# ON THE USE OF LOCAL COHOMOLOGY IN ALGEBRA AND GEOMETRY

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## Contents

## Introduction

Local cohomology is a useful tool in several branches of commutative algebra and algebraic geometry. The main aim of this series of lectures is to illustrate a few of these techniques. The material presented in the sequel needs some basic knowledge about commutative resp. homological algebra. The basic chapters of the textbooks [9], [28], and [48] are a recommended reading for the preparation. The author's intention was to present applications of local cohomology in addition to the examples in these textbooks as well as those of [7].

Several times the author applies spectral sequence techniques for the proofs. Often people claim that it is possible to avoid spectral sequence arguments in the proofs for certain results. The present author believes that these techniques are quite natural. They will give deep insights in the underlying structure. So he forced these kinds of arguments even in cases where he knows more 'elementary' proofs. He has the hope to interest more researchers working in commutative algebra for such a powerful technique. As an introduction to spectral sequences he suggests the study of the corresponding chapters in the textbooks [9] and [48].

In the first section there is an introduction to local duality and dualizing complexes. There is a consequent use of the Čech complexes. In the main result, see 1.6, there is a family of dualities, including Matlis duality and duality for a dualizing complex of a complete local ring. This approach does not use 'sophisticated' prerequisites like derived categories. It is based on a few results about complexes and flat resp. injective modules. As applications there are a proof of the local duality theorem and vanishing theorems of the local cohomology of the canonical module. In particular it follows that a factorial domain is a Cohen-Macaulay ring provided it is a 'half way' Cohen-Macaulay ring. The first section concludes with a discussion of the cohomological annihilators $\operatorname{Ann} H^n_{\mathfrak{a}}(M)$ of a finitely generated $A$-module $M$ and an ideal $\mathfrak{a}$. The consideration of these annihilators provides more subtle information than vanishing results.

Section 2 is concerned with the structure of the local cohomology modules in 'small' resp. 'large' homological dimensions. The 'small' homological dimension has to do with ideal transforms. To this end there is a generalization of Chevalley's theorem about the equivalence of ideal topologies. This is applied in order to prove Grothendieck's finiteness result for ideal transforms. The structure of particular cases of ideal transforms of certain Rees rings is a main technical tool for the study of asymptotic prime divisors. On the other side of the range, i.e. the 'large' homological dimensions, there is a proof of the Lichtenbaum-Hartshorne vanishing theorem for local cohomology. In fact the non-vanishing of the $d$-dimensional local cohomology of a $d$-dimensional local ring is the obstruction for the equivalence of a certain topology to the adic topology. The Lichtenbaum-Hartshorne vanishing theorem is a helpful tool for the proof of a connectedness result invented by G. Faltings. We do not relate our considerations to a more detailed study of the cohomological dimension of an ideal. For results on cohomological dimensions see R. Hartshorne's article [16]. For more recent developments compare C. Huneke's and G. Lyubeznik's work in [22].

The third Section is devoted to the study of finite free resolutions of an $A$-module $M$ in terms of its local cohomology modules. There are length estimates for $\operatorname{Ext}_A^n(M, N)$ and $\operatorname{Tor}_n^A(M, N)$ for two finitely generated $A$-modules $M, N$ such that $M \otimes_A N$ is of finite length. This leads to an equality of the Auslander-Buchsbaum type, first studied by M. Auslander in [1], and a Cohen-Macaulay criterion. Moreover there are estimates of the Betti numbers of $M$ in terms of the Betti numbers of the modules of deficiency of $M$. More subtle considerations are included in the case of graded modules over graded rings. This leads to the study of the Castelnuovo-Mumford regularity and a generalization of M. Green's duality result for certain Betti numbers of $M$ and its canonical module $K_M$.

The author's aim is to present several pictures about the powerful tool of local cohomology in different fields of commutative algebra and algebraic geometry. Of course the collection of known applications is not exhausted. The reader may feel a challenge to continue with the study of local cohomology in his own field. In most of the cases the author tried to present basic ideas of an application. It was not his goal to present the most sophisticated generalization. The author expects further applications of local cohomology in the forthcoming textbook [6].

The present contribution has grown out of the author's series of lectures held at the Summer School on Commutative Algebra at CRM in Ballaterra, July 16 - 26, 1996. The author thanks the organizers of the Summer School at Centre de Recerca Matemàtica for bringing together all of the participants at this exciting meeting. For the author it was a great pleasure to present a series of lectures in the nice and stimulating atmosphere of this Summer School. During the meeting there were a lot of opportunities for discussions with sevaral people; this made this School so exciting for the author. Among them the author wants to thank Luchezar Avramov, Hans-Bjørn Foxby, José-Maria Giral, Craig Huneke, David A. Jorgensen, Ruth Kantorovitz, Leif Melkersson, Claudia Miller, who drew the author's attention to several improvements of his original text. The author wants to thank also the staff members of the Centre de Recerca Matemàtica for their effort to make the stay in Ballaterra so pleasant. Finally he wants to thank R. Y. Sharp for a careful reading of the manuscript and several suggestions for an improvement of the text.

