size subsets and odd size subsets.

B2) We want to buy 20 fruits in the market: Apples, Bananas, Coconuts and Durians. We can tell apart different kind of fruits, but not fruits of the same kind. We must buy at least 4 Apples and 3 Coconuts, but we must not buy more than 5 Bananas. Also, we must not buy more than 4 Durians. In how many different ways this can shopping be carried out?

B3) Show with a bijections:

(a) the number of compositions of n into terms that can be only 1 and 2 equals to the number of compositions of $n+1$ into terms that has to be odd.

(b) the number in part (a) equals to the number of compositions of $n+2$ into terms greater than 1 i.e. 2,3,4,...

B4) Assume that the integers $k_i \geq 0$ $(0 \leq i \leq n)$ satisfy $\sum_{i=1}^{n} ik_i = n$. Show that the number of partitions of an n-element set into k_1 singletons, k_2 doubletons, ... , k_i i-element sets, ... , k_n n-element set is equal to

$$\frac{n!}{\prod_{i=1}^{n} k_i!(i!)^{k_i}}.$$

B5) [Generalized Leibniz Rule] Let $f^{(i)}(x)$ denote the i^{th} derivative of the function $f(x)$, with $f^{(0)}(x) = f(x)$. Show that if $f(x)$ and $g(x)$ are differentiable as many times as needed, then

$$\left(\frac{d}{dx}\right)^n f(x)g(x) = \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(x)g^{(n-i)}(x).$$

B6) [Generalized Chain Rule] Let $D_x^k u$ denote the k^{th} derivative of u with with respect to the variable x. Assume that $u = u(x)$ and $w = w(u)$ are differentiable as many times as needed. The familair Chain Rule states $D_x^1 w = D_u^1 w D_x^1 u$. Differentiating one more time we obtain $D_x^2 w = D_u^2 w (D_x^1 u)^2 + D_u^1 w D_x^2 u$. Show that in general

$$D_x^n w = \sum_{0 \leq j \leq n} \sum_{\substack{k_1 + \ldots k_n = j \\ k_1 + 2k_2 + \ldots nk_n = n \\ k_1, \ldots, k_n \geq 0}} D_u^j w \frac{n!}{\prod_{i=1}^{n} k_i!(i!)^{k_i}} (D_x^1 u)^{k_1} (D_x^2 u)^{k_2} \cdots (D_x^n u)^{k_n}.$$

B7) Let D denote the differential operator $t\frac{d}{dt}$. Show that for an m times differentiable function $y(t)$, we have

$$D^m(y) = \sum_{i=1}^{m} S(m,i) t^i y^{(i)}.$$

(The power of a differential operator means repeating the action of the differential operator. $S(m,i)$ stands for the Stirling number of the second kind.)

B8) The Bell number B_m is the number of all partitions of an m-element set, i.e. $B_m = \sum_{i=1}^{m} S(m,i)$. Apply the result of the previous problem to the exponential function $y(t) = e^t$, to obtain Dobinski's formula:

$$B_m = \sum_{i=1}^{m} S(m,i) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{n^m}{n!}.$$

3 Fibonacci numbers.

The Fibonacci numbers F_n are defined recursively by $F_1 = F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \tag{3.1}$$ fibrec

for $n \geq 2$.

F1) Extend the definition of the Fibonacci numbers for zero and negative integer subscripts, such that recurrence relation (3.1) remains valid for all subscripts.

F2) Show that the number of length n words made of 2 letters, R and W, where consecutive R's are forbidden, is F_{n+2}.

F3) Prove the following identity using combinatorial interpretation from the previous problem for $n, m \geq 2$:

$$F_{n+m} = F_m F_{n+1} + F_{m-1} F_n. \tag{3.2}$$ tolucas

F4) Show that for all $n \geq 1$, $gcd(F_n, F_{n+1}) = 1$.

F5) Show that for all $n, m \geq 1$, $gcd(F_n, F_m) = F_{gcd(n,m)}$.

