Erdős-Ko-Rado and Hilton-Milner type theorems for intersecting chains in posets

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Abstract

We prove Erdős-Ko-Rado and Hilton-Milner type theorems for t-intersecting k-chains in posets using the kernel method. These results are common generalizations of the original EKR and HM theorems, and our earlier results for intersecting k-chains in the Boolean algebra. For intersecting k-chains in the c-truncated Boolean algebra we also prove an exact EKR theorem (for all n) using the shift method. An application of the general theorem gives a similar result for t-intersecting chains if n is large enough.

1 Introduction

One of the basic results in extremal set theory is the Erdős-Ko-Rado (EKR) theorem [5]: if $\mathcal{F}$ is an *intersecting* family of k-element subsets of $[n] = \{1, 2, \ldots, n\}$ (i.e. every two members of $\mathcal{F}$ have at least one element in common) and $n \geq 2k$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and this bound is attained. A similar result holds for t-intersecting k-element subsets (Wilson, [18]): if $n \geq (k-t+1)(t+1)$ and $\mathcal{F}$ is a t-intersecting family, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. Since k-subsets of $[n]$ can be considered as length-k chains in the (total) order $1 < 2 < \cdots < n$; using this terminology, the EKR theorem is a result about intersecting k-chains in a special partially ordered set.

Let B_n^c denote the inclusion poset of the sets $\{X \subset [n] : \ c \leq |X| \leq n-c\}$. A *$k$-chain* in B_n^c is an $\mathcal{L} = (L_1, L_2, \ldots, L_k)$ such that $L_i \subset L_{i+1}$, but $L_i \neq L_{i+1}$ for $i = 1, 2, \ldots, k-1$. Note that B_n^0 is the Boolean algebra, while B_n^1 is the truncated Boolean algebra where the empty set and the universe are eliminated. A family $\mathcal{F}$ of k-chains in B_n^c is *t-intersecting* if any two elements of $\mathcal{F}$ have at

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least t elements in common. A 1-intersecting family is simply called *intersecting.* In an earlier paper [6] we proved an EKR theorem for intersecting k-chains in B_n^c for $c = 0, 1$ and promised generalizations for general c and t-intersecting chains, and also a Hilton-Milner (HM) (see [13]) type theorem. We also promised a common generalization of the EKR theorem and our theorem. In this paper we deliver all these results.

The paper is organized as follows. Section 2 generalizes our earlier EKR result on intersecting k-chains for every c. The proof is based on the shifting technique [9], and in most parts it is applicable for general t as well. In Section 3 we prove EKR and HM theorems for t-intersecting chains in posets. The proofs utilize the kernel method [12] and therefore work from some threshold. These results likely have a number of applications, however, in this paper, we focus on B_n^c. Application of our general method to B_n^c are given in Section 4. It is worth mentioning that since the EKR theorem for t-intersecting chains does not hold for all possible n, k, c and t, the method of Section 2 alone is probably not appropriate to prove an exact EKR theorem for general t.

We give a brief account on the history of our problems. M. Simonovits and V. T. Sós proposed a research program on "structured intersection theorems" [16, 17], which has developed a fairly large literature. They investigated the maximum number of graphs on n vertices such that any two intersect in a prescribed graph, e.g. a path or cycle. The following problem fits into their scheme: given a graph G, what is the maximum number of pairwise intersecting complete k-subgraphs? In this paper we study the latter problem if G is the comparability graph of a poset.

P. L. Erdős, Faigle, and Kern [4] pointed out that certain results of Deza, Frankl [3, Thm. 5.8] and Frankl, Füredi [10] on intersecting sequences of integers may be interpreted as results on intersecting families of chains in some partially ordered sets. They posed the problem of finding the largest number of pairwise intersecting k-chains in the truncated Boolean algebra B_n^c. Füredi solved this problem first, using the kernel method, for $c = 0, 1$ and $n > 6k \log k$ (personal communication). We solved the problem for $c = 0, 1$ and every n [6]. Ahlswede and Cai [1] also solved the problem for $c = 0$.

We refer to two good surveys on EKR type theorems: Deza and Frankl [3], Frankl [9].

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2 Exact EKR theorem for intersecting chains

In this section we generalize our previous results in [6]: we prove an exact EKR theorem for intersecting k-chains in B_n^c for every meaningful value of n, c and k. Our earlier results covered only the cases $c = 0$ and 1. We give here a complete proof for $c \geq 1$. We think the results about shifting t-intersecting chains may be interesting for their own sake.

2.1 Shifting families of chains in B_n^c

In this subsection we introduce the shifting of chains, a tool that we need to prove tight EKR theorems for intersecting chains in B_n^c. This method can be also useful in studying t-intersecting chains, therefore we present it in generality exceeding our needs.

We reduce the EKR problem to the examination of so-called *compressed* sets of chains and prove that compressed sets of chains satisfy a strong intersection property. This subsection is a more or less straightforward generalization of shifting in B_n [6]. Let's start with some notations.

Definition. For $c \leq m_1 < m_2 < \cdots < m_t \leq n-c$, let $\mathcal{T}^c_{n,k}(m_1, m_2, \ldots, m_t)$ denote the set of those k-chains in B^c_n, which contain as elements the initial segments $[m_1],\ [m_2], \ldots, [m_t]$. (We say that $M \in B^c_n$ is an *initial segment* if $M = [m]$ for some $1 \leq m \leq n$ or $M = \emptyset$.) Set $T^c_{n,k}(m_1, m_2, \ldots, m_t) = |\mathcal{T}^c_{n,k}(m_1, m_2, \ldots, m_t)|$. Clearly $T^c_{n,k}(m_1, m_2, \ldots, m_t)$ is also the cardinality of the set of those k-chains in $B^c_{n,k}$ which contain *any* specified subchain of length t with specified sizes $m_1, m_2, \ldots, m_t$.

Let $\mathcal{F}$ be a family of pairwise t-intersecting k-chains from B^c_n and let $1 \leq i < j \leq n$ be integers. The (i,j) *chain-shift* $S_{ij}(\mathcal{F})$ of the family $\mathcal{F}$ is defined as follows.

For every k-chain $\mathcal{L} = (L_1, \ldots L_k) \in \mathcal{F}$, let $S_{ij}(\mathcal{L}) = (L'_1, \ldots, L'_k)$ where

$$L'_l = \begin{cases} L_l \setminus \{j\} \cup \{i\} & \text{if } j \in L_l \text{ and } i \notin L_l, \\ L_l & \text{otherwise.} \end{cases}$$

We say that L'_l is the *shift* of L_l. Shifting preserves set containment, so $S_{ij}(\mathcal{L})$ is a k-chain. The shifted family $S_{ij}(\mathcal{F})$ is obtained by the following rule: replace every k-chain $\mathcal{L} \in \mathcal{F}$ by $S_{ij}(\mathcal{L})$ if and only if (1) $S_{ij}(\mathcal{L}) \neq \mathcal{L}$ and (2) $S_{ij}(\mathcal{L}) \notin \mathcal{F}$.

It is clear from the definition that $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$. Moreover, shifting preserves the t-intersection property.

Lemma 2.1 *If $\mathcal{F}$ is a t-intersecting family of k-chains in B^c_n then $S_{ij}(\mathcal{F})$ is also t-intersecting.*

Proof. Let $\mathcal{L}_1, \mathcal{L}_2 \in S_{ij}(\mathcal{F})$; we have to prove that they contain t common elements. We distinguish three cases:

Case 1: $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{F}$. In this case it is obvious that $\mathcal{L}_1$ and $\mathcal{L}_2$ t-intersect.