## 1. A Guide to Duality

1.1. **Local Duality.** Let $A$ denote a commutative Noetherian ring. Let $C$ denote a complex of $A$-modules. For an integer $k \in \mathbb{Z}$ let $C[k]$ denote the complex $C$ shifted

$k$ places to the left and the sign of differentials changed to $(-1)^k$, i.e.

$$(C[k])^n = C^{k+n} \quad \text{and} \quad d_{C[k]} = (-1)^k d_C.$$

Moreover note that $H^n(C[k]) = H^{n+k}(C)$.

For a homomorphism $f : C \to D$ of two complexes of $A$-modules let us consider the mapping cone $M(f)$. This is the complex $C \oplus D[-1]$ with the boundary map $d_{M(f)}$ given by the following matrix

$$\begin{pmatrix} d_C & 0 \\ -f & -d_D \end{pmatrix}$$

where $d_C$ resp. $d_D$ denote the boundary maps of $C$ and $D$ resp. Note that $(M(f), d_{M(f)})$ forms indeed a complex.

There is a natural short exact sequence of complexes

$$0 \to D[-1] \xrightarrow{i} M(f) \xrightarrow{p} C \to 0,$$

where $i(b) = (0, -b)$ and $p(a, b) = a$. Clearly these homomorphisms make $i$ and $p$ into homomorphisms of complexes. Because $H^{n+1}(D[-1]) = H^n(D)$ the connecting homomorphism $\delta$ provides a map $\delta^{\cdot} : H^{\cdot}(C) \to H^{\cdot}(D)$. By an obvious observation it follows that $\delta^{\cdot} = H^{\cdot}(f)$. Note that $f : C \to D$ induces an isomorphism on cohomology if and only if $M(f)$ is an exact complex.

Let $M, N$ be two $A$-modules considered as complexes concentrated in homological degree zero. Let $f : M \to N$ be a homomorphism. Then the mapping cone of $f$ is

$$M(f) : \quad \ldots \to 0 \to M \xrightarrow{-f} N \to 0 \to \ldots$$

with the cohomology modules given by

$$H^n(M(f)) \simeq \begin{cases} \ker f & i = 0, \\ \operatorname{coker} f & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This basic observation yields the following result:

**Lemma 1.1.** *Let $f : C \to D$ be a homomorphism of complexes. Then there is a short exact sequence*

$$0 \to H^1(M(H^{n-1}(f))) \to H^n(M(f)) \to H^0(M(H^n(f))) \to 0$$

*for all $n \in \mathbb{Z}$.*

*Proof.* This is an immediate consequence of the long exact cohomology sequence. Recall that the connecting homomorphism is $H^{\cdot}(f)$. $\square$

For a complex $C$ and $x \in A$ let $C \xrightarrow{x} C$ denote the multiplication map induced by $x$, i.e. the map on $C^n$ is given by multiplication with $x$. Furthermore let $C \to C \otimes_A A_x$ denote the natural map induced by the localization, i.e. the map on $C^n$ is given by $C^n \xrightarrow{i} C^n \otimes_A A_x$, where for an $A$-module $M$ the map $i$ is the natural map $i : M \to M \otimes_A A_x$.

In the following let us use the previous consideration in order to construct the Koszul and Čech complexes with respect to a system of elements $\underline{x} = x_1, \ldots, x_r$ of $A$. To this end we consider the ring $A$ as a complex concentrated in degree zero. Then define

$$K^\cdot(x; A) = M(A \xrightarrow{x} A) \text{ and } K^\cdot_x(A) = M(A \to A_x).$$

Note that both of these complexes are bounded in degree 0 and 1. Inductively put

$$\begin{aligned} K^\cdot(\underline{x}; A) &= M(K^\cdot(\underline{y}; A) \xrightarrow{x} K^\cdot(\underline{y}; A)) \quad \text{and} \\ K^\cdot_{\underline{x}}(A) &= M(K^\cdot_{\underline{y}}(A) \to K^\cdot_{\underline{y}}(A) \otimes_A A_x), \end{aligned}$$

where $\underline{y} = x_1, \ldots, x_{r-1}$ and $x = x_r$. For an $A$-module $M$ finally define

$$K^\cdot(\underline{x}; M) = K^\cdot(\underline{x}; A) \otimes_A M \text{ and } K^\cdot_{\underline{x}}(M) = K^\cdot_{\underline{x}}(A) \otimes_A M.$$

Call them (co-) Koszul complex resp. Čech complex of $\underline{x}$ with respect to $M$. Obviously the Čech complex is bounded. It has the following structure

$$0 \to M \to \oplus_i M_{x_i} \to \oplus_{i<j} M_{x_i x_j} \to \ldots \to M_{x_1 \cdots x_r} \to 0$$

with the corresponding boundary maps.