F6) Show $F_{n+2} = \sum_k \binom{n-k+1}{k}$.

F7) Show that for every positive integer m, there are infinitely many Fibonacci numbers that happen to be multiples of m.

4 Catalan numbers

Recall that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the number of lattice paths using unit horizontal and unit vertical steps from $(0,0)$ to (n,n), such that the lattice path has no point above the line $y = x$, and also the number of lattice paths using unit horizontal and unit vertical steps from $(0,0)$ to $(n+1, n+1)$, such that the lattice path is always under the line $y = x$ except in the two endpoints.

C1) Show by a bijective argument that the number of parenthesizations of the $(n+1)$-term product $a_1 \cdot a_2 \cdot \ldots \cdot a_{n+1}$ is C_n. (Hint: use n pairs of parentheses to specify the n multiplications. Use left parenthesis for horizontal step and multiplication dot for vertical step.)

C2) Give a bijective argument to show that the number of binary planted plane trees with n leaves is the same as the number of planted plane trees with n non-root vertices.

C3) Count the number of ways of filling in with integers $1, 2, ..., 2n$ the entries of an $2 \times n$ matrix, such that both rows and all n columns have decreasing entries.

C4) We have a yardstick and we want to chop it into 1-inch pieces. How many ways we can do it, if
a) at a step we chop one piece, which is longer than 1 inch, into two
b) at a step we chop simultaneously all pieces, which are longer than 1 inch, into two pieces.

5 Labelled trees

Recall Menon's theorem: the number of labelled trees on n vertices that have degree d_i in vertex i for $i = 1, 2, ..., n$, is $\binom{n-2}{(d_1-1)!\cdots(d_n-1)!}$, if all $d_i \geq 1$ and $\sum_i d_i = 2(n-1)$; and is 0 otherwise.

T1) What is the number of labelled trees on n vertices, in which the degree of v is d? (A single degree is prescribed, all others are arbitrary.)

T2) We are given vertex disjoint subtrees on $f_1, f_2, ..., f_m$ vertices in the complete graph K_n. Show that the number of spanning trees that contain all of the m given subtrees is $f_1 f_2 ... f_m n^{n-2-\sum_i (f_i-1)}$.

T3) Show that the number of spanning trees of the complete bipartite graph $K_{n,m}$ is $n^{m-1}m^{n-1}$. Can you derive a formula for the number of spanning trees of the complete tripartite graph $K_{n,m,p}$?

6 Abel and Hurwitz binomial theorems

A1) [Abel's Binomial Theorem] Prove that for variables x, y, z the following polynomial identity holds:

$$\sum_{k=0}^{n} \binom{n}{k} x(x+kz)^{k-1}(y+(n-k)z)^{n-k} = (x+y+nz)^n \tag{6.3}$$ binom

(Interpret the $k=0$ term as $(y=nz)^n$ to make clear that all terms are polynomials indeed. Do such a step for all identities in this section, when a^{-1} powers occur.) Hint: call $f_n(x,y,z)$ the LHS of 6.3 and call $g_n(x,y,z)$ the RHS of (6.3). Show by induction on n that $f_n(x,y,z) = g_n(x,y,z)$. Use for base case $n=1$. Build your induction step on the identities

$$\frac{\partial}{\partial y} f_n(x,y,z) = \frac{\partial}{\partial y} g_n(x,y,z); \tag{6.4}$$

$$f_n(x,-x-nz,z) = g_n(x,-x-nz,z). \tag{6.5}$$

A2) Prove that for variables x, y, z the following polynomial identity holds:

$$\sum_{k=0}^{n} \binom{n}{k} x(x+kz)^{k-1} y(y+(n-k)z)^{n-k-1} = (x+y)(x+y+nz)^{n-1} \tag{6.6}$$ binom1

Hint: derive first from (A1) the special case $z=1$ of (6.6), and then go for the general case.