Case 2: $\mathcal{L}_1, \mathcal{L}_2 \notin \mathcal{F}$. In this case, there are $\mathcal{L}_3, \mathcal{L}_4 \in \mathcal{F}$ such that $\mathcal{L}_1 = S_{ij}(\mathcal{L}_3)$ and $\mathcal{L}_2 = S_{ij}(\mathcal{L}_4)$. Let $\{M_1, M_2, \ldots, M_t\} \subset \mathcal{L}_3 \cap \mathcal{L}_4$. Then the shift of M_i (which may be M_i itself) is a common element of $\mathcal{L}_1$ and $\mathcal{L}_2$ for $i = 1, 2, \ldots, t$. Note that the shifts of the M_i's are distinct, since they make a t-chain which is shifted into a t-chain.

Case 3: $\mathcal{L}_1 \notin \mathcal{F}$ and $\mathcal{L}_2 \in \mathcal{F}$. Then let $\mathcal{L}_3 \in \mathcal{F}$ such that $\mathcal{L}_1 = S_{ij}(\mathcal{L}_3)$. There may be two reasons why $\mathcal{L}_2$ was not replaced. If $\mathcal{L}_2 = S_{ij}(\mathcal{L}_2)$ then let $\{M_1, M_2, \ldots, M_t\} \subset \mathcal{L}_2 \cap \mathcal{L}_3$. The shift of M_s $(s = 1, 2, \ldots, t)$ is itself (since $\mathcal{L}_2 = S_{ij}(\mathcal{L}_2)$) so $M_s \in \mathcal{L}_2 \cap S_{ij}(\mathcal{L}_3) = \mathcal{L}_2 \cap \mathcal{L}_1$ as well.

The other reason is that $\mathcal{L}_2 \neq S_{ij}(\mathcal{L}_2)$ but $S_{ij}(\mathcal{L}_2) \in \mathcal{F}$. In this subcase, let $\{M_1, M_2, \ldots, M_t\} \subset \mathcal{L}_3 \cap S_{ij}(\mathcal{L}_2)$. It is impossible that $j \in M_s$ and $i \notin M_s$ since M_s is the shift of some element of $\mathcal{L}_2$. Also, it is impossible that $i \in M_s$ and $j \notin M_s$ because there is some $K \in \mathcal{L}_3$ such that $j \in K$ and $i \notin K$ (because $S_{ij}(\mathcal{L}_3) \neq \mathcal{L}_3$) and one of K, M_s must contain the other. So M_s $(s = 1, 2, \ldots, t)$ is a set containing either both of i, j or neither of i, j. In either case, from $M_s \in S_{ij}(\mathcal{L}_2)$ we have $M_s \in \mathcal{L}_2$ so $M_s \in \mathcal{L}_1 \cap \mathcal{L}_2$. $\square$

We say that the family $\mathcal{F}$ of t-intersecting k-chains is *compressed* if $\mathcal{F}$ is invariant for all chain-shift operations S_{ij}, $1 \leq i < j \leq n$. By Lemma 2.1, for any intersecting family $\mathcal{F}$, repeated applications of chain-shifts result in a compressed family of the same size.

Compressed families satisfy a strong intersection property.

Lemma 2.2 *Let $\mathcal{F}$ be a compressed family of t-intersecting k-chains. Then for any $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{F}$, there are at least t initial segments in their intersection $\mathcal{L}_1 \cap \mathcal{L}_2$.*

Proof. Suppose that the lemma is not true and let $\mathcal{L}_1 \in \mathcal{F}$ be a minimal counterexample in the sense that

(i) there exists $\mathcal{L}_2 \in \mathcal{F}$ such that $\mathcal{L}_1 \cap \mathcal{L}_2$ contains fewer than t initial segments

(ii) $\sum_{L \in \mathcal{L}_1} \sum_{x \in L} x$ is minimal among all $\mathcal{L}_1$ satisfying (i).

Take a set $M \in \mathcal{L}_1 \cap \mathcal{L}_2$ which is not an initial segment. Since M is not an initial segment, there exist $1 \leq i < j \leq n$ such that $i \notin M$ and $j \in M$. Then $S_{ij}(\mathcal{L}_1) \neq \mathcal{L}_1$, so $S_{ij}(\mathcal{L}_1)$ is not a counterexample. Therefore, there exist t initial segments $\{K_1, K_2, \ldots, K_t\} \subset S_{ij}(\mathcal{L}_1) \cap \mathcal{L}_2$. It is impossible that $j \in K_s$ and $i \notin K_s$ since K_s is an initial segment ($s = 1, 2, \ldots, t$). Also, it is impossible that $i \in K_s$ and $j \notin K_s$ because $K_s, M \in \mathcal{L}_2$ and so one of them must contain the other. So K_s is a set containing both of i, j or neither of i, j. In either case $K_s \in \mathcal{L}_1$ ($s = 1, 2, \ldots, t$), and $\{K_1, K_2, \ldots, K_t\} \subset \mathcal{L}_1 \cap \mathcal{L}_2$, a contradiction. $\square$

2.2 Exact EKR theorems for B_n^c

We give a tight upper bound for the number of intersecting k-chains in B_n^c, which works for all n and c, using the shift method. This method, however, fails to give characterization of the extremes. The case $c = 0$ were already proved by shifting in our previous paper [6] and we omit it here. The proof of cases $c \geq 1$ is a generalization of the original shifting proof of the case $c = 1$, but it is described in the language of injections instead of estimates.

Recall that for $c \leq m \leq n-c$, $\mathcal{T}_{n,k}^c(m)$ denotes the set of those k-chains in B_n^c, which contain as element the initial segment $[m]$, and that $T_{n,k}^c(m) = |\mathcal{T}_{n,k}^c(m)|$. Clearly $T_{n,k}^c(m)$ is also the cardinality of the set of those k-chains in $B_{n,k}^c$ which contain *any* specified subchain of length 1 with specified sizes m.

We prove a slightly stronger result which is more appropriate for induction:

Theorem 2.1 *Let $c \geq 1$ and let $\mathcal{F}$ be a family of intersecting k-chains in B_n^c. Then $|\mathcal{F}| \leq T_{n,k}^c(c)$, and there is an injection $\phi : \mathcal{F} \to \mathcal{T}_{n,k}^c(c)$ such that every chain $\mathcal{L} = (L_1, L_2, \ldots, L_k) \in \mathcal{F}$ and its image $\phi(\mathcal{L}) = \mathcal{H} = (H_1, H_2, \ldots, H_k) \in \mathcal{T}_{n,k}^c(c)$ satisfy*

$$|L_k| \geq |H_k|. \tag{1}$$

Proof. We use induction on n and k. If $k = 1$ or $n = 2c$ then $|\mathcal{F}| \leq 1$ and it is trivial to check that the theorem holds. These simple facts are the base cases of the induction. Assume the hypothesis for $n' < n$ and $k' \leq k$, and also for $n' = n$ and $k' < k$. We may assume that $\mathcal{F}$, a family of intersecting k-chains in B_n^c, is already compressed. We distinguish two cases:

Case 1: For all $\mathcal{L} \in \mathcal{F}$, $n-c \notin L_1$. Define

$$\begin{aligned} \mathcal{F}_i &= \{\mathcal{L} \in \mathcal{F} : L_{i+1} \setminus L_i = \{n-c\}\}, \qquad (i = 1, 2, \ldots, k-1) &(2)\\ \mathcal{F}_k &= \{\mathcal{L} \in \mathcal{F} : |L_k| = n-c \text{ and } n-c \notin L_k\}, &(3)\\ \mathcal{F}_0 &= \mathcal{F} - \bigcup_{j=1}^{k} \mathcal{F}_j. &(4)\end{aligned}$$

We use the shorthand notation $\mathcal{I}(n,k) = \mathcal{T}^c_{n,k}(c)$. Similarly define

$$\begin{aligned} \mathcal{I}(n,k)_i &= \{\mathcal{H} \in \mathcal{I}(n,k) : H_{i+1} \setminus H_i = \{n-c\}\}, \qquad (i = 1, 2, \ldots, k-1)\\ \mathcal{I}(n,k)_k &= \{\mathcal{H} \in \mathcal{I}(n,k) : |H_k| = n-c \text{ and } n-c \notin H_k\},\\ \mathcal{I}(n,k)_0 &= \mathcal{I}(n,k) - \bigcup_{j=1}^{k} \mathcal{I}(n,k)_j.\end{aligned}$$

Deleting $n-c$ from each element of each chain of $\mathcal{F}_0$ we obtain a family $\mathcal{F}_0'$ of intersecting k-chains in B^c_{n-1} on the underlying set $\hat{[n]} = [n] \setminus \{n-c\}$. We obtain the family $\mathcal{I}(n,k)_0'$ similarly. Now it is clear that $\mathcal{I}(n,k)_0'$ coincides with $\mathcal{I}(n-1,k)$ on the underlying set $\hat{[n]}$. Applying our inductive hypothesis, there exists an injection $\phi_0' : \mathcal{F}_0' \to \mathcal{I}(n,k)_0'$ not increasing the size of the k^{th} elements of the chains.

This injection can be lifted to a suitable injection $\phi_0 : \mathcal{F}_0 \to \mathcal{I}(n,k)_0$ the following way. Assume that L_j is the first element of $\mathcal{L} \in \mathcal{F}_0$ which contains the number $(n-c)$ ($j \in \{2, \ldots, k\}$, or such a j does not exist at all). Assume that the deletion of $(n-c)$ turns $\mathcal{L}$ into $\mathcal{L}' \in \mathcal{F}_0'$. If $\phi_0'(\mathcal{L}') = (H_1', H_2', \ldots, H_k')$, then define

$$\phi_0(\mathcal{L}) = (H_1', \ldots, H_{j-1}', H_j' \cup \{n-c\}, \ldots, H_k' \cup \{n-c\}),$$

if such a j existed, and $\phi_0(\mathcal{L}) = \phi_0'(\mathcal{L}')$ otherwise. Now the inequality (1) obviously holds for the map ϕ_0.

Deleting $n-c$ from every set in every chain in $\mathcal{F}_i$ for $i = 1, 2, \ldots, k-1$, we obtain a family $\mathcal{F}_i'$ of intersecting $(k-1)$-chains in B^c_{n-1} on the underlying set $\hat{[n]}$. We similarly obtain $\mathcal{I}(n,k)_i'$, which coincides with $\mathcal{I}(n-1,k-1)_i$ for every i. By the inductive hypothesis, there exist injections $\phi_i' : \mathcal{F}_i' \to \mathcal{I}(n,k)_i'$ with property (1). These injections can be lifted to suitable injections ϕ_i ($i = 1, 2, \ldots, k-1$) from $\mathcal{F}_i$ into $\mathcal{I}(n,k)_i$ the following way. We know that L_{i+1} is the first element of $\mathcal{L} \in \mathcal{F}_i$ which contains the number $(n-c)$. Assume that the deletion of $(n-c)$ turns $\mathcal{L}$ into $\mathcal{L}' \in \mathcal{F}_i'$. If $\phi_i'(\mathcal{L}') = (H_1', H_2', \ldots, H_{k-1}')$, then define

$$\phi_i(\mathcal{L}) = (H_1', \ldots, H_i', H_i' \cup \{n-c\}, \ldots, H_{k-1}' \cup \{n-c\}).$$

Now the inequality (1) obviously holds for the map ϕ_i.

Finally, define the family of chains $\mathcal{F}_k'$ by deleting the largest set L_k from every chain in $\mathcal{F}_k$. (Remember that for all $\mathcal{L} \in \mathcal{F}_k$, $|L_k| = n-c$.) We obtain the family $\mathcal{I}_k(n,k)'$ similarly, by deleting the k^{th} element of every chain in $\mathcal{I}_k(n,k)$. Observe that $\mathcal{F}_k'$ is a family of intersecting $(k-1)$-chains in B^c_{n-1} on the underlying set $\hat{[n]}$, since the sets that we dropped are not initial segments in the original underlying set $[n]$. Furthermore, $\mathcal{I}(n,k)_k'$ coincides with $\mathcal{I}(n-1,k-1)$

on the underlying set $\hat{[n]}$. Therefore, by hypothesis, there exists an injection $\phi_k^* : \mathcal{F}_k' \to \mathcal{I}(n,k)_k' = \mathcal{I}(n-1,k-1)$ satisfying inequality (1).

Now we lift ϕ_k^* into a suitable $\phi_k : \mathcal{F}_k \to \mathcal{I}(n,k)_k$ by a greedy procedure. By the inductive hypothesis, we have a map $\phi_k^* : \mathcal{F}_k' \to \mathcal{I}(n,k)_k'$, which for every $\mathcal{L}' \in \mathcal{F}_k'$ assigns a $\phi_k^*(\mathcal{L}') = \mathcal{H} \in \mathcal{I}(n,k)_k'$, such that $|L_{k-1}| \geq |H_{k-1}|$. We want to define an injection $\phi_k : \mathcal{F}_k \to \mathcal{I}(n,k)_k$ such that $(\phi_k(\mathcal{L}))' = \phi_k^*(\mathcal{L}')$. Such a definition is possible if any $\mathcal{L}'$ has at most as many pre-images under $'$ than $\phi_k^*(\mathcal{L}')$. This is the case, since the number of pre-images of $\mathcal{L}'$ under $'$ is at most

$$\binom{n-1-|L_{k-1}|}{n-c-|L_{k-1}|}, \tag{5}$$

and the number of pre-images of $\phi_k^*(\mathcal{L}')$ under $'$ is exactly

$$\binom{n-1-|H_{k-1}|}{n-c-|H_{k-1}|}. \tag{6}$$

It is easy to see, that (6) is at least as big as (5), since $|L_{k-1}| \geq |H_{k-1}|$.

Finally, $\phi = \phi_0 \cup (\bigcup_{i=1}^k \phi_i)$ is an appropriate injection from $\mathcal{F}$ into $T_{n,k}^c(c)$, satisfying (1). The reason is that the ϕ_i's were such injections by construction, and their ranges are disjoint.