It is well known that there is an isomorphism of complexes

$$K^\cdot_{\underline{x}}(M) \simeq \varinjlim K^\cdot(\underline{x}^{(n)}; M),$$

where $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$, see [14]. The direct maps in the direct system are induced in a natural way by the inductive construction of the complex. The proof follows by induction on the number of elements.

The importance of the Čech complex is its close relation to the local cohomology. For an ideal $\mathfrak{a}$ of $A$ let $\Gamma_\mathfrak{a}$ denote the section functor with respect to $\mathfrak{a}$. That is, $\Gamma_\mathfrak{a}$ is the subfunctor of the identity functor given by

$$\Gamma_\mathfrak{a}(M) = \{m \in M \mid \operatorname{Supp} Am \subseteq V(\mathfrak{a})\}.$$

The right derived functors of $\Gamma_\mathfrak{a}$ are denoted by $H^i_\mathfrak{a}$, $i \in \mathbb{N}$. They are called the local cohomology functors with respect to $\mathfrak{a}$.

**Lemma 1.2.** *Let $\mathfrak{a}$ resp. $S$ be an ideal resp. a multiplicatively closed set of $A$. Let $E$ denote an injective $A$-module. Then*

a) $\Gamma_\mathfrak{a}(E)$ *is an injective $A$-module,*
b) *the natural map $E \to E_S, e \mapsto \frac{e}{1}$, is surjective, and*

c) *the localization $E_S$ is an injective $A$-module.*

*Proof.* The first statement is an easy consequence of Matlis' Structure Theorem about injective modules. In order to prove b) let $\frac{e}{s} \in E_S$ be an arbitrary element. In the case $s$ is $A$-regular it follows easily that there is an $f \in E$ such that $e = fs$ and $\frac{f}{1} = \frac{e}{s}$, which proves the claim. In the general case choose $n \in \mathbb{N}$ such that $0 :_A s^n = 0 :_A s^{n+1}$. Recall that $A$ is a Noetherian ring. Then consider the injective $(A/0 :_A s^n)$-module $\operatorname{Hom}_A(A/0 :_A s^n, E)$. Moreover

$$\operatorname{Hom}_A(A/0 :_A s^n, E) \simeq E/0 :_E s^n$$

because $A/0 :_A s^n \simeq s^n A$. Therefore it turns out that $E/0 :_E s^n$ has the structure of an injective $A/0 :_A s^n$-module. But now $s$ acts on $A/0 :_A s^n$ as a regular element. Therefore for $e + 0 :_E s^n$ there exists an $f + 0 :_E s^n$, $f \in E$, such that $e - fs \in 0 :_E s^n$. But this proves $\frac{e}{s} = \frac{f}{1}$, i.e. the surjectivity of the considered map.

By Matlis' Structure Theorem and because the localization commutes with direct sums it is enough to prove statement c) for $E = E_A(A/\mathfrak{p})$, the injective hull of $A/\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} A$. In the case $S \cap \mathfrak{p} \neq \emptyset$ it follows that $E_S = 0$. So let $S \cap \mathfrak{p} = \emptyset$. Then any $s \in S$ acts regularly on $E$. Therefore the map in b) is an isomorphism. $\square$

For simplicity put $H^n_{\underline{x}}(M) = H^n(K_{\underline{x}}(M))$. This could give a misunderstanding to $H^n_{\mathfrak{a}}(M)$. But in fact both are isomorphic as shown in the sequel.

**Theorem 1.3.** *Let $\underline{x} = x_1, \ldots, x_r$ denote a system of elements of $A$ with $\mathfrak{a} = \underline{x}A$. Then there are functorial isomorphisms $H^n_{\underline{x}}(M) \simeq H^n_{\mathfrak{a}}(M)$ for any $A$-module $M$ and any $n \in \mathbb{Z}$.*

*Proof.* First note that $H^0_{\underline{x}}(M) \simeq \Gamma_{\mathfrak{a}}(M)$ as is easily seen by the structure of $K^{\cdot}_{\underline{x}}(M)$. Furthermore let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $A$-modules. Because $K^{\cdot}_{\underline{x}}(A)$ consists of flat $A$-modules the induced sequence of complexes

$$0 \to K^{\cdot}_{\underline{x}}(M') \to K^{\cdot}_{\underline{x}}(M) \to K^{\cdot}_{\underline{x}}(M'') \to 0$$

is exact. That is, $H^n_{\underline{x}}(\cdot)$ forms a connected sequence of functors. Therefore, in order to prove the claim it will be enough to prove that $H^n_{\underline{x}}(E) = 0$ for $n > 0$ and an injective $A$-module $E$. This will be proved by induction on $r$. For $r = 1$ it is a particular case of 1.2. Let $r > 1$. Put $\underline{y} = x_1, \ldots, x_{r-1}$ and $x = x_r$. Then 1.1 provides a short exact sequence

$$0 \to H^1_x(H^{n-1}_{\underline{y}}(E)) \to H^n_{\underline{x}}(E) \to H^0_x(H^n_{\underline{y}}(E)) \to 0.$$

For the case $n > 2$ the claim follows by the induction hypothesis. In the remaining case $n = 1$ note that $H^0_{\underline{y}}(E) = \Gamma_{\underline{y}A}(E)$ is an injective $A$-module, see 1.2. So the induction hypothesis applies again. $\square$

Together with 1.1 the previous result provides a short exact sequence describing the behaviour of local cohomology under enlarging the number of generators of an ideal.

