The Gap between Crossing Numbers and Convex Crossing Numbers

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Abstract

A convex drawing of an n-vertex graph $G = (V(G), E(G))$ is a drawing in which the vertices are placed on the corners of a convex n-gon in the plane and each edge is drawn using one straight line segment. We derive a general lower bound on the number of crossings in any convex drawings of G, using isoperimetric properties of G. The lower bound implies that convex drawings of many planar graphs have at least $\Omega(n \log n)$ crossings. Moreover, for any drawing of G with c crossings in the plane, we construct a convex drawing with at most $O((c + \sum_{v \in V} d_v^2) \log n)$ crossings, where d_v is the degree of v. This upper bound is asymptotically tight. For planar graphs, a convex drawing with the required properties can be constructed in $O(n \log n)$ time.

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1 Introduction

Throughout this paper $G = (V(G), E(G))$ denotes a graph on n vertices and m edges. Let d_v denote the degree of $v \in V$. A *drawing* of G is a placement of the vertices into distinct points of the plane and a representation of edges uv by simple continuous curves connecting the corresponding points and not passing through any point corresponding to a vertex other than u and v. A *crossing* is a common interior point of two edges of G. We also assume that any two curves representing the edges of G have at most one interior point in common and that two curves incident to the same vertex do not cross. If it leads to no confusion, we make no distinction between the vertices (edges) of G and the points (resp. curves) representing them. Let $\mathrm{cr}(G)$ denote the *crossing number* of G, i.e. the minimum number of crossings over all possible drawings of G in the plane with the above properties. Although the concept of crossing numbers has played a crucial role in settling many problems in combinatorial and computational geometry [S, PST, D], and also in VLSI [L], many interesting problems involving crossing numbers themselves, remain unresolved or even untouched. An important application area of crossing numbers is automated graph drawing. The number of crossings greatly influences the aesthetical properties and readability of graphs [DETT, P]. A *rectilinear drawing* of G is a drawing in which each edge is drawn using a single straight line segment. A *convex drawing* of G is a rectilinear drawing in which the vertices are placed in the corners of a convex n-gon, see Fig.1.

Let $\overline{\mathrm{cr}}(G)$ and $\mathrm{cr}^*(G)$ denote the rectilinear crossing number and the convex crossing numbers of G, respectively. *Convex crossing numbers* (also called *outerplanar crossing numbers*) were first introduced by Kainen [K] in connection with the book thickness problem. Clearly $\mathrm{cr}(G) \leq \overline{\mathrm{cr}}(G) \leq \mathrm{cr}^*(G)$. In particular, it is well known that $\mathrm{cr}(K_8) = 18 < \overline{\mathrm{cr}}(K_8) = 19$. (Note that $\mathrm{cr}^*(K_8) = \binom{8}{4} = 70$.) In terms of the k-page crossing number ν_k [SSSV1, SSSV2], it is obvious that $\mathrm{cr}^*(G) = \nu_1(G)$ for every graph G.

The main result in this paper is a general lower bound on the convex crossing number. It is easy to see, that $\mathrm{cr}^*(G) \geq m - 2n + 2$. Consequently, $\mathrm{cr}^*(G) \geq \frac{1}{27} \cdot \frac{m^3}{n^2}$, for $m \geq 3n$, using standard methods such as those in [SSSV1]. Let $B(G)$ denote the minimum size of a $(1/3, 2/3)$ edge separator in G, i.e. $B(G) = \min_{U \subset V(G)} |E(U, \bar{U})|$, where $n/3 \leq |U| \leq 2n/3$ is required in the minimization. It is well known that $\mathrm{cr}(G) = \Omega(B^2(G) - 2.5 \sum_{v \in V} d_v^2)$, and hence $\mathrm{cr}^*(G) = \Omega(B^2(G) - 2.5 \sum_{v \in V} d_v^2)$. Our lower bound presented in Section 2 involves isoperimetric properties of G, and in certain cases is much stronger than the lower bound above. Using our lower bound we exhibit classes of graphs for which $\mathrm{cr}^*(G) = \Omega\big((\overline{\mathrm{cr}}(G) + \sum_{v \in V} d_v^2) \log n\big)$. A surprising consequence is that in any convex drawing of an n-vertex 2-dimensional grid (which is a planar graph), there are $\Omega(n \log n)$ crossings. Similar results hold for hexagonal and triangular grids.

