Counting rooted spanning forests in complete multipartite graphs

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Abstract

Jin and Liu discovered an elegant formula for the number of rooted spanning forests in the complete bipartite graph $K_{a_1, a_2}$, with $b_1$ roots in the first vertex class and $b_2$ roots in the second vertex class. We give a simple proof to their formula, and a generalization for complete $m$-partite graphs, using the multivariate Lagrange inverse.

Y. Jin and C. Liu [3] give a formula for $f(m, l; n, k)$, the number of spanning forests of the labelled complete bipartite graph $K_{n,m}$, where in the forest every tree is rooted, there are $k$ roots in the first vertex class (among the $n$ vertices) and $l$ roots in the second vertex class (among the $m$ vertices), and the trees in the forest are not ordered. They discovered the elegant formula

$$f(m, l; n, k) = \binom{m}{l} \binom{n}{k} n^{m-l-1} m^{n-k-1} (kn + ln - lk). \quad (1)$$

The goal of the present note is generalization of (1) from complete bipartite to complete multipartite graphs, through a simple proof using the multivariate Lagrange inverse.

Let $f(a_1, b_1; \ldots; a_m, b_m)$ denote the number of spanning forests of the labelled complete multipartite graph $K_{a_1, a_2, \ldots, a_m}$, where in the forest every tree is rooted, there are $b_i$ roots in the $i^{th}$ vertex class for $i = 1, 2, \ldots, m$, and the trees in the forest are not ordered. Let $w_i(t_1, \ldots, t_m)$ denote the multivariate exponential generating function (EGF) of the numbers $f(a_1, 0; \ldots; a_i, 1; \ldots; a_m, 0)$ (the number of rooted spanning trees of the complete multipartite graph $K_{a_1, a_2, \ldots, a_m}$, if the root has to be in the $i^{th}$ class), i.e.

$$w_i(t_1, \ldots, t_m) = \sum_{a_1 = 0}^{\infty} \cdots \sum_{a_{i-1} = 0}^{\infty} \sum_{a_m = 0}^{\infty} f(a_1, 0; \ldots; a_i, 1; \ldots; a_m, 0) \prod_{k=1}^{m} \frac{t_k^{a_k}}{a_k!}. \quad (2)$$

\[1\]Research partially supported by the NSF Grant 0072187 and the Hungarian NSF Grant T 0324155.
The key identity for our argument is
\[
t_i e^{(w_1 + \cdots + w_m) - w_i} = w_i \quad \text{for } i = 1, 2, \ldots, m. \tag{3}
\]

The proof of formula (3) is based on the following combinatorial decomposition. Given a rooted spanning tree of the complete multipartite graph \(K_{a_1, a_2, \ldots, a_m}\), where the root is in the \(i^{th}\) class, remove the root vertex from the tree to obtain a spanning forest of \(K_{a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m}\), and mark the former neighbors of the eliminated root vertex as roots in the forest. This decomposition establishes a bijection between the following two sets:

the set of rooted spanning trees of the complete multipartite graph \(K_{a_1, a_2, \ldots, a_m}\), where the root is in the \(i^{th}\) vertex class,

and

the set of some ordered pairs, where the first entry of the ordered pair is one of the vertices of the \(i^{th}\) vertex class, the second element of the ordered pair is a rooted spanning forest of \(K_{a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m}\), where the vertex from the first entry is removed from the \(i^{th}\) vertex class, and the trees of the forest are not ordered.

Now \(t_i e^{(w_1 + w_2 + \cdots + w_m) - w_i}\) is the EGF of the set of ordered pairs in question, according to the Exponential Formula; and \(w_i\) is the same EGF by the bijection. Set \(\Phi_i(w_1, w_2, \ldots, w_m) = e^{(w_1 + w_2 + \cdots + w_m) - w_i}\).