**Corollary 1.4.** *Let $\mathfrak{a}$ resp. $x$ denote an ideal resp. an element of $A$. Then for $n \in \mathbb{N}$ there is a functorial short exact sequence*

$$0 \to H^1_{xA}(H^{n-1}_{\mathfrak{a}}(M)) \to H^n_{(\mathfrak{a},xA)}(M) \to H^0_{xA}(H^n_{\mathfrak{a}}(M)) \to 0$$

*for any $A$-module $M$.*

The previous theorem provides a structural result about local cohomology functors with support in $\mathfrak{m}$ in the case of a local ring $(A, \mathfrak{m})$. To this end let $E = E_A(A/\mathfrak{m})$ denote the injective hull of the residue field. Furthermore it provides also a change of ring theorem.

**Corollary 1.5.** a) *Let $(A, \mathfrak{m})$ denote a local ring. Then $H^n_{\mathfrak{m}}(M)$, $n \in \mathbb{N}$, is an Artinian $A$-module for any finitely generated $A$-module $M$.*
b) *Let $A \to B$ denote a homomorphism of Noetherian rings. Let $\mathfrak{a}$ be an ideal of $A$. For a $B$-module $M$ there are $A$-isomorphisms $H^n_{\mathfrak{a}}(M) \simeq H^n_{\mathfrak{a}B}(M)$ for all $n \in \mathbb{N}$. Here in the first local cohomology module $M$ is considered as an $A$-module.*

*Proof.* a) Let $E^{\cdot}(M)$ denote the minimal injective resolution of $M$. Then by Matlis' Structure Theorem on injective $A$-modules it follows that $\Gamma_{\mathfrak{m}}(E^{\cdot}(M))$ is a complex consisting of finitely many copies of $E$ in each homological degree. Therefore $H^n_{\mathfrak{m}}(M)$ is – as a subquotient of an Artinian $A$-module – an Artinian module.
b) Let $\underline{x} = x_1, \ldots, x_r$ denote a generating set of $\mathfrak{a}$. Let $\underline{y} = y_1, \ldots, y_r$ denote the images in $B$. Then there is the following isomorphism $K^{\cdot}_{\underline{x}}(M) \simeq K^{\cdot}_{\underline{y}}(B) \otimes_B M$, where both sides are considered as complexes of $A$-modules. This proves the claim. □

In the following let $T(\cdot) = \operatorname{Hom}_A(\cdot, E)$ denote the Matlis duality functor for a local ring $(A, \mathfrak{m})$. An exceptional rôle is played by the complex $D^{\cdot}_{\underline{x}} = T(K^{\cdot}_{\underline{x}})$ with $K^{\cdot}_{\underline{x}} = K^{\cdot}_{\underline{x}}(A)$ as follows by the following theorem. In some sense it extends the Matlis duality.

For two complexes $C, D$ consider the single complex $\operatorname{Hom}_A(C, D)$ associated to the corresponding double complex. To be more precise let

$$\operatorname{Hom}_A(C, D)^n = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_A(C^i, D^{i+n}).$$

The $n$-th boundary map restricted to $\operatorname{Hom}_A(C^i, D^{i+n})$ is given by

$$\operatorname{Hom}_A(d^{i-1}_C, D^{i+n}) + (-1)^{n+1} \operatorname{Hom}_A(C^i, d^{i+n}_D).$$

Note that this induces a boundary map on $\operatorname{Hom}_A(C, D)$. Moreover, it is easy to see that $H^0(\operatorname{Hom}_A(C, D))$ is isomorphic to the homotopy equivalence classes of homomorphisms of the complexes $C$ and $D$.