We also derive a general upper bound on $\mathrm{cr}^*(G)$. In particular, given any drawing of G with c crossings, one can construct a convex drawing with $O((c + \sum_{v \in V} d_v^2) \log n)$ crossings. Moreover, if the original drawing is represented by a

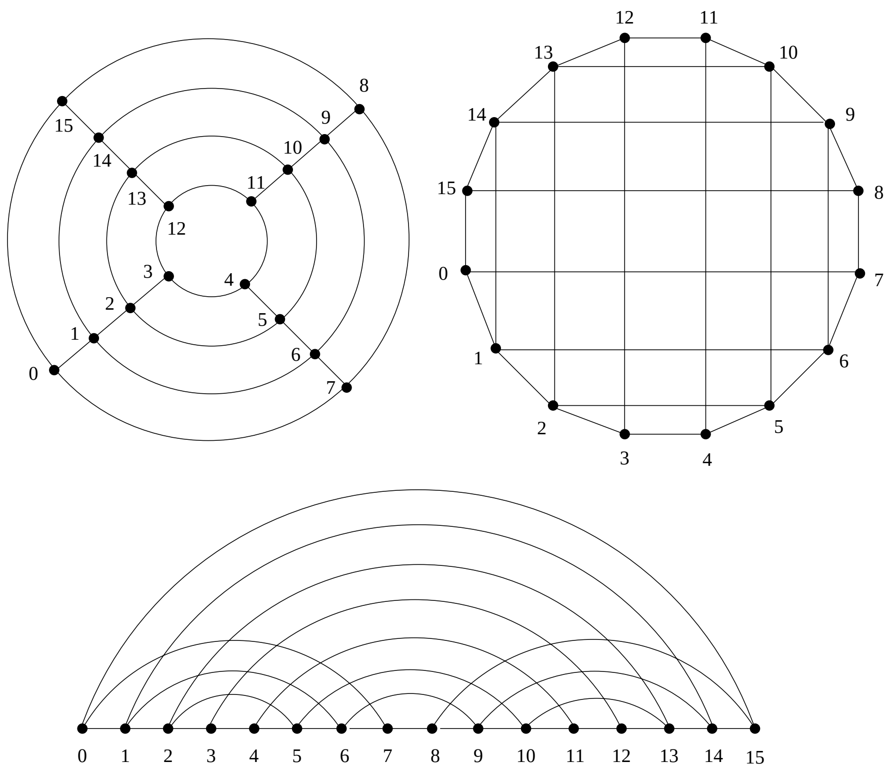

Figure 1: The 4×4 cylindrical grid and its convex and one-page drawings

planar graph, where each crossing is replaced by a vertex of degree 4, then our construction takes only $O((c+n)\log n)$ time. Previously Bienstock and Dean [BD] had proved that $\overline{\text{cr}}(G) = O(\Delta \cdot \text{cr}^2(G))$, where Δ denotes the maximum degree of G. We improved their result in [SSSV2] by showing that

$$\text{cr}^*(G) = O((\text{cr}(G) + \sum_{v \in V} d_v^2) \log^2 n). \tag{1}$$

Very recently, Even et al. [EGS], proved that for every degree bounded graph $\text{cr}^*(G) = O((\text{cr}(G)+n)\log n))$. Our new upper bound extends the construction in [EGS] from degree bounded graphs to arbitrary graphs, and improves our previous bound (1) in [SSSV2] by a $\log n$ factor. The upper bound is tight, within a constant multiplicative factor, for many interesting graphs including grids, and hence it can not be improved in general. Our upper bound implies that if $m \geq 4n$, and $\Delta = O((\frac{m}{n})^2)$, then, $\text{cr}^*(G) = O(\text{cr}(G)\log n)$, and therefore, $\overline{\text{cr}}(G) = O(\text{cr}(G)\log n)$. Thus, when G is "semi-regular" and not too sparse, $\text{cr}^*(G)$ is a good approximation for both $\overline{\text{cr}}(G)$ and $\text{cr}(G)$.

This paper is an extended version of the conference paper [SSSV3]. The authors are indebted to Éva Czabarka for her useful comments.