Case 2: There exists a chain $\mathcal{L} \in \mathcal{F}$ such that $n-c \in L_1$. We claim that $L_k = [n-c]$ and this is the only initial segment in $\mathcal{L}$. Since each L_i contains $n-c$ and $|L_i| \leq n-c$, $[n-c]$ is the only initial segment which may occur in $\mathcal{L}$. On the other hand, by Lemma 2.2, $\mathcal{L}$ contains at least one initial segment, proving our claim. Note that the inequality

$$|\mathcal{F}| \leq T_{n,k}^c(c) \tag{7}$$

is sufficient to finish the proof of Case 2 since, by Lemma 2.2, for every $\mathcal{L}' \in \mathcal{F}$, $[n-c] \in \mathcal{L}'$. Therefore any injection $\phi : \mathcal{F} \to \mathcal{T}_{n,k}^c(c)$ would be suitable. But this inequality clearly holds since the dual of $\mathcal{F}$ is a subset of $\mathcal{T}_{n,k}^c(c)$.

□

3 Results for chains of posets

In this section we prove EKR and HM theorems for chains of posets. The basic technique is the kernel method introduced by Hajnal and Rothschild [12]. The limitation of this method is that it works just from some threshold.

3.1 Review of sunflowers

In this subsection we review facts about sunflowers that we use in the kernel method. A set system $\{A_1, A_2, \ldots, A_m\}$ is called a *sunflower* or *delta-system*, if $A_i \cap A_j = \bigcap_{l=1}^m A_l$ for all $1 \leq i < j \leq m$. The sets A_i are called the *petals* and $\bigcap_{l=1}^m A_l$ is called the *kernel* of the sunflower.

We say that a set system is of *rank* k, if $|H| \leq k$ for all $H \in \mathcal{H}$; and $\mathcal{H}$ is t-*intersecting*, if $|H_1 \cap H_2| \geq t$ for all $H_1, H_2 \in \mathcal{H}$. For $t \geq 1$, we say that $\mathcal{H}$ is *non-trivially* t-intersecting, if it is t-intersecting, and $|\bigcap \mathcal{H}| < t$. We say that

$\mathcal{H}$ is *critically* t-intersecting, if it is t-intersecting, and deleting any $x \in H$ from any $H \in \mathcal{H}$, the resulting set system $\mathcal{H} \setminus \{H\} \cup \{H \setminus \{x\}\}$ is not t-intersecting.

Estimates in the kernel method are usually based on the following simple observation.

Lemma 3.1 *Let $\mathcal{H}$ be a critically t-intersecting system ($t \geq 1$) of rank k. Then $\mathcal{H}$ does not contain a sunflower with $k+1$ petals.*

Proof. Indeed, if $\{H_1, H_2, \ldots, H_{k+1}\}$ is a sunflower in $\mathcal{H}$, then any $H \in \mathcal{H}$ must intersect the kernel K of the sunflower in at least t elements, since a $\leq k$-element set cannot intersect each of the $k+1$ disjoint sets $H_1 \setminus K$, $H_2 \setminus K$,..., $H_{k+1} \setminus K$. Hence the deletion of $H_1 \setminus K$ from H_1 (if $H_1 \neq K$) results a t-intersecting set system, contradicting the minimality of $\mathcal{H}$. $\square$

We will also need the Erdős-Rado theorem [7]:

Lemma 3.2 *For every i and l, there exists a number $f(i,l)$, such that any family of $f(i,l)$ sets of size i each, contains a sunflower with l petals.* $\square$

3.2 EKR and HM theorems for chains in posets

Throughout Subsection 3.2, let us be given a fixed k and a sequence of posets P_n. A *t-chain* $\mathcal{L}$ in P_n is a strict chain of elements $\mathcal{L} = (x_1 < x_2 < \cdots < x_t)$. For a given t-chain $\mathcal{L} = (x_1 < x_2 < \cdots < x_t)$, let $\mathcal{T}_{n,k}(x_1, x_2, \ldots, x_t)$ denote the set of k-chains in P_n which contain $\mathcal{L}$ as a subset. Define $T_{n,k}(x_1, x_2, \ldots, x_t) = |\mathcal{T}_{n,k}(x_1, x_2, \ldots, x_t)|$. Sometimes we write T instead of $T_{n,k}$, when it does not cause ambiguity. Also define $r_t(n) = \max T_{n,k}(x_1, x_2, \ldots, x_t)$, where the maximum is taken for t-chains $x_1 < x_2 < \cdots < x_t$ in P_n. It follows from the definition that

$$r_i(n) \geq r_{i+1}(n). \tag{8}$$

A family $\mathcal{F}$ of k-chains in P_n is *t-intersecting*, if any two k-chains of $\mathcal{F}$ share at least t elements of the poset.

Theorem 3.1 *For fixed $1 \leq t < k$, and a sequence of posets P_n, let us be given a family $\mathcal{F}_n$ of t-intersecting k-chains in P_n. Assume that*

$$\lim_{n \to \infty} r_{t+1}(n)/r_t(n) = 0. \tag{9}$$

Then, for n sufficiently large, $|\mathcal{F}_n| \leq r_t(n)$, and equality implies that the elements of $\mathcal{F}_n$ share a t-subchain.

Proof. Let us be given a family $\mathcal{F}_n$ of t-intersecting k-chains. We reduce $\mathcal{F}_n$ to a critically t-intersecting family $\mathcal{H}$ as follows: we repeatedly delete an element x_i of a chain $\mathcal{L}$ if the chain $\mathcal{L} \setminus \{x_i\}$ still intersects all other chains in at least t elements. We neglect the possible multiplicities with which chains arise. We write $\mathcal{H} = \mathcal{H}_t \cup \mathcal{H}_{t+1} \cup \cdots \cup \mathcal{H}_k$, where $\mathcal{H}_i$ contains the i-element chains from $\mathcal{H}$. If $\mathcal{H}_t \neq \emptyset$, then $\mathcal{H} = \mathcal{H}_t$, and $\mathcal{H}_t$ is contained by every $\mathcal{L} \in \mathcal{F}_n$, hence $|\mathcal{F}_n| \leq r_t(n)$.

If $\mathcal{H}_t = \emptyset$, we argue the following way. By Lemmas 3.1 and 3.2 we have $|\mathcal{H}_i| \leq f(i, k+1)$, and $\mathcal{F}$ has at most $r_i(n)$ chains containing any element of $\mathcal{H}_i$.

Combining these observations with (8) and (9), we obtain $|\mathcal{F}_n| \leq \sum_{i=t+1}^{k} f(i,k+1)r_i(n) = O(r_{t+1}(n)) = o(r_t(n))$. $\square$

Note that for the poset $1 < 2 < \cdots < n$, we have $r_t(n) = \binom{n-t}{k-t}$. We get back the original EKR theorem for n large enough, with the kernel method proof [12].