**Theorem 1.6.** *There is a functorial map*

$$M \otimes_A \hat{A} \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D^{\cdot}_{\underline{x}}), D^{\cdot}_{\underline{x}}),$$

*which induces an isomorphism in cohomology for any finitely generated $A$-module $M$.*

*Proof.* Let $M$ be an arbitrary $A$-module. First note that for two $A$-modules $X$ and $Y$ there is a functorial map

$$\begin{array}{ccc} M \otimes_A \operatorname{Hom}_A(X, Y) & \to & \operatorname{Hom}_A(\operatorname{Hom}_A(M, X), Y), \\ m \otimes f & \mapsto & (m \otimes f)(g) = f(g(m)) \end{array}$$

for $m \in M, f \in \operatorname{Hom}_A(X, Y)$ and $g \in \operatorname{Hom}_A(M, X)$. It induces an isomorphism for a finitely generated $A$-module $M$ provided $Y$ is an injective $A$-module. Because $D^{\cdot}_{\underline{x}}$ is a bounded complex of injective $A$-modules it induces a functorial isomorphism of complexes

$$M \otimes_A \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, D^{\cdot}_{\underline{x}}) \xrightarrow{\sim} \operatorname{Hom}_A(\operatorname{Hom}_A(M, D^{\cdot}_{\underline{x}}), D^{\cdot}_{\underline{x}})$$

for any finitely generated $A$-module $M$.

Now consider the complex $\operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, D^{\cdot}_{\underline{x}})$. It is isomorphic to $T(K^{\cdot}_{\underline{x}} \otimes_A T(K^{\cdot}_{\underline{x}}))$. Continue with the investigation of the natural map

$$f : K^{\cdot}_{\underline{x}} \otimes_A T(K^{\cdot}_{\underline{x}}) \to E$$

defined in homological degree zero. In the following we abbreviate

$$C := K^{\cdot}_{\underline{x}} \otimes_A T(K^{\cdot}_{\underline{x}}).$$

We claim that $f$ induces an isomorphism in cohomology. To this end consider the spectral sequence

$$E_1^{ij} = H^i(K^{\cdot}_{\underline{x}} \otimes_A T(K^{-j}_{\underline{x}})) \Rightarrow E^{i+j} = H^{i+j}(C).$$

Because $T(K^{-j}_{\underline{x}})$ is an injective $A$-module it follows that

$$E_1^{ij} = H^i_{\mathfrak{a}}(T(K^{-j}_{\underline{x}})) = 0$$

for all $i \neq 0$ as shown in 1.3. Here $\mathfrak{a}$ denotes the ideal generated by $\underline{x}$. Let $i = 0$. Then we have $E_1^{0j} = \Gamma_{\mathfrak{a}}(T(K^{-j}_{\underline{x}}))$. This implies that

$$E_1^{0j} = \varinjlim \operatorname{Hom}_A(A/\mathfrak{a}^n \otimes K^{-j}_{\underline{x}}, E).$$

Therefore $E_1^{0j} = 0$ for $j \neq 0$ because of $(A/\mathfrak{a}^n) \otimes_A A_x = 0$ for an element $x \in \mathfrak{a}$. Finally

$$E_1^{00} = \varinjlim \operatorname{Hom}_A(A/\mathfrak{a}^n, E) = E,$$

because $E$ is an Artinian $A$-module. This proves the claim as is easily seen.

Moreover $C$ is a complex of injective $A$-modules as follows by view of 1.2. Therefore the mapping cone $M(f)$ is an exact complex of injective $A$-modules. Furthermore let $g = T(f)$ denote the natural map

$$g : \hat{A} \to \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, D^{\cdot}_{\underline{x}}).$$

This induces an isomorphism in cohomology because the mapping cone $M(g) = T(M(f))$ is an exact complex. But now $\operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, D^{\cdot}_{\underline{x}})$ is a complex of flat $A$-modules. So there is a sequence of functorial maps

$$M \otimes_A \hat{A} \to M \otimes_A \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, D^{\cdot}_{\underline{x}}) \xrightarrow{\sim} \operatorname{Hom}_A(\operatorname{Hom}_A(M, D^{\cdot}_{\underline{x}}), D^{\cdot}_{\underline{x}}).$$

In order to prove the statement it is enough to show that the first map induces an isomorphism in cohomology. To this end note that the mapping cone $M(g)$ is a bounded exact complex of flat $A$-modules. But now $M \otimes_A M(g) \simeq M(1_M \otimes g)$ is an exact complex. Therefore the map

$$1_M \otimes g : M \otimes_A \hat{A} \to M \otimes_A \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, D^{\cdot}_{\underline{x}})$$

induces an isomorphism in cohomology. $\square$

In the case of a complete local ring $(A, \mathfrak{m})$ and a system $\underline{x} = x_1, \ldots, x_r$ of elements such that $\mathfrak{m} = \operatorname{Rad} \underline{x}A$ it follows that $D^{\cdot}_{\underline{x}}$ is a bounded complex of injective $A$-modules with finitely generated cohomology such that the natural map

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D^{\cdot}_{\underline{x}}), D^{\cdot}_{\underline{x}})$$

induces an isomorphism. Such a complex is called a dualizing complex. By virtue of this observation call $D^{\cdot}_{\underline{x}}$ a quasi-dualizing complex with support in $V(\mathfrak{a})$, where $\mathfrak{a} = \operatorname{Rad} \underline{x}A$. While the dualizing complex does not exist always, there are no restrictions about the existence of quasi-dualizing complexes with support in $V(\mathfrak{a})$. The isomorphisms in 1.6 are a common generalization of the Matlis duality obtained for $r = 0$ and the duality for a dualizing complex.