A3) [Hurwitz' Binomial Theorem] Prove that for variables $x, y, z_1, ..., z_n$ the following polynomial identities

$$\sum_{\epsilon_i=0,1} x(x+\sum_{i=1}^{n}\epsilon_i z_i)^{\sum_{i=1}^n \epsilon_i - 1}(y+\sum_{i=1}^{n}(1-\epsilon_i)z_i)^{n-\sum_{i=1}^n \epsilon_i} = \tag{6.7}$$ H1'

$$(x+y+z_1+...+z_n)^n$$

and

$$\sum_{\epsilon_i=0,1} x(x+\sum_{i=1}^{n}\epsilon_i z_i)^{\sum_{i=1}^n \epsilon_i - 1}(y-\sum_{i=1}^{n}\epsilon_i z_i)^{n-\sum_{i=1}^n \epsilon_i} = (x+y)^n, \tag{6.8}$$ H1

imply each other.

A5) Show that the polynomial identity (6.9) below implies (6.7)

$$\sum_{\epsilon_i=0,1} x(x+\sum_{i=1}^{n}\epsilon_i z_i)^{\sum_{i=1}^n \epsilon_i - 1} y(y+\sum_{i=1}^{n}(1-\epsilon_i)z_i)^{n-1-\sum_{i=1}^n \epsilon_i} = \tag{6.9}$$ H2

$$(x+y)(x+y+z_1+...+z_n)^{n-1}.$$

A6) Given two sets, $U = \{1, ..., p, p+1, ..., q\}$, and $V = \{v_1, ..., v_r\}$. Take a $g: V \to U$ function and an $f: \{p+1, ..., q\} \to V$ function. Create a directed graph G by adding all $(u, f(u))$ and $(v, g(v))$ edges to the vertex set.

a) Given a fixed f, show that the number of g functions such that the resulting directed graph has no directed cycle, is $q^{r-1}p$. (For $r=0$, this value is 1.)

b) Given a fixed f, show that the number of g injections such that the resulting directed graph has no directed cycle, is $(q-1)_{r-1}p$. (For $r=0$, this value is 1.)

Hint for (a): create a bijection between the set of g functions and sequences $(a_1, a_2, ..., a_r)$, where for $1 \le k \le r-1$, $1 \le a_k \le q$ and $1 \le a_r \le p$. Call a vertex v_k free, if there is no directed path from any other element of V to v_k. Define b_1 as the least integer k such that v_k is free. If $b_1, ..., b_t$ has already been selected, define b_{t+1} as the smallest integer $k \notin \{b_1, ..., b_t\}$, such that $v_{b_{t+1}}$ is free in the graph that results after the deletion of the edges $(v_{b_k}, g(v_{b_k}))$ for $k = 1, 2, ..., t$. Assign to g the sequence $(g(v_{b_1}), g(v_{b_2}), ..., g(v_{b_r}))$ and show that a this is a bijection.

A7) Consider $x + y + n + z_1 + ... + z_n$ vertices in the following way: we have an r_i vertex for $1 \le i \le x + y$, a t_j vertex for $1 \le j \le n$, and s_{jk} vertices for $1 \le j \le n$, $1 \le k \le z_j$. For all j, k, the directed egde from s_{jk} to t_j is given. We want to send from every t_j a single edge to an r or s vertex, such that we do not create a directed cycle. Count such directed graphs in two ways to prove (6.9). Hint. One counting is based on A6). For the other counting, associate a $C \subseteq \{1, 2, ..., n\}$ to your graph by

$$j \in C \iff \exists i \le x \text{ and a directed path from } t_j \text{ to } r_i.$$

A8) Make the convention $x(x-1)_{-1} = 1$. Show the Hurwitz-Hagen-Rothe identity that involves falling factorials:

$$\sum_{\epsilon_i=0,1} x(x-1+\sum_{i=1}^{n} \epsilon_i z_i)_{\sum_{i=1}^{n} \epsilon_i - 1} y(y-1+\sum_{i=1}^{n}(1-\epsilon_i) z_i)_{n-1-\sum_{i=1}^{n} \epsilon_i} = \tag{6.10}$$

