1 Infinite problems

K1) For $x \geq 0, y \geq 0$, define a rectangle $R_{x,y}$, whose vertices are $(0,0),(x,0),(0,y),(x,y)$.
(a) Show that if $\{(x_i, y_i) : i \in \mathbb{N}\}$ is an infinite set of lattice points (i.e $x_i, y_i \in \mathbb{N}$), then there is an infinite subsequence of rectangles R_{x_i,y_i} such that for any two of them, one contains the other.
(b) Is the conclusion still valid, if we allow any $x_i, y_i \in \mathbb{Q}^+$?

K2) Assume that $g, f_1, f_2, ..., f_k : X \to \mathbb{R}$ functions such that there exist $\epsilon > 0$ and $\delta > 0$ such that for all $x, y \in X$, if $g(x) - g(y) > \epsilon$, then there exists an i $(1 \leq i \leq k)$ such that $f_i(x) - f_i(y) > \delta$. Show that if all f_i functions are bounded, then g is bounded.

K12) Color natural numbers with two colors, such that none of the colorclasses contains an an infinite arithmetic progression.

K13) Is there a continuum cardinality (i.e. as many as real numbers are) family of subsets of a countable set, such that any two of these sets have finite intersection?

K3) (a) Show that for any $f : \mathbb{R}^2 \to [3]$ coloration, some two points unit distance apart have the same color.
(b) Show that for some $f : \mathbb{R}^2 \to [7]$ coloration the conclusion above does not hold.

(The problem above is known as the chromatic number of $\mathbb{R}^2$ problem. Consider a graph, whose vertices are the points in $\mathbb{R}^2$, and two of them are joined if their distance is 1. What is the chromatic number of this graph? Problem K3 contains the best known results.)

K4) Estimate the chromatic number of $\mathbb{R}^3$.

K5) Show that for some $f : \mathbb{R}^2 \to [2]$ coloration no unit regular triangle has monochromatic vertices.

K6) Show that for any $f : \mathbb{R}^2 \to [2]$ coloration, there is a triangle with unit hypothenus, $90°, 60°, 30°$ angles with monochromatic vertices.

2 Finite problems

K7) Show that $R_2^2(3,3) = 6$.

K8) Show that $R_2^2(3,4) = 9$.

K9) Show that $R_3^2(3,3,3) = 17$.

K10) $R_k^2(3,3,...,3) \leq \lfloor k!e \rfloor + 1$.

K11) (a) Show that for any sequence $a_1, a_2, ..., a_{nk+1}$ of real numbers, there is a strictly increasing subsequence of length $n+1$ or there is a decreasing subsequence of length $k+1$.
(b) Show that this conclusion is not true for all sequences of just nk real numbers.