The most important feature of the dualizing complex is the local duality theorem first proved by A. Grothendieck. As an application of our considerations let us derive another proof.

**Theorem 1.7.** (Local Duality) *Let $(A, \mathfrak{m})$ denote a local ring. Let $\underline{x} = x_1, \ldots, x_r$ be a system of elements such that $\mathfrak{m} = \operatorname{Rad} \underline{x}A$. Then there are functorial isomorphisms*

$$H^n_{\mathfrak{m}}(M) \simeq \operatorname{Hom}_A(H^{-n}(\operatorname{Hom}_A(M, D^{\cdot}_{\underline{x}})), E), \ n \in \mathbb{Z},$$

*for a finitely generated $A$-module $M$.*

*Proof.* First note that $H^n_{\mathfrak{m}}(M) \simeq H^n(K^{\cdot}_{\underline{x}} \otimes_A M)$ by 1.3. Now $D^{\cdot}_{\underline{x}}$ is a bounded complex of injective $A$-modules. As in the proof of 1.6 there is a functorial map

$$M \otimes_A \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, E) \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D^{\cdot}_{\underline{x}}), E),$$

which is an isomorphism of complexes for any finitely generated $A$-module $M$. Now consider the functorial map

$$K^{\cdot}_{\underline{x}} \to \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, E) = T^2(K^{\cdot}_{\underline{x}}).$$

By 1.5 and the Matlis duality it induces an isomorphism in cohomology. Because $K^{\cdot}_{\underline{x}}$ and $\operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, E)$ are complexes of flat $A$-modules the natural map

$$M \otimes_A K^{\cdot}_{\underline{x}} \to M \otimes_A \operatorname{Hom}_A(D^{\cdot}_{\underline{x}}, E)$$

induces an isomorphism in cohomology by the same argument as in 1.6. The proof follows now by putting together both of the maps. $\square$

In this form of local duality the complex $D^{\cdot}_{\underline{x}}$ plays the rôle of the dualizing complex. In the next section there are a few more statements about dualizing complexes.

1.2. **Dualizing Complexes and Some Vanishing Theorems.** Let $(A, \mathfrak{m})$ denote a local ring. For a non-zero finitely generated $A$-module $M$ there are the well-known vanishing results $\operatorname{depth} M = \min\{n \in \mathbb{Z} \mid H^n_{\mathfrak{m}}(M) \neq 0\}$ and $\dim M = \max\{n \in \mathbb{Z} \mid H^n_{\mathfrak{m}}(M) \neq 0\}$ shown by A. Grothendieck. In the following we recall two more subtle vanishing results on $H^n_{\mathfrak{m}}(M)$. To this end let us first investigate a few consequences of local duality.

**Theorem 1.8.** *Suppose that the local ring $(A, \mathfrak{m})$ is the factor ring of a Gorenstein ring $(B, \mathfrak{n})$ with $r = \dim B$. Then there are functorial isomorphisms*

$$H^n_{\mathfrak{m}}(M) \simeq \operatorname{Hom}_A(\operatorname{Ext}^{r-n}_B(M, B), E), \quad n \in \mathbb{Z},$$

*for any finitely generated $A$-module $M$, where $E$ denotes the injective hull of the residue field.*

*Proof.* By 1.5 one may assume without loss of generality that $A$ itself is a Gorenstein ring. Let $\underline{x} = x_1, \ldots, x_r$ denote a system of parameters of $A$. Under this additional Gorenstein assumption it follows that $K^{\cdot}_{\underline{x}}$ is a flat resolution of $H^r_{\mathfrak{m}}(A)[-r] \simeq E[-r]$, where $E$ denotes the injective hull of the residue field. Therefore $D^{\cdot}_{\underline{x}}$ is an injective resolution of $\hat{A}[r]$. By definition it turns out that

$$H^{-n}(\operatorname{Hom}(M, D^{\cdot}_{\underline{x}})) \simeq \operatorname{Ext}^{r-n}_A(M, \hat{A})$$

for all $n \in \mathbb{Z}$. Because of $T(\operatorname{Ext}^{r-n}_A(M, \hat{A})) \simeq T(\operatorname{Ext}^{r-n}_A(M, A))$ this proves the claim. $\square$

In the situation of 1.8 introduce a few abbreviations. For $n \in \mathbb{Z}$ put

$$K_M^n = \operatorname{Ext}_B^{r-n}(M, B).$$

Moreover for $n = \dim M$ we often write $K_M$ instead of $K_M^{\dim M}$. The module $K_M$ is called the canonical module of $M$. In the case of $M = A$ it coincides with the classical definition of the canonical module of $A$. By the Matlis duality and by 1.8 the modules $K_M^n$ do not depend – up to isomorphisms – on the presentation of the Gorenstein ring $B$. Clearly $K_M^n = 0$ for all $n > \dim M$ and $n < 0$. Moreover we have the isomorphism

$$K_M^n \otimes_A \hat{A} \simeq H^{-n}(\operatorname{Hom}_A(M, D_{\underline{x}}^{\cdot})),\ n \in \mathbb{Z},$$

as follows by view of 1.7. The advantage of $K_M^n$ lies in the fact that it is – in contrast to $H^{-n}(\operatorname{Hom}_A(M, D_{\underline{x}}^{\cdot}))$ – a finitely generated $A$-module.