For a t-chain $\mathcal{X} \subset P_n$ and $y \notin \mathcal{X}$, let $T(\mathcal{X}, y)$ denote the number of k-chains which contain X and y. For a t-chain $\mathcal{X}$ and a k-chain $\mathcal{L}$ in P_n, such that $|\mathcal{X} \cup \mathcal{L}| = k+1$, let $y^*_{\mathcal{L}} \in \mathcal{L} \setminus \mathcal{X}$ such that $T(\mathcal{X}, y^*_{\mathcal{L}})$ minimize $T(\mathcal{X}, y_{\mathcal{L}})$ for the elements $y \in \mathcal{L} \setminus \mathcal{X}$, and set

$$\tau(\mathcal{X}, \mathcal{L}) = \sum_{y \in \mathcal{L} \setminus \mathcal{X},\ y \neq y^*_{\mathcal{L}}} T(\mathcal{X}, y). \tag{10}$$

Also define

$$M_\tau(n) = \max_{\mathcal{X}, \mathcal{L}} \ \tau(\mathcal{X}, \mathcal{L}), \tag{11}$$

and

$$M^*_\tau(n) = \max_{\substack{\mathcal{X}, \mathcal{L}: \\ \tau(\mathcal{X}, \mathcal{L}) = M_\tau(n)}} T(\mathcal{X}, y^*_{\mathcal{L}}). \tag{12}$$

Now the following Hilton-Milner type theorem holds:

Theorem 3.2 *For fixed $1 \leq t < k$, and a sequence of posets P_n, let us be given a maximum sized family $\mathcal{F}_n$ of non-trivially t-intersecting k-chains in P_n. Assume further that*

$$\lim_{n \to \infty} r_{t+2}(n)/M^*_\tau(n) = 0. \tag{13}$$

Then, for n sufficiently large, $\mathcal{F}_n$ has one of the following two descriptions:

(i) there exists a t-chain $\mathcal{X}$ and a $(k+1-t)$-chain $\mathcal{Y}$, such that $\mathcal{X} \cap \mathcal{Y} = \emptyset$; and $\mathcal{F}_n$ is the following set of k-chains:

$$\{\mathcal{L} :\ \mathcal{X} \subseteq \mathcal{L} \text{ and } \mathcal{L} \cap \mathcal{Y} \neq \emptyset\} \cup \{\mathcal{L} :\ \mathcal{Y} \subseteq \mathcal{L} \text{ and } |\mathcal{L} \cap \mathcal{X}| = t-1\}, \tag{14}$$

where the second set of chains is non-empty;

(ii) there exists a $(t+2)$-chain $\mathcal{Z}$, and $\mathcal{F}_n$ is the following set of k-chains:

$$\{\mathcal{L} :\ |\mathcal{L} \cap \mathcal{Z}| \geq t+1\}, \tag{15}$$

and $|\bigcap_{\mathcal{L} \in \mathcal{F}_n} \mathcal{L} \cap \mathcal{Z}| \leq t-1$.

Proof. We reduce $\mathcal{F}_n$ to a critically t-intersecting family $\mathcal{H} = \mathcal{H}_t \cup \mathcal{H}_{t+1} \cup \cdots \cup \mathcal{H}_k$ as we did in the proof of Theorem 3.1. Note that $\mathcal{H}_t = \emptyset$ by assumption.

First we observe that $\mathcal{X}$ and $\mathcal{L}$ that defines $M^*_\tau(n)$ in (12), provides for a feasible $\mathcal{X}$ and $\mathcal{Y} = \mathcal{L} \setminus \mathcal{X}$ in (14), yielding a construction for non-trivially t-intersecting k-chains. Inclusion-exclusion shows that this construction has at least

$$M_\tau(n) + M^*_\tau(n) - O(r_{t+2}(n)) \tag{16}$$

k-chains, and so has the maximum sized $\mathcal{F}_n$.

We partition $\mathcal{F}_n = \mathcal{F}' \cup \mathcal{F}''$, where

$$\begin{aligned} \mathcal{F}' &= \{\mathcal{L} \in \mathcal{F} :\ \exists H \in \mathcal{H}_i \text{ such that } H \subset \mathcal{L} \text{ for some } i \geq t+2\}, \\ \mathcal{F}'' &= \mathcal{F}_n \setminus \mathcal{F}'. \end{aligned}$$

First we estimate $|\mathcal{F}'|$. By Lemmas 3.1 and 3.2 we have $|\mathcal{H}_i| \le f(i,k+1)$, and $\mathcal{F}_n$ has at most $r_i(n)$ chains containing any element of $\mathcal{H}_i$. Using (8),

$$|\mathcal{F}'| \le \sum_{i=t+2}^{k} f(i,k+1) r_i(n) = O(r_{t+2}(n)). \tag{17}$$

We distinguish cases.
Case 1: Either $|\mathcal{H}_{t+1}| = 2$ or there exist $H_1,\ H_2,\ H_3 \in \mathcal{H}_{t+1}$, such that $|H_1 \cap H_2 \cap H_3| = t$.
It easily follows that $\mathcal{H}_{t+1}$ is a sunflower with kernel $\mathcal{X} = \bigcap \mathcal{H}_{t+1}$ of size t.

We have $|\mathcal{H}_{t+1}| \le k+1-t$, otherwise for every $\mathcal{L} \in \mathcal{F}_n$, $\mathcal{X} \subset \mathcal{L}$, contradicting our assumption.

If $|\mathcal{H}_{t+1}| = k+1-t$, then any $\mathcal{L} \in \mathcal{F}_n$ not containing $\mathcal{X}$, contains $\mathcal{Y} = \bigcup \mathcal{H}_{t+1} \setminus \bigcap \mathcal{H}_{t+1}$. It is easy to see that $\mathcal{H}_{t+2} = \cdots = \mathcal{H}_{k-1} = \emptyset$, and $H \in \mathcal{H}_k$ implies $|H \cap \mathcal{X}| = t-1$ and $|H \cap \mathcal{Y}| = k+1-t$. $\mathcal{X}$ and $\mathcal{Y}$ are chains, since they are contained in some k-chains, and $\mathcal{X} \cap \mathcal{Y} = \emptyset$ (by definition). We are in the situation described in Part (i) of the Theorem.

If $|\mathcal{H}_{t+1}| = l < k+1-t$, then any $\mathcal{L} \in \mathcal{F}_n$ not containing $\mathcal{X}$, contains $\{y_1, \ldots, y_l\} = \bigcup \mathcal{H}_{t+1} \setminus \bigcap \mathcal{H}_{t+1}$. Also $|\mathcal{L} \cap \mathcal{X}| \le t-1$, and hence equal to $t-1$, since otherwise $\mathcal{L}$ cannot t-intersect the members of $\mathcal{H}_{t+1}$. Hence $\mathcal{X}$ and $\mathcal{L}$ are chains, and $|\mathcal{X} \cup \mathcal{L}| = k+1$. Therefore $\mathcal{X}$ and $\mathcal{L}$ were considered in the definition of $M_\tau(n)$ in (11). Using (10), (11), and (17), respectively, we have

$$\begin{aligned} |\mathcal{F}''| &\le \sum_{i=1}^{l} T(\mathcal{X}, y_i) \le M_\tau(n), \\ |\mathcal{F}'| &= O(r_{t+2}(n)). \end{aligned}$$