HHR

$$(x+y)(x+y+z_1+...+z_n-1)_{n-1}.$$

A9) Show the following analogue of the binomial theorem:

$$(x+y)_n = \sum_k \binom{n}{k} (x)_k (y)_{n-k}.$$

A10) Also, defining the rising factorial polynomial by $x^{\overline{0}} = 1$ and $x^{\overline{k}} = x(x+1)\cdots(x+k-1)$, show another analogue:

$$(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}.$$

7 Recurrences, odds and ends

R1) Solve the following recurrence in explicit form: $a_0 = 1, a_1 = 3$, and for $n \ge 1$, $a_{n+1} = a_n + 2a_{n-1} - 2n + 3$.

R2) Solve the following recurrence in explicit form: $a_0 = 6, a_1 = 11, a_2 = 23$, and for $n \ge 2$, $a_{n+1} = 5a_n - 8a_{n-1} + 4a_{n-2}$.

A positive integer k is a *perfect power*, if it can be written as $k = n^m$, with integer $n \ge 2, m \ge 2$.

R3) Show that for every t, there are t consecutive positive integers, such that none of them is a perfect power.

R4) Show that $\sum_{k \in P} \frac{1}{k-1} = 1$, where P denotes the set of perfect powers. Verify R3).

R5) Show that for $0 \le k < m$ integers

$$\binom{n}{k} + \binom{n}{m+k} + \binom{n}{2m+k} + ... = \frac{1}{m} \sum_{0 \le j < m} \left(2\cos\frac{j\pi}{m}\right)^n \cos\frac{j(n-2k)\pi}{m}.$$

8 Generating functions

O1) For n, m positive integers, prove

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k} 2^k$$

using the snake oil method.

Let $b_{n,m}$ count the number of lattice walks from $(0,0)$ to (n,m), where every step must be unit length horizontal to the right, unit length up.

O2) Let $B(x,y)=\sum_i\sum_j b_{i,j}x^iy^j$. Verify that $B(x,y)=\frac{1}{1-x-y}$ by establishing a recurrence for the $b_{i,j}$ numbers and evaluting $(1-x-y)B(x,y)$.

O3) Verify that $B(x,y)=\sum_i\sum_j\binom{i+j}{j}x^iy^j$ using the snake oil method. (Pretend that you do not already know from elementary reasoning $b_{i,j}$.)

Let $a_{n,m}$ count the number of lattice walks from $(0,0)$ to (n,m), where every step must be unit length horizontal to the right, unit length up, or $\sqrt{2}$ length parallel with the $y=x$ line, going to up and right.

O4) Show that $a_{n,n}=\sum_j\binom{n}{j}^2 2^j$.

O5) Show that $\sum_i\sum_j a_{i,j}x^iy^j=\frac{1}{1-x-y-xy}$.

O6) Derive O4) from O5).

O7) Define the n^{th} Legendre polynomial by $P_n(x)=\frac{1}{n!2^n}\left(\frac{d}{dx}\right)^n(x^2-1)^n$. Verify that $a_{n,n}=P_n(3)$.

O8) We have a particle at time zero with the following property: at every integer hour the particle explodes randomly into i pieces for $i=1,2,\ldots.$ Let p_i denote the probability that it explodes into i particles, and set $G(x)=\sum)_i p_ix^i$. Every successor particle explodes independently of its history with the same probability distribution. Let q_i denote the probability that at time i hours and 2 minutes we have exactly i particles. Express $\sum_i q_ix^i$ in terms of $G(x)$.

O9) Let a_n denote the number of $n\times n$ 0-1 matrices with constant row and column sum 2, and define $a_0=1$. Show that $\sum_n\frac{a_n}{(n!)^2}x^n=\frac{e^{-\frac{x}{2}}}{\sqrt{1-x}}$.

9 Some number theoretic results

N1) (a) Assume that $p\geq 3$ is a prime. Color the vertices of a regular p-gon with k colors. Two colourations are considered equivalent, if there is a rotation of the regular p-gon that moves one colouration into the other. What is the number of non-equivalent colourations?