2 Lower Bound

Let us be given a non-negative function $f(x)$ defined on non-negative integers (or sometimes on all non-negative real numbers). We say that G satisfies the $f(x)$-*isoperimetric inequality* if for any $k \leq n/2$, and any k-element subset $U \subseteq V$, there are at least $f(k)$ edges between U and $V \backslash U$. Define the difference function of f, denoted by Δf as

$$\Delta f(i) = f(i+1) - f(i)$$

for any $i = 0, 1, ..., \lfloor \frac{n}{2} \rfloor - 1$, and set

$$\Delta^2 f(i) = (\Delta(\Delta f))(i)$$

for any $i = 0, 1, ..., \lfloor \frac{n}{2} \rfloor - 2$. Next, we derive a general lower bound for the number of crossings in convex drawings of G.

Theorem 2.1. *Assume that $G = (V(G), E(G))$ satisfies an $f(x)$-isoperimetric inequality so that Δf is non-negative and decreasing for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, where $n = |V(G)| \geq 4$. Then we have*

$$\mathrm{cr}^*(G) \geq -\frac{n}{8} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 2} f(j)\Delta^2 f(j) - \frac{1}{2}\sum_{v \in V} d_v^2. \tag{2}$$

Proof. Let D be a convex drawing of G. Without loss of generality we may assume that the vertices in D form a regular n-gon. Label the vertices by $0, 1, 2, ..., n-1$ in counter-clockwise order. For simplicity, we will often identify a vertex with the corresponding integer and all computations will be taken modulo n. Define the *distance* $l(u,v)$ between $u, v \in V$ as $\min\{|u-v|, n-|u-v|\}$, see Fig.2.

For any $uv \in E$, let $c(u,v)$ denote the number of crossings of the edge uv with other edges in D, and $c(D)$ denote the number of crossings in the drawing D. Observe that $c(u,v) \geq f(l(u,v)+1) - d_u - d_v$ if $l(u,v) < \lfloor \frac{n}{2} \rfloor$; and $c(u,v) \geq f(l(u,v)) - d_u - d_v$, if $l(u,v) = \lfloor \frac{n}{2} \rfloor$. We conclude that

$$\begin{aligned} c(D) &= \frac{1}{2}\sum_{uv \in E} c(u,v) \geq \frac{1}{2}\sum_{uv \in E} [f(l(u,v)) - d_u - d_v] \\ &= \frac{1}{2}\sum_{uv \in E} f(l(u,v)) - \frac{1}{2}\sum_{v \in V} d_v^2. \end{aligned} \tag{3}$$

We say that an edge $uv \in E$ in the drawing D *covers* a vertex i if the unique shortest path between u and v (using only the edges along the boundary of the convex n-gon) contains i. If the shortest path is not unique (this happens if

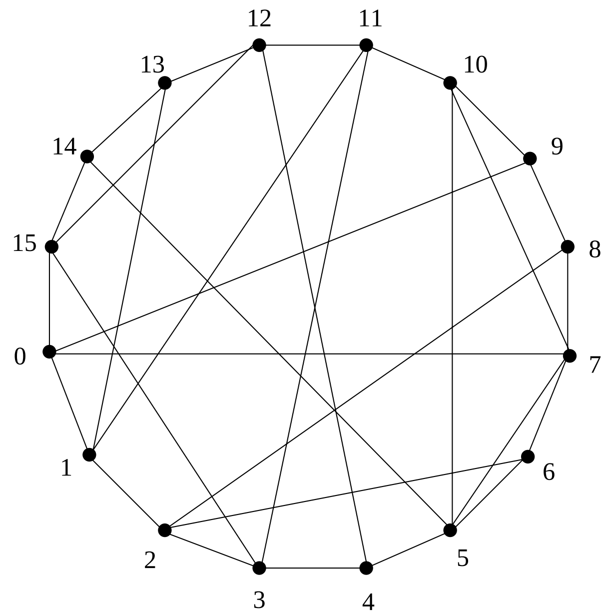

Figure 2: A convex drawing of a 16-vertex graph. E.g. the distance of 0 and 9 is 7.