Our $\mathcal{F}_n$ has at most $|\mathcal{F}'| + |\mathcal{F}''| \le M_\tau(n) + O(r_{t+2}(n))$ k-chains, and hence is short of optimum by (16) and (13).
Case 2: $|\mathcal{H}_{t+1}| \ge 3$ and for all distinct $H_1,\ H_2,\ H_3 \in \mathcal{H}_{t+1}$, we have $|H_1 \cap H_2 \cap H_3| < t$.
We fix $H_1,\ H_2,\ H_3 \in \mathcal{H}_{t+1}$. It is not difficult to see that $|H_1 \cap H_2 \cap H_3| = t-1$ and $|H_1 \cup H_2 \cup H_3| = t+2$. We show that the choice $\mathcal{Z} = H_1 \cup H_2 \cup H_3$ is appropriate to exhibit that we are in Part (ii) of the Theorem. For any $H \in \mathcal{H}$, $|H \cap \mathcal{Z}| \ge t+1$, otherwise H cannot intersect all of H_1, H_2, H_3 in at least t elements. We use that $\mathcal{H}$ is critically t-intersecting. Assume that for some $H' \in \mathcal{H}$, $H' \setminus \mathcal{Z} \ne \emptyset$. Then H' can be changed to $H'' = \mathcal{Z} \cap H'$, keeping the t-intersection property, and contradicting the criticality. Similarly, assume that for some $H' \in \mathcal{H}$, $|H' \cap \mathcal{Z}| = t+2$. Then any element of H' can be deleted, keeping the t-intersection property, and contradicting the criticality. The last claim to prove is that $\mathcal{Z}$ is a chain. Note that H_1, H_2, H_3 were chains, and any two elements of $\mathcal{Z}$ are contained by some H_i.
Case 3: $\mathcal{H}_{t+1} = \emptyset$.
In this case $\mathcal{F}_n = \mathcal{F}'$, and its size is estimated by (17). By (16) and (13), the optimal choice for Part (i) in the theorem beats this size for n large enough.
Case 4: $|\mathcal{H}_{t+1}| = 1$.
Then $\mathcal{H}_{t+1} = \{H\}$. For any $y_1 \in H$, define $\mathcal{X} = H \setminus \{y_1\}$. From here the situation is identical with the $|\mathcal{H}_{t+1}| = l < k+1-t$ subcase of Case 1. ◻

Note that for the poset $1 < 2 < \cdots < n$, we have $r_{t+2}(n) = \binom{n-t-2}{k-t-2}$, $T(\mathcal{X}, y) = \binom{n-t-1}{k-t-1}$, $\tau(\mathcal{X}, \mathcal{L}) = (k-t)\binom{n-t-1}{k-t-1}$, and $M^*_\tau(n) = \binom{n-t-1}{k-t-1}$, and

(13) holds. Therefore we get back the old t-intersecting Hilton-Milner theorem (Hilton and Milner, [13] for $t = 1$ and Frankl [8]), for n large enough.

We also remark, that the previous proof may generalize for a much more general situation: it uses only the fact, that chains with length at most k in a poset form a down-ideal. Therefore Theorem 3.2 has close connection to the Chvátal conjecture (see, for example, Miklós, [15]). We shall return to this issue in a forthcoming paper.

4 t-intersecting chains in B_n^c

In this section we apply the general results we just proved for EKR and Hilton-Milner type theorems to t-intersecting k-chains in B_n^c for n large enough.

We show an example below to point out that the t-intersecting EKR theorem in B_n^c will not hold for all values of n, k, t, i.e. the largest family is not all chains containing a particular t-chain. This is much like the case of the ordinary t-intersecting EKR theorem, and therefore we may expect a t-intersecting EKR theorem in B_n^c for large values of n only. Hence we may not expect the use of shifting and have to use the kernel method.

Take a system $\mathcal{F}$ of $(n-3)$-intersecting $(n-1)$-chains (i.e. maximal chains) in B_n^1. We have $|\mathcal{T}^1_{n,n-3}(1, 2, \ldots, n-3)| = O(1)$. On the other hand, if $\mathcal{F}$ contains the chain $([1], [2], \ldots, [n-1])$ and the chains $([1], [2], \ldots, [i-1], [i-1] \cup \{i+1\}, [i+1], \ldots, [n-1])$ for all $i = 1, 2, \ldots, n-1$, then $\mathcal{F}$ is $(n-3)$-intersecting and $|\mathcal{F}| = n$.

4.1 Technicalities on Stirling numbers and B_n^c

This subsection will characterize which t-chains in B_n^c are contained by the largest number of k-chains. This characterization in Theorem 4.1 will be obtained from a sequence of lemmas.

For a chain $\mathcal{L} = (L_1, L_2, \ldots, L_k)$ in B_n^c, we define its *dual* by $\mathcal{L}^* = ([n] \setminus L_k, \ldots, [n] \setminus L_1)$. For a family of k-chains $\mathcal{F}$, define $\mathcal{F}^* = \{\mathcal{L}^* : \mathcal{L} \in \mathcal{F}\}$. If $\mathcal{F}$ was t-intersecting, then so is $\mathcal{F}^*$. Clearly, $|\mathcal{T}^c_{n,k}(m_1, m_2, \ldots, m_t)^*| = |\mathcal{T}^c_{n,k}(n - m_t, n - m_{t-1}, \ldots, n - m_1)|$. Hence we have

Lemma 4.1 $T^c_{n,k}(m_1, m_2, \ldots, m_t) = T^c_{n,k}(n - m_t, n - m_{t-1}, \ldots, n - m_1)$. $\square$

Lemma 4.2 *Observe that there is a bijection*

$$\mathcal{T}^c_{n,k}(m_1, m_2, \ldots, m_1 + t - 2, m_t) \longleftrightarrow \mathcal{T}^c_{n-t+2,k-t+2}(m_1, m_t - t + 2).$$

Proof. $(L_1, \ldots, L_k) \mapsto (L_1 \setminus [m_1 + 1, m_1 + t - 2], \ldots, L_k \setminus [m_1 + 1, m_1 + t - 2])$, and the chain becomes shorter by $(t - 2)$. $\square$

Each chain $\mathcal{L} = (L_1, \ldots, L_k)$ defines an ordered partition $[n] = L_1 \cup (L_2 \setminus L_1) \cup \cdots \cup (L_k \setminus L_{k-1}) \cup ([n] \setminus L_k)$. For $c \geq 1$, all parts are non-empty, and $\mathcal{L}$ corresponds to a surjection from $[n]$ to $[k + 1]$. This hints that we have to deal with Stirling numbers of the second kind. Let $S(n, k)$ denote the Stirling number of the second kind. We need the basic recurrence

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k). \tag{18}$$

The following results are easy exercises:

$$T^1_{n,k} = (k+1)!S(n,k+1), \tag{19}$$

where $T^1_{n,k}$ denotes the number of all k-chains in B^1_n; and for $t \geq 2$ using (18) we obtain

$$T^0_{n,k}(0,1,\ldots,t-2,n) = (k-t+1)!S(n-t+2,k-t+1) \tag{20}$$

$$T^0_{n,k}(n) = (k-1)!S(n,k-1) + k!S(n,k) = (k-1)!S(n+1,k). \tag{21}$$

We slightly generalize the notation $\mathcal{T}^c_{n,k}$ and $T^c_{n,k}$ to $\mathcal{T}^{c_1,c_2}_{n,k}$ and $T^{c_1,c_2}_{n,k}$, by allowing chains whose smallest element is at least c_1 and whose largest element is at most $n-c_2$ by size.