For a finitely generated $A$-module $M$ say it satisfies *Serre's condition* $S_k$, $k \in \mathbb{N}$, provided

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min\{k, \dim M_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \operatorname{Supp} M.$$

Note that $M$ satisfies $S_1$ if and only if it is unmixed. $M$ is a Cohen-Macaulay module if and only if it satisfies $S_k$ for all $k \in \mathbb{N}$.

**Lemma 1.9.** *Let $M$ denote a finitely generated $A$-module. The finitely generated $A$-modules $K_M^n$ satisfy the following properties:*

a) $(K_M^n)_{\mathfrak{p}} \simeq K_{M_{\mathfrak{p}}}^{n-\dim A/\mathfrak{p}}$ *for any* $\mathfrak{p} \in \operatorname{Supp} M$, *i.e.* $\dim K_M^n \leq n$ *for all* $n \in \mathbb{Z}$.
b) *If* $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$ *for some* $\mathfrak{p} \in \operatorname{Supp} M$, *then* $(K_M)_{\mathfrak{p}} \simeq K_{M_{\mathfrak{p}}}$.
c) $\operatorname{Ass} K_M = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} = \dim M\}$, *i.e.* $\dim M = \dim K_M$.
d) *Suppose that $M$ is equidimensional. Then $M$ satisfies condition $S_k$ if and only if* $\dim K_M^n \leq n - k$ *for all* $0 \leq n < \dim M$.
e) $K_M$ *satisfies* $S_2$.

*Proof.* Let $\mathfrak{p} \in \operatorname{Supp} M$ denote a prime ideal. Then $(K_M^n)_{\mathfrak{p}} \simeq K_{M_{\mathfrak{p}}}^{n-\dim A/\mathfrak{p}}$, as is easily seen by the presentation as an Ext-module. Therefore $\dim K_M^n \leq n$.

Let $E^{\cdot}(B)$ denote the minimal injective resolution of $B$ as a $B$-module. Then $(\operatorname{Hom}_B(M, E^{\cdot}(B)))^n = 0$ for all $n < r - \dim M$ and

$$\operatorname{Ass}_A H^{r-\dim M}(\operatorname{Hom}_B(M, E^{\cdot}(B))) = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} = \dim M\}.$$

Putting this together the proofs of a), b), and c) follow immediately.

In order to prove d) first note that $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$ for all $\mathfrak{p} \in \operatorname{Supp} M$ since $M$ is equidimensional. Suppose there is an integer $n$ with $0 \leq n < \dim M$ and $\mathfrak{p} \in \operatorname{Supp} K_M^n$ such that $\dim A/\mathfrak{p} > n - k \geq 0$. This implies $H_{\mathfrak{p}A_{\mathfrak{p}}}^{n-\dim A/\mathfrak{p}}(M_{\mathfrak{p}}) \neq 0$, see 1.8. Therefore

$$\begin{aligned} \operatorname{depth} M_{\mathfrak{p}} &\leq n - \dim A/\mathfrak{p} < \dim M - \dim A/\mathfrak{p} = \dim M_{\mathfrak{p}} \text{ and} \\ \operatorname{depth} M_{\mathfrak{p}} &\leq n - \dim A/\mathfrak{p} < k, \end{aligned}$$

in contradiction to $S_k$. Conversely suppose there is a $\mathfrak{p} \in \operatorname{Supp} M$ such that $\operatorname{depth} M_\mathfrak{p} < \min\{k, \dim M_\mathfrak{p}\}$. Then $(K_M^n)_\mathfrak{p} \neq 0$ for $n = \dim A/\mathfrak{p} + \operatorname{depth} M_\mathfrak{p}$ and $\dim A/\mathfrak{p} = n - \operatorname{depth} M_\mathfrak{p} > n - k$, a contradiction. This finishes the proof of the statement in d).

In order to prove e) it is enough to show that $\operatorname{depth} K_M \geq \min\{2, \dim K_M\}$. Note that $(K_M)_\mathfrak{p} \simeq K_{M_\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp} K_M$. This follows because $K_M$ is unmixed by c) and because $\operatorname{Supp} K_M$ is catenerian, i. e. $\dim M_\mathfrak{p} + \dim A/\mathfrak{p} = \dim M$ for all $\mathfrak{p} \in \operatorname{Supp} K_M$.