$n = 2l(u, v)$), then we pick arbitrarily one of the two shortest paths, and declare its vertices be covered by the uv edge. (Note that when uv covers i, we may have $i = u$ or $i = v$.) For any edge $e = uv$ and any vertex i, define $load_{u,v}(i)$ as

$$load_{u,v}(i) = \begin{cases} \Delta f\Big(\min\{l(u,i), l(i,v)\}\Big) & \text{if } e \text{ covers } i, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for any $uv \in E$,

$$\sum_{i \in V} load_{u,v}(i) \leq 2 \sum_{j=0}^{\left\lfloor \frac{l(u,v)}{2} \right\rfloor} \Delta f(j) \leq 2f\left(\left\lfloor \frac{l(u,v)}{2} \right\rfloor + 1\right). \tag{4}$$

We conclude using (4) and (3) that

$$\begin{aligned} \frac{1}{4} \sum_{uv \in E} \sum_{i \in V} load_{u,v}(i) &\leq \frac{1}{2} \sum_{uv \in E} f\left(\left\lfloor \frac{l(u,v)}{2} \right\rfloor + 1\right) \\ &\leq \frac{1}{2} \sum_{uv \in E} f(l(u,v)) \\ &\leq c(D) + \frac{1}{2} \sum_{v \in V} d_v^2, \end{aligned} \tag{5}$$

and therefore it is sufficient to bound from below the sum involving loads.

Let $i \in V$. For $0 \leq j$, define $E_{i,j}$ to be the set of all edge $uv \in E$ covering vertex i in D such that $\min\{l(i,u), l(i,v)\} \leq j$. Observe that $E_{i,j-1} \subseteq E_{i,j}$. Note that for any $i \in V$ and for any $uv \in E_{i,j} \setminus E_{i,j-1}$, we have that i is at distance j from one of u and v, and at distance at least j from the other one. Therefore, for any $i \in V$ and for any $uv \in E_{i,j} \setminus E_{i,j-1}$, we have $load_{u,v}(i) = \Delta f(j)$, according to the definition of the load. Let k_j denote $\sum_{i \in V} |E_{i,j}|$. For any $s \leq \lfloor \frac{n}{2} \rfloor - 1$, we have

$$\begin{aligned}
\sum_{i \in V} \sum_{uv \in E_{i,s}} load_{u,v}(i) &= \sum_{i \in V} \sum_{j=0}^{s} \sum_{uv \in E_{i,j} \setminus E_{i,j-1}} load_{u,v}(i) \\
&= \sum_{i \in V} \sum_{uv \in E_{i,0}} load_{u,v}(i) + \sum_{j=1}^{s} \sum_{i \in V} \sum_{uv \in E_{i,j} \setminus E_{i,j-1}} load_{u,v}(i) \\
&\geq k_0 \Delta f(0) + \sum_{j=1}^{s} (k_j - k_{j-1}) \Delta f(j),
\end{aligned}$$

where the last inequality is obtained by observing that the number of terms in the sum $\sum_{i \in V} \sum_{uv \in E_{i,j} \setminus E_{i,j-1}} load_{u,v}(i)$, is $k_j - k_{j-1}$. It follows that

$$\sum_{i \in V} \sum_{uv \in E_{i,s}} load_{u,v}(i) \geq k_s \Delta f(s) - \sum_{j=0}^{s-1} k_j \Delta^2 f(j). \tag{6}$$

Note that up to (6) we did not use the assumption that Δf is decreasing, we used only that Δf is non-negative. Since $k_s \Delta f(s) \geq 0$, we can drop the first term from the lower bound in (6). We also have for all $j \leq n/2$,

$$k_j \geq \frac{1}{2} n f(j).$$

To see this, consider any j consecutive integers $i, i+1, ..., i+j-1$. The number of edges leaving this j-element set is at least $f(j)$, and each of them must cover either i or $i+j-1$. We may have counted some ordered pairs (edge, vertex covered by the edge) twice, since any vertex i is an endpoint of two intervals of length (distance) j, and if an edge f goes from the first interval to the second, then the ordered pair (f, i) is counted twice. Observe that if an edge uv covers a vertex i, then $uv \in E_{i, \lfloor \frac{n}{2} \rfloor - 1}$, if $n \geq 4$. Using $s = \lfloor \frac{n}{2} \rfloor - 1$ we conclude that

$$\sum_{i \in V} \sum_{uv \in E} load_{u,v}(i) = \sum_{i \in V} \sum_{uv \in E_{i, \lfloor \frac{n}{2} \rfloor - 1}} load_{u,v}(i) \geq -\frac{n}{2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 2} f(j) \Delta^2 f(j).$$

Note that in the last inequality we did use the condition that $\Delta f(j) - \Delta f(j+1) \geq 0$, i.e. that Δf is decreasing. In view of (5), this completes the proof. $\square$