Lemma 4.3 *For all n,k, $c_1 \leq c_2$, and $c_1 \leq m \leq n-c_2$, we have $T^{c_1,c_2}_{n,k}(m) \leq T^{c_1,c_2}_{n,k}(c_1)$.*

Proof. Let us be given any sequence $c_1 = l_1 < l_2 < \cdots < l_i \leq m < l_{i+1} < \cdots < l_k \leq n-c_2$. We claim that the number of chains in $\mathcal{T}^{c_1,c_2}_{n,k}(c_1)$ with $|L_j| = l_j$ is at least as large as the number of chains in $\mathcal{T}^{c_1,c_2}_{n,k}(m)$ with $|L'_j| = l_j + m - l_i$ for $j \leq i$ and $|L'_j| = l_j$ for $j \geq i+1$. Routine calculations show that the number of the first type of chains is

$$\frac{(n-c_1)!}{(l_2-l_1)!(l_3-l_2)!\cdots(l_k-l_{k-1})!(n-l_k)!,}$$

and the number of chains of the second type is

$$\frac{m!(n-m)!}{(l_1+m-l_i)!(l_2-l_1)!\cdots(l_i-l_{i-1})!(l_{i+1}-m)!(l_{i+2}-l_{i+1})!\cdots(l_k-l_{k-1})!(n-l_k)!}.$$

Hence our claim boils down to proving

$$(n-c_1)!(c_1+m-l_i)!(l_{i+1}-m)! \geq m!(n-m)!(l_{i+1}-l_i)!. \tag{22}$$

Either $c_1 + m \geq l_{i+1}$ or $c_1 + m < l_{i+1}$. In the first case (22) is equivalent to

$$\binom{n}{m}\binom{c_1+m-l_i}{m-l_i} \geq \binom{n}{c_1}\binom{l_{i+1}-l_i}{m-l_i}.$$

This inequality holds termwise. In the other case (22) is equivalent to

$$(n-c_1)_{n-c_1-m} \geq (n-m)_{n-l_{i+1}}(l_{i+1}-l_i)_{l_{i+1}-c_1-m}.$$

It is easy to see that $(n-c_1)_{n-c_1-m} = (n-c_1)_{n-l_{i+1}}(l_{i+1}-c_1)_{l_{i+1}-c_1-m}$. Finally, $n-c_1 \geq n-m$ and therefore $(n-c_1)_{n-l_{i+1}} \geq (n-m)_{n-l_{i+1}}$; $l_{i+1}-c_1 \geq l_{i+1}-l_i$, and therefore $(l_{i+1}-c_1)_{l_{i+1}-c_1-m} \geq (l_{i+1}-l_i)_{l_{i+1}-c_1-m}$.

Using the claim we may partition $\mathcal{T}^{c_1,c_2}_{n,k}(c_1)$ and $\mathcal{T}^{c_1,c_2}_{n,k}(m)$ such that we have bijection between the sets of classes and the classes in $\mathcal{T}^{c_1,c_2}_{n,k}(c_1)$ are at least as big as the corresponding classes. $\square$

Lemma 4.4 *Assume $2 \leq t \leq k-1$. For $c \geq 2$*

$$T^c_{n,k}(c, c+1, \dots, c+t-2, n-c) \leq T^c_{n,k}(c, c+1, \dots, c+t-1),$$

for $c = 1$ equality holds, and for $c = 0$ the inequality turns over.

Proof. Assume $c \geq 2$. Observe that the RHS counts surjections from $[n-c-t+1]$ to $[k-t+1]$, such that the size of the pre-image of $(k-t+1)$ is at least c. The LHS counts surjections from $[n-2c-t+2]$ to $[k-t+1]$. There is an injection from the second set of surjections into the first set of surjections: extend the function to the points $n-2c-t+3, n-2c-t+4, \dots, n-c-t+1$ with value $k-t+1$.

For $c = 1$ the two sets of surjections described above are identical.

For $c = 0$ the RHS is $S(n-t+1, k-t) \cdot (k-t)! + S(n-t+1, k-t+1) \cdot (k-t+1)!$, the LHS is $S(n-t+2, k-t+1) \cdot (k-t+1)!$. Due to the recurrence $S(n-t+2, k-t+1) = S(n-t+1, k-t) + (k-t+1)S(n-t+1, k-t+1)$ we have the inequality claimed. $\square$

Theorem 4.1 *For all $c \geq 1$, n, k, c, t and $m_1, \dots, m_t$ holds*

$$T^c_{n,k}(m_1, m_2, \dots, m_t) \leq T^c_{n,k}(c, c+1, \dots, c+t-1).$$

For $1 \leq t < k$ and $n \geq k + 2c - 1$ equality holds if and only if $m_i = c+i-1$ for $i = 1, 2, \dots, t$ or $m_i = n-c-t+i$ for $i = 1, 2, \dots, t$, or $c = 1$, $t \geq 2$ and $m_i = c+i-1$ for $i = 1, 2, \dots, j$ and $m_i = n-c-t+i$ for $i = j+1, \dots, t$ for some j.

For $c = 0$, the case $t \leq 1$ is like above. For $c = 0$, for all $t \geq 2$, n, k and $m_1, \dots, m_t$ holds

$$T^0_{n,k}(m_1, m_2, \dots, m_t) \leq T^0_{n,k}(0, 1, \dots, t-2, n).$$

For $2 \leq t < k$ and $n \geq k+1$ equality holds if and only if $m_i = c+i-1$ for $i = 1, 2, \dots, j$ and $m_i = n-c-t+i$ for $i = j+1, \dots, t$ for some $1 \leq j < t$.

Proof. We focus on the proof of the inequality and leave the characterization of equalities as an exercise to the Reader. The dual extremities are explained by Lemma 4.1.

Given a sequence $m_1, m_2, \dots, m_t$, using Lemma 4.3 with $c_1 = c_2 = 1$ shows that $T^c_{n,k}(m_1, m_2, \dots, m_t)$ does not decrease when we change m_i for $m_{i-1}+1$ or $m_{i+1}-1$ for any $i = 2, 3, \dots, t-1$. Iterating this estimate yields

$$T^c_{n,k}(m_1, m_2, \dots, m_t) \leq T^c_{n,k}(m_1, m_1+1, \dots, m_1+t-2, m_t).$$

Lemma 4.2 yields that the latter is equal to $T^c_{n-t+2,k-t+2}(m_1, m_t-t+2)$. Another application of Lemma 4.3 shows that $T^c_{n-t+2,k-t+2}(m_1, m_t-t+2)$ is maximized if m_1, m_t-t+2 are the smallest and largest set sizes allowed to be in a chain, or are the smallest two or largest two consecutive set sizes allowed to be in a chain. Transforming this result back by Lemma 4.2, we see that the maximum size arising in one of the expressions compared in Lemma 4.1. Hence Lemma 4.1 finishes the proof. $\square$

4.2 Asymptotic results for B_n^c

In this subsection we prove asymptotic EKR and Hilton-Milner theorems for B_n^c.
For fixed k, we have the following asymptotics ([2], p. 293):

$$S(n,k) \sim k^n/k!. \tag{23}$$

Using (23), it is easy to prove the following lemma:

Lemma 4.5 *Assume that k is fixed and $c = O(n/\log n)$. Then almost all, i.e. $(1-o(1))S(n,k)k!$ ordered partitions of n elements into k non-empty parts have the property, that all classes have sizes at least c.*

Proof. The number of ordered partitions not having the required property is at most

$$k\sum_{i=1}^{c-1}\binom{n}{i}S(n-i,k-1)(k-1)!. \tag{24}$$

In order to estimate (24), note that $S(n-i,k-1)(k-1)! = O((k-1)^n)$ and that $\binom{n}{1} + ... + \binom{n}{c-1} < \binom{n+c}{c-1}$ by the identity $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}$. In view of (23), one has to verify that $\binom{n+c}{c-1}(k-1)^n = o(k^n)$. We estimate $\binom{n+c}{c-1}$ by $(\frac{(n+c)e}{c-1})^{c-1}$. Since $(\frac{(n+c)e}{c-1})^{c-1}$ is an increasing function in c, we just have to check $(\frac{(n+c)e}{c-1})^{c-1}(k-1)^n = o(k^n)$ for $c = \frac{c'n}{\log n}$, which is an easy exercise. $\square$

Given a sequence $c \le m_1 < m_2 < \cdots < m_t \le n-c$, let $g_1 \ge g_2 \ge \cdots \ge g_{t+1}$ denote the sequence $m_1, m_2-m_1, m_3-m_2, \ldots, m_t-m_{t-1}, n-m_t$ after sorting. We refer to these numbers as *gaps*.