According to the multiplication rule of EGF’s, \(\prod_{k=1}^{m} u_k^{b_k}\) is the multivariate exponential generating function of the number of rooted spanning forests of complete \(m\)-partite graphs, with \(b_k\) roots in the \(k^{th}\) vertex class, where the trees rooted in the same part are ordered; hence
\[
f(a_1, b_1; \ldots; a_m, b_m) = \frac{a_1!a_2! \cdots a_m!}{b_1!b_2! \cdots b_m!} \left[ e^{a_1 u_1} \cdots e^{a_m u_m} \right] \prod_{k=1}^{m} u_k^{b_k}. \tag{4}
\]

According to Part 1 of Theorem 1.2.9 (Multivariate Lagrange Formula) from [2], (3) implies
\[
\left[ e^{a_1 u_1} \cdots e^{a_m u_m} \right] \prod_{k=1}^{m} u_k^{b_k} = \left[ \lambda_1^a \cdots \lambda_m^a \right] \left\{ \det \left[ \delta_{ij} - \frac{\lambda_j}{\Phi_i} \frac{\partial \Phi_i}{\partial \lambda_j} \right] \right\} \prod_{k=1}^{m} \lambda_k^{b_k} \prod_{k=1}^{m} e^{a_k (w_1 + \cdots + w_m) - a_k w_i} \right\}, \tag{5}
\]
where \(\Phi_i\) is a short-hand notation for \(\Phi_i(\lambda_1, \ldots, \lambda_m)\). Observe that \(\frac{\lambda_j}{\Phi_i} = (1 - \delta_{ij}) \lambda_j\), and the for the determinant in (5) we have the well-known evaluation
\[
\det \left[ \delta_{ij} - (1 - \delta_{ij}) \lambda_j \right] = (\lambda_1 + 1) \cdots (\lambda_m + 1) \left( 1 - \frac{\lambda_1}{\lambda_1 + 1} - \cdots - \frac{\lambda_m}{\lambda_m + 1} \right) \tag{7}
\]
(see for example Exercise 225 in [1]). Now (7) is easily rewritten as
\[
1 - \sum_{j=2}^{m} (j - 1) \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j}, \tag{8}
\]
and (8) is rewritten as
\[ \frac{1}{l_1} \sum_{l_1=0}^{l_2} \ldots \sum_{l_m=0}^{l_m} (1 - l_1 - l_2 - \ldots - l_m)(\lambda_1)_{l_1}(\lambda_2)_{l_2} \cdots (\lambda_m)_{l_m}, \quad (9) \]

where \((x)_l\) stands for the falling factorial, \((x)_0 = 1\) and \((x)_1 = x\). Introducing the notation \(A = a_1 + a_2 + \ldots + a_m\) and using (9), we find that (5) and (6) are equal to
\[
[\lambda^a_{1-b_1} \lambda^{a_2-b_2} \ldots \lambda^{a_m-b_m}] \frac{1}{l_1} \sum_{l_1=0}^{l_2} \ldots \sum_{l_m=0}^{l_m} (1 - l_1 - l_2 - \ldots - l_m) \\
\times (\lambda_1)_{l_1}(\lambda_2)_{l_2} \cdots (\lambda_m)_{l_m} e^{(A-a_1)\lambda_1} e^{(A-a_2)\lambda_2} \ldots e^{(A-a_m)\lambda_m} \\
= \left( \prod_{k=1}^{m} \frac{(A - a_k)^{a_k-b_k-1}}{(a_k - b_k)!} \right) \frac{1}{l_1} \sum_{l_1=0}^{l_2} \ldots \sum_{l_m=0}^{l_m} (1 - l_1 - l_2 - \ldots - l_m) \\
\times \left( \prod_{j=1}^{m} (A - a_j)^{1-l_j} (a_j - b_j)_{l_j} \right). \quad (10)\]

Combining (10), (11), and (4), we obtain the main result:

**Theorem 1**

\[ f(a_1, b_1; \ldots; a_m, b_m) = \left( \prod_{k=1}^{m} \frac{a_k}{b_k} \right) (A - a_k)^{a_k-b_k-1} \]
\[ \times \frac{1}{l_1} \sum_{l_1=0}^{l_2} \ldots \sum_{l_m=0}^{l_m} (1 - l_1 - l_2 - \ldots - l_m) \left( \prod_{j=1}^{m} (A - a_j)^{1-l_j} (a_j - b_j)_{l_j} \right). \quad (12) \]

For the case \(m = 2\), formula (12), (13) specializes to the formula of Jin and Liu (1), and formula (12), (13) yields a closed formula for every fixed \(m\). Note that for the case \(m = 2\) we do not even have to evaluate the determinant in general, since for \(m = 2\) simply
\[ \det \left| \delta_{ij} - (1 - \delta_{ij}) \lambda_j \right| = 1 - \lambda_1 \lambda_2. \]

**References**