Corollary 2.2. *Let G be an $N \times N$ grid. Then, as $n = N^2$ tends to infinity, we have*

$$\mathrm{cr}^*(G) = \Omega(n \log n).$$

Proof. According to Bollobás and Leader [BL], $f(x) = \sqrt{2x}$, for $x \leq n/2$, for the $N \times N$ grid. The result follows from Theorem 2.1 by standard calculations. □

Theorem 2.3. *Let H be one of the infinite {square, hexagonal, triangular} lattices in the plane and let C be any bounded open convex domain. For any $\lambda > 1$ real number, let H_λ denote the induced subgraph of H on the vertex set $V(H) \cap \lambda C$, where λC is an enlarged copy of C. Then, if $\Omega(.)$ refers to $\lambda \to \infty$, we have*

$$\mathrm{cr}^*(H_\lambda) = \Omega(|V(H_\lambda)| \cdot \log |V(H_\lambda)|). \tag{7}$$

Proof. (Sketch.) Since cr^* is monotone for the subgraph relation, it is sufficient to prove an analogue of (7) for any particular infinite family of finite subgraphs H_n of H, which has the property

$$\exists k > 0 \exists \lambda_0 \forall \lambda > \lambda_0 \exists n: \ H_n \text{ is a subgraph of } H_\lambda \text{ and } |V(H_n)| \geq k|V(H_\lambda)|. \tag{8}$$

If H is the square lattice, then $H_n = n \times n$ grid is a choice of satisfying (8), and this proves Theorem 2.3 for the square lattice.

It is clear that (8) holds for every "reasonably rounded" family of subgraphs of the hexagonal or triangular lattice. Assume for a while that Theorem 2.3 holds for the hexagonal lattice, and we show it for the triangular lattice. Just observe that starting with a unit edge length triangular lattice, colouring the vertices of the lattice with three colours, and removing vertices of one colour class, we end up with a unit edge length hexagonal lattice. Therefore $\mathrm{cr}^*(H_\lambda^{tri}) \geq \mathrm{cr}^*(H_\lambda^{hex})$.

Therefore, it is sufficient to prove Theorem 2.3 for the hexagonal lattice.

The edges of the hexagonal lattice fall into 3 parallel classes, denoted by 1,2, and 3, as shown on Fig. 2. Simultaneously contracting all edges of a fixed type (say, type 2), one obtains a square lattice. Therefore, in a "reasonably rounded" chunk of the hexagonal lattice one finds a square grid represented, whose size is linear in the size of the chunk of the hexagonal lattice. In such a representation, some vertices of the square lattice correspond to a pair of vertices of the hexagonal lattice. Fix now a convex drawing of a chunk of the hexagonal lattice with c crossings. Next, we transform it into a convex drawing of the chunk of the square grid contained in it so that the number of crossings remains $O(c)$. For this purpose, we contract every edge uv in the first drawing, whose endpoints correspond to the same vertex of the square grid. Put the new vertex corresponding this edge to the same location where u or v used to be. Every new vertex has one or two pre-images. Assume that $e = ab$ and $f = uv$ are crossing edges of square lattice in the convex drawing. Each of the 4 vertices a, b, u, v has 1 or 2 pre-images before contraction. The pre-images of the edges e and f (induced subgraphs on the pre-image vertex set) are 1-paths, 2-paths, or 3-paths. Observe that if the edges $e = ab$ and $f = uv$ cross each other, then