Lemma 4.6 *Assume that $1 \le t \le k-2$ are fixed, $1 \le c \le n/\log n$, and $(g_1-g_2) \to \infty$. Then*

$$T_{n,k}^c(m_1,m_2,\ldots,m_t) \sim (k+1-t)^{g_1}. \tag{25}$$

Proof. Observe that $T_{n,k}^c(m_1,m_2,\ldots,m_t) =$

$$\sum_{\substack{a_1+a_2+\cdots+a_{t+1}=k-t\\ a_i\ge 0}} T_{m_1,a_1}^{c,1}\cdot T_{m_2-m_1,a_2}^{1}\cdots T_{m_t-m_{t-1},a_t}^{1}\cdot T_{n-m_t,a_{t+1}}^{1,c}, \tag{26}$$

where $T_{n,k}^{c_1,c_2}$ and $T_{n,k}^c$ count all chains in the respective truncated Boolean algebra. These T's count ordered partitions. It is easy to see that $T_{u,v}^1 = S(u,v+1)(v+1)!$ and hence $T_{u,v}^1 \sim (v+1)^u$ for any fixed v by (23). The first and last factors cannot be expressed explicitely, since in them a certain class has size at least c. If $m_1 = g_1$ then, in particular, $m_1 \ge n/(t+1)$ and by Lemma 4.5, $T_{m_1,a_1}^{c,1} \sim (a_1+1)^{m_1}$. If $m_1 < g_1$ then we use the upper estimate $T_{m_1,a_1}^{c,1} \le T_{m_1,a_1}^{1}$. The term $T_{n-m_t,a_{t+1}}^{1,c}$ is handled similarly.

Working out asymptotic formula for a finite sum like (26), only the dominant term counts, if there is a single dominant term. A single dominant term is achieved when the largest possible base meets the largest possible exponent. $\square$

Theorem 4.2 *Assume that $\mathcal{F}$ is a maximum size family of t-intersecting k-chains in B_n^c. Then, for fixed $1 \le t < k$ and $(n-c)$ sufficiently large, $\mathcal{F}$ consists of all k-chains containing a specific t-chain $M_1, M_2, \dots, M_t$, such that $|M_i| = m_i$, and $m_1, m_2, \dots, m_t$ maximizes $T_{n,k}^c(m_1, m_2, \dots, m_t)$, as described in Theorem 4.1.*

Proof. For $(n-c)$ large, the application of Theorem 3.1 is possible, since Theorem 4.1 explicitly gives the size of $r_t(n)$. We have to check that condition (9) from Theorem 3.1 holds. For $c \ge 1$, we have $r_{t+1}(n) = T_{n,k}^c(c, c+1, \dots, c+t)$, $r_t(n) = T_{n,k}^c(c, c+1, \dots, c+t-1)$, and $r_{t+1}(n)/r_t(n) < 1/(n-c-t+1)$, since there are $n-c-t+1$ ways to choose a $(c+t)$-element set containing $[c+t-1]$. For $c = 0$, Theorem 4.1 yields $r_{t+1}(n) = T_{n,k}^0(0, 1, \dots, t-1, n)$, $r_t(n) = T_{n,k}^0(0, 1, \dots, t-2, n)$. For $t \ge 2$, $r_{t+1}(n)/r_t(n) < 1/(n-t+2)$, since there are $n-t+2$ ways to choose a $(t-1)$-element set containing $[t-2]$. For $t = 1$, $r_2(n) = T_{n,k}^0(0, n) = (k-1)!S(n, k-1)$ and $r_1(n) = T_{n,k}^0(n) = (k-1)!S(n, k-1) + k!S(n, k)$, from Theorem 4.1 and (20), (21). By (23), $S(n, k-1) = o(S(n, k))$ as n goes to infinity. It implies $\lim_{n\to\infty} r_2(n)/r_1(n) = 0$, and hence condition (9) from Theorem 3.1 holds. (We note that the extreme cases for $c = 0$, $t = 1$ were already characterized for all n in our previous paper [6].) $\square$

Theorem 4.3 *For fixed $1 \le t \le k-3$, n large, and $c \le n/\log n$, any maximum sized family of non-trivially t-intersecting k-chains in B_n^c is described by (15) in Part (ii) of Theorem 3.2, where the sizes of $\mathcal{Z}$ are*

$$(c, c+1, \dots, c+t+1) \text{ or } (n-c-t-1, \dots, n-c)$$

for $c \ge 2$; and the sizes of $\mathcal{Z}$ are

$$(c, c+1, \dots, c+i, n-c-t+i, \dots, n-c-1, n-c)$$

for $c = 0, 1$, with any $0 \le i < t$.

Proof. We treat here only the case $c \ge 2$, and leave $c = 0, 1$ to the Reader. First we show that condition (13) of Theorem 3.2 holds. We have

$$r_{t+2}(n) = T_{n,k}^c(c, c+1, \dots, c+t+1) \sim (k-t-1)^{n-c-t-1}$$

by Theorem 4.1 and Lemma 4.6. An $\mathcal{X}, \mathcal{L}$ pair defining $M_\tau(n)$ must have the property, that the sizes of its elements are within $O(1)$ distance either from c or from $(n-c)$. Hence $M_\tau^*(n) \ge (k-t)^{n-c-O(1)}$, and (13) holds.

Looking at possible extremal families of type (i) or (ii) from Theorem 3.2, one realizes that they decompose to a union of constant number of terms of families $\mathcal{T}_{u,v}^{c_1,c_2}(m_1, ..., m_s)$. Lemma 4.6 applies to each of these families.

It is not difficult to see that the best candidates to realize Part (i) of Theorem 3.2 are $\mathcal{X} = \{[c], \dots, [c+t-1]\}$ and $\mathcal{Y} = \{[c+t], \dots, [k+c]\}$, while the best candidate to realize Part (ii) is $\mathcal{Z}$ as given above. The reason is that the family of t-intersecting k-chains $\mathcal{F}_n$ that they define beats gapwise all other constructions of the respective type. Finally, we have to decide if $|\mathcal{F}_n(\mathcal{X}, \mathcal{Y})|$ or $|\mathcal{F}_n(\mathcal{Z})|$ is bigger. Using Lemma 4.6, we obtain

$$\begin{aligned} |\mathcal{F}_n(\mathcal{Z})| &\sim (t+1)(k-t)^{n-t-c-1} + (k-t)^{n-t-c} = (k+1)(k-t)^{n-t-c-1}, \\ |\mathcal{F}_n(\mathcal{X}, \mathcal{Y})| &\sim \sum_{i=1}^{k+1-t} (k-t)^{n+1-c-t-i}. \end{aligned}$$

$|\mathcal{F}_n(\mathcal{Z})|$ wins. $\square$

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