(b) What kind of number theoretic result is implied by your result? (The result holds for $p=2$ as well.)

N2) (a) Assume $p\geq 3$ is a prime. We draw a digraph in the plane such that its vertices are the vertices of a regular p-gon. Draw in straight line segments a directed Hamiltonian cycle on this vertex set. We consider two directed Hamiltonian cycles equivalent, if there is a rotation of the regular p-gon that moves one directed Hamiltonian cycle into the other. What is the number of non-equivalent directed Hamiltonian cycles?

(b) What kind of number theoretic result is implied by your result? (The result holds for $p=2$ as well.)

Let $n \mod m$ denote the unique *integer* between 0 and $m-1$ that is congruent to n modulo m. (So this is an integer, not a residue class!)

N3) Show that for a prime p

$$\binom{n}{k}\equiv\binom{\lfloor n/p\rfloor}{\lfloor k/p\rfloor}\binom{n \mod p}{k \mod p}\text{ modulo } p.$$

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N5) Assume that p is a prime, and that n and k are written in base p representation as $n=a_rp^r+\ldots+a_1p+a_0$ and $k=b_rp^r+\ldots+b_1p+b_0$. Show that

$$\binom{n}{k}\equiv\binom{a_r}{b_r}\cdots\binom{a_1}{b_1}\binom{a_0}{b_0}\text{ modulo } p.$$

10 Partitions

Call a partition *self-dual*, if the Ferrers diagram of the partition is symmetric to its diagonal. E.g. 5+2+1+1+1 is symmetric. The *Durfee square* of a partition is the largest square that fits into the Ferrers diagram of the partition. The partition above has a 2×2 Durfee square.

P1) Show that the generating function of the number of self-dual partitions of n that have Durfee square size k is $\frac{x^{k^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}$.

P2) Show that $[x^n]\frac{x^{k^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}$ = the number of partitions of $n-k^2$ into even terms, with all terms at most $2k$.

P3) Show that the number of partitions of n into distinct odd terms equals the number of self-dual partitions of n.

P4) Show

$$(1+x)(1+x^3)\cdots(1+x^{2i+1})\cdots = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}.$$

P5) Show that for $k = o(n^{1/3})$, $\lim_{n\to\infty} \frac{\binom{n+\binom{k}{2}-1}{k-1}}{\binom{n-1}{k-1}} = 1$.

11 Exponential generating functions and Bell numbers

E1) Find the number of connected graphs on 5 labelled vertices.

E2) In a game, children can form pairs or circles in the playground. A child belongs to exactly one thing, which can be a cycle or a pair. How many games are possible for n children? (A one-variable summation suffices for solution.)

E3) Let B^*_n denote the number of partitions of an n-element set into classes that are all at least 2 in size. Show $B_n = B^*_{n+1} + B^*_n$.

E4) Show $B^*_{n+1} = \sum_{i=1}^{n} (-1)^{n-i} B_i$.

E5) Find the exponential generating function of the sequence B^*_n.

E6)(a) Find the exponential generating function of the number of directed n-cycles on n vertices (allow loops and 2-cycles).

(b) Give every cycle the same real weight w. Modify the generating function above so that the coefficient of $[x^n]/n!$ is the sum of weights of directed n-cycles on n vertices.

(c) Let $c(n,k)$ denote the number of permutations of n elements that have exactly k cycles. What is the exponential generating function of the sequence $n \to \sum_{k=1}^{n} c(n,k)w^k$? (Hint: $(1-x)^{-w}$.)

(d) Show that $(w+n-1)_n = \sum_k c(n,k)w^k$. It also holds if w is a variable, not a number.

(e) Recall that that Stirling numbers of the first kind are defined by $(x)_n = \sum_k s(n,k)x^k$. Show that $(-1)^{n-k}s(n,k) = |s(n,k)| = c(n,k)$.

E7) (a) For a positive n, show that the equation $\lambda(n)\log(\lambda(n)) = n$ has a unique real solution $\lambda(n)$.