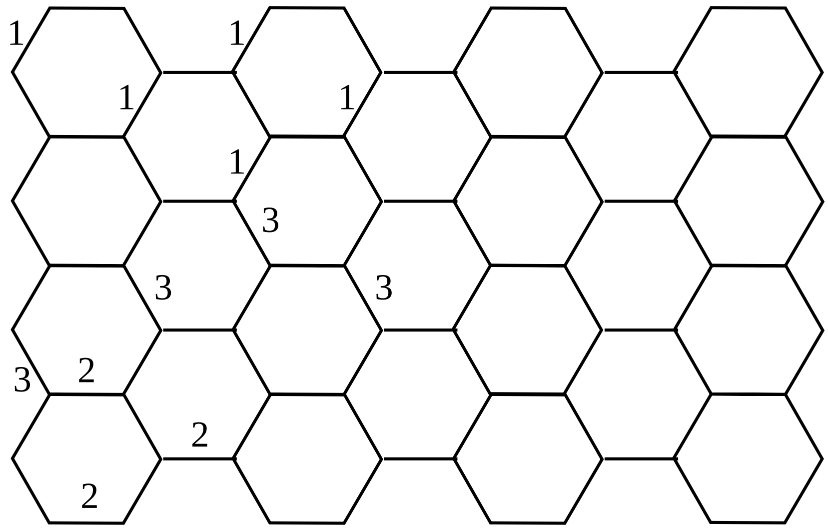

Figure 3: Hexagonal lattice

their their pre-images are two crossing paths. A simple calculation shows that the number of crossing i-paths and j-paths ($1 \leq i, j \leq 3$) in the convex drawing of a chunk of the hexagonal lattice is at most $O(c)$, since every degree is at most 3 in the hexagonal graph. This proves the claim, since the contraction kept at least 50% of the number of vertices, and Corollary 2.2 applies to a sufficently large subgraph of the result of the contraction. □

3 Upper bound

Theorem 3.1. *If G is drawn in the plane with c crossings, then a convex drawing of G with $O\left((c + \sum_{v \in V} d_v^2) \log n\right)$ crossings can be constructed. Moreover, if the original drawing is represented as a plane embedding of a planar graph, where new vertices of degree 4 represent crossings, then the order in which the vertices of G appear in the convex drawing can be determined in $O\left((c+n)\log n\right)$ time.*

Proof. To some extent we follow the arguments of Even, Guha, and Schieber [EGS], and therefore we do not go deep into the details. Consider any drawing of G in the plane with c crossings and let the set of crossings be denoted by C. Construct a planar graph denoted by $\widehat{G}$, on the vertex set $\widehat{V} = V \cup C$ by inserting vertices of degree 4 at the crossings.

Recall that in a *one-page drawing* of $\widehat{G}$ all vertices are placed on a straight line l and any edge is drawn using a semicircle above the line [SSSV2], see Fig. 1. The crucial part of the proof will be to construct a one-page drawing of $\widehat{G}$ with $O((c + \sum_{v \in \widehat{V}} d_v^2) \log n)$ crossings. Accept, for the moment, that we have such a one-page drawing of $\widehat{G}$. We then modify this drawing to obtain a one-page drawing of G: $\widehat{G}$ provides the order of the vertices of V, and the edges of G are represented by semicircles in the halfplane. It is easy to see that the number of crossings in this one-page drawing of G does not exceed by more than c the number of crossings in the one-page drawing of $\widehat{G}$. Finally, a convex drawing of G with the same number of crossings is easy to obtain from the one-page drawing.

To obtain the desired drawing of $\widehat{G}$ we construct a partition tree T [SSSV2] of $\widehat{G}$. The root of T corresponds to $\widehat{G}$, and any non-leaf node in T corresponds to a subgraph of $\widehat{G}$ with at least 2 vertices. To describe the tree, it is sufficient to indicate how to construct the left and right children of $\widehat{G}$, denoted by $\widehat{G}_1$ and $\widehat{G}_2$, respectively; the procedure recursively extends to the entire tree.

Assign a weight of $w(v) = \frac{d_v^2}{\sum_{y \in \widehat{V}} d_y^2}$, to any vertex v of $\widehat{V} = V(\widehat{G})$, where d_v is the degree of v in $\widehat{G}$. Recall a well known theorem of Gazit and Miller [GM] that any $\widehat{G}$ has a $(1/3, 2/3)$ edge separator of size at most

$$1.6\sqrt{\sum_{v \in \widehat{V}} d_v^2},$$

if for all v, $w(v) \leq 2/3$.

• Case 1. Assume that $w(v) \leq 2/3$ for any vertex $v \in \widehat{V}$. Apply the theorem cited above to find an $(1/3, 2/3)$ edge separator of size at most

$$1.6\sqrt{\sum_{v \in \widehat{V}} d_v^2}.$$

Now define $\widehat{G}_1$ and $\widehat{G}_2$ to be the two components of $\widehat{G}$ that are the obtained by the removal of the (1/3,2/3) separator.

• Case 2. Assume that there is a vertex v in $\widehat{G}$ with $w(v) \geq 2/3$. Define $V(\widehat{G}_1) = \{v\}$ and $V(\widehat{G}_2) = V(\widehat{G}) - \{v\}$, and let $\widehat{G}_1$ and $\widehat{G}_2$ be graphs induced by $\widehat{G}$ on these vertex sets.