(b) Show that for $n \to \infty$,

$$\log\lambda(n) = \log n - O(\log\log n).$$

(c) Try harder to show that

$$\log\lambda(n) = \log n - \log\log n + o(1).$$

E8) Observe that

$$e^{e^x-1} \geq \frac{B_n}{n!}x^n \tag{11.11}$$ oneterm

for all $0 < x < \infty$.

(a) Show from this observation that $n!e^{\lambda(n)-1-n\log\log\lambda(n)} \geq B_n$.

(b) Show that this is the best possible bound from (11.11).

E9) Moser and Wyman proved the asymptotic formula, $B_n \sim \frac{1}{\sqrt{n}}\lambda(n)^{n+\frac{1}{2}}e^{\lambda(n)-1-n}$. Give a tight $O()$ estimation for the fraction of our upper bound over B_n in E8), and B_n as $n\to\infty$. (You will need Stirling's Formula.)

E10) Show that if $x>1$ is a real number sufficiently close to one, then the sequence x, x^x, x^{x^x},... converges.

12 Inclusion-exclusion

Let $A_1,...,A_n$ be events in a probability space. Let σ_j denote $\sum_{1\le i_1<\dots<i_j} P(\cap_{\ell=1}^{j} A_{i_\ell})$.

I1) Show that the probability of *exactly* q of these $A_1,...,A_n$ events occuring is $\sum_{j=q}^{n}\binom{j}{q}(-1)^{q+j}\sigma_j$. Interpret this result for $q=0$.

I2) Show that for *any* $\ell\ge 1$ natural number,

$$\binom{\ell+1}{2}P\Big(\bigcup_{i=1}^{n}A_i\Big)\ge \ell\sigma_1-\sigma_2.$$

I3) Assume that no more than d of the $A_1,...,A_n$ events can occur simultaneously. Show that

$$P\Big(\bigcup_{i=1}^{n}A_i\Big)\le \sigma_1-\frac{2}{d}\sigma_2.$$

I4) (a) Compute by inclusion-exclusion the number of surjections from an n-element set to an m-element set.
(b) Interpret your result for $m=n$.

I5) Let us be given a fixed simple graph G on n labelled vertices. A *good colouration* of the vertices with k colours assigns to every vertex a colour from the set $\{1,2,...,k\}$, such that endvertices of any edge get different colours. Show that the number of good colourations of G with k colours is a polynomial of k.

I6) (a) How many m-element families of k-subsets of an N-element set have the property, that their union is the underlying N-element set?
(b) Show the identity

$$\sum_{i\ge1}\binom{2n}{i}(-1)^{i+1}\binom{\binom{2n-i}{2}}{n}=(2n-1)!!.$$

I7) Let A_1, A_2,..., A_n be subsets of an n-element set, such that for all $i<j$ we have $|A_i\cap A_j|\le 1$. Show that $\sum_{i=1}^{n}|A_i|=O(n^{3/2})$.

I8) Assume that the prime factorization of n is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with different prime factors. Compute the number of integers in $[1,n]$ that are relative primes to n.

I9) There are n points on a straight line. Assume that any distance among these points occurs only once or twice. Show that at least $(n-1)/2$ distances occur only once.

I10) (Area of a parallelogram problem - was shown in class with drawing.) Let us be given an $ABCD$ paralleogram of unit area. Divide the sides into three equal pieces by adding new points in the cyclic order $AEE'BFF'CGG'DHH'$. Define four new points as intersection of segments: $I=\overline{CE}\cap\overline{BH}$, $J=\overline{CE}\cap\overline{DF}$, $K=\overline{GA}\cap\overline{DF}$, $L=\overline{GA}\cap\overline{HB}$. What is the area of the $IJKL$ parallelogram?

I11) n persons deposited their hats in the cloakroom, and by bookkeeping error, they get back one hat each, such that any distribution of the n hats happens with the same probability.
(a) What is the probability of the event that nobody gets back his own hat?
(b) What is the limit of this probability as n goes to infinity?