A one-page drawing of $\widehat{G}$ is obtained by placing a one-page drawing of $\widehat{G}_1$ to the left of a one-page drawing of $\widehat{G}_2$, and then drawing the removed edges between $\widehat{G}_1$ and $\widehat{G}_2$ as semicircles between the corresponding vertices. Let $b(\widehat{G})$ denote the number of edges that have one endpoint in $\widehat{G}_1$ and the other endpoint in $\widehat{G}_2$. Similarly, define $b(\widehat{G}_i)$, $i = 1, 2$. It follows from cases 1 and 2 and from the recursive definition that

$$b(\widehat{G}) \leq 1.6\sqrt{\sum_{v \in \widehat{V}} d_v^2},$$

and

$$b(\widehat{G}_i) \leq 1.6\sqrt{\sum_{v \in V(\widehat{G}_i)} d_{i,v}^2},$$

where $d_{i,v}$ denotes the degree of $v \in \widehat{V}_i$ in $\widehat{G}_i$, $i = 1, 2$. It follows that

$$b(\widehat{G}_i) \le 1.6\sqrt{\frac{2\sum_{v\in\widehat{V}} d_v^2}{3}},$$

$i = 1, 2$. Let $S(\widehat{G})$ denote the maximum number of edges that go above any vertex in the obtained one-page drawing of $\widehat{G}$. Similarly, define $S(\widehat{G}_i)$, $i = 1, 2$. Also note that,

$$S(\widehat{G}) \le b(\widehat{G}) + \max\{S(\widehat{G}_1), S(\widehat{G}_2)\},$$

and therefore

$$S(\widehat{G}) = O\Big(\sqrt{\sum_{v\in\widehat{V}} d_v^2}\,\Big). \tag{9}$$

Now let $c(\widehat{G})$ and $c(\widehat{G}_i)$ denote the number of crossings in the one-page drawing for $\widehat{G}$, and for $\widehat{G}_i$, $i = 1, 2$, respectively. Observe that

$$c(\widehat{G}) \le c(\widehat{G}_1) + c(\widehat{G}_2) + 2b(\widehat{G})S(\widehat{G}),$$

and thus

$$c(\widehat{G}) \le c(\widehat{G}_1) + c(\widehat{G}_2) + O\left(\sum_{v\in\widehat{V}} d_v^2\right). \tag{10}$$

This implies the claimed upper bound, since the depth of the partition tree is logarithmic in $\sum_v d_v^2$, and hence in n, and the sum of the square of degrees is superadditive over the subgraphs. To finish the proof, assume that the planar graph $\widehat{G}$ is given. Then, the claim regarding the time complexity follows from the fact that computing the edge separators in [GM] can be done in the linear time for any planar graph, and hence the partition tree T can be constructed in $O((c + n)\log n)$ time. □

4 An Elementary Extremal Problem

Consider an $N \times N$ chessboard, and fill in the fields with the numbers $1, 2, 3, \ldots,$ N^2, using every number once. If two fields adjacent along an edge contain the numbers a and b, we put a weight of $\sqrt{|a-b|}$ on this edge. The goal is to fill in the numbers so as to minimize the total weight of all edges. The solution to this problem is not known. Let $C(N)$ be the value of the optimal solution. We do not know the exact value of $C(N)$. However, the results in the paper set the lower bound $C(N) = \Theta(N^2 \log N)$, as follows. Define $C^o(N)$ like $C(N)$, but use the weight $\min(\sqrt{|a-b|}, \sqrt{n^2 - |a-b|})$ instead of $\sqrt{|a-b|}$. Equation (3) in Theorem 2.1 sets a lower bound for $C^o(N)$, $C^o(N) = \Omega(N^2 \log N)$ in the following way. View the chessboard as a planar grid, and let G denote its dual, i.e., another weighted grid whose vertices correspond to the fields and are associated with the numbers sitting in them. Assign to each edge the number associated with the corresponding edge of the original grid. Consider now a convex drawing of G, where the the cyclic order of the vertices is determined by the increasing order of their weights. Note that the sum of the weights is identical to the lower bound for $cr^*(G)$. It is not difficult to see that $C(N) = \Omega(C^o(N))$.

On the other hand, Theorem 3.1 applies to G and provides the upper bound $C(N) = O(N^2 \log N)$, and hence $C(N) = \Theta(N^2 \log N)$. Finally, note that it takes only $O(N^2 \log N)$ time to fill in the fields of the $N \times N$ chessboard achieving the approximation above for $C(N)$.

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