Without loss of generality we may assume that there is an $M$-regular element $x \in \mathfrak{m}$. Then the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ induces an injection $0 \to K_M/xK_M \to K_{M/xM}$, which proves the claim. $\square$

Another reading of a) in 1.9 is that $H_\mathfrak{m}^{i+\dim A/\mathfrak{p}}(M) \neq 0$ provided $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. This is true for an arbitrary local ring as shown by R. Y. Sharp, see [46, Theorem (4.8)].

**Proposition 1.10.** *Let $\mathfrak{p}$ be a $t$-dimensional prime ideal in a local ring $(A, \mathfrak{m})$. Let $M$ denote a finitely generated $A$-module such that $H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ for a certain $i \in \mathbb{N}$. Then $H^{i+t}_\mathfrak{m}(M) \neq 0$.*

*Proof.* Choose $P \in V(\mathfrak{p}\widehat{A})$ a prime ideal such that $\dim \widehat{A}/P = \dim \widehat{A}/\mathfrak{p}\widehat{A} = t$. In particular this implies $P \cap A = \mathfrak{p}$ and that $\mathfrak{p}\widehat{A}_P$ is a $P\widehat{A}_P$-primary ideal. These data induce a faithful flat ring homomorphism $A_\mathfrak{p} \to \widehat{A}_P$. It yields that

$$0 \neq H^i_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) \otimes_{A_\mathfrak{p}} \widehat{A}_P \simeq H^i_{\mathfrak{p}\widehat{A}_P}(M_\mathfrak{p} \otimes_{A_\mathfrak{p}} \widehat{A}_P).$$

But now there is the following canonical isomorphism $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} \widehat{A}_P \simeq (M \otimes_A \widehat{A})_P$. Because $\mathfrak{p}\widehat{A}_P$ is a $P\widehat{A}_P$-primary ideal it follows that $0 \neq H^i_{P\widehat{A}_P}((M \otimes_A \widehat{A})_P)$. By Cohen's Structure Theorem $\widehat{A}$ is a homomorphic image of a Gorenstein ring. By the faithful flatness of $A \to \widehat{A}$ and by the corresponding result for a homomorphic image of a Gorenstein ring, see 1.9, it turns out that

$$H^{i+t}_\mathfrak{m}(M) \otimes \widehat{A} \simeq H^{i+t}_{\mathfrak{m}\widehat{A}}(M \otimes_A \widehat{A}) \neq 0,$$

which finally proves the claim. $\square$

Note that the previous result has the following consequence. Let $\mathfrak{p} \in \operatorname{Supp} M$ denote a prime ideal. Then

$$\operatorname{depth} M \leq \dim A/\mathfrak{p} + \operatorname{depth} M_\mathfrak{p}$$

for a finitely generated $A$-module $M$. This follows by the non-vanishing of the local cohomology for the depth of a module.

As above let $(A, \mathfrak{m})$ denote a local ring which is the factor ring of a Gorenstein ring $(B, \mathfrak{n})$. Let $E^\cdot(B)$ denote the minimal injective resolution of the Gorenstein ring $B$ as a $B$-module. The complex $D_A = \operatorname{Hom}_B(A, E^\cdot(B))$ is a bounded complex of injective $A$-modules and finitely generated cohomology modules $H^n(D_A) \simeq \operatorname{Ext}_B^n(A, B)$.

**Theorem 1.11.** *The complex $D_A$ is a dualizing complex of $A$. That is, there is a functorial map*

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A), D_A)$$

*that induces an isomorphism in cohomology for any finitely generated $A$-module $M$.*

*Proof.* Because $D_A$ is a bounded complex of injective $A$-modules there is an isomorphism of complexes

$$M \otimes_A \operatorname{Hom}_A(D_A, D_A) \xrightarrow{\sim} \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A), D_A)$$

for any finitely generated $A$-module $M$ as shown above. By similar arguments as before there is a natural map $A \to \operatorname{Hom}_A(D_A, D_A)$. Because both of the complexes involved – $A$ as well as $\operatorname{Hom}_A(D_A, D_A)$ – are complexes of flat $A$-modules it will be enough to show that this map induces an isomorphism in cohomology in order to prove the statement.

Next consider the isomorphism of complexes

$$\operatorname{Hom}_A(D_A, D_A) \xrightarrow{\sim} \operatorname{Hom}_B(\operatorname{Hom}_B(A, E^\cdot(B)), E^\cdot(B)).$$

Therefore, in order to show the claim it will be enough to prove the statement for the Gorenstein ring $B$. First it will be shown that the natural map

$$j_B : B \to \operatorname{Hom}_B(E^\cdot(B), E^\cdot(B))$$

induces an isomorphism in cohomology. To this end consider the natural map $i_B : B \to E^\cdot(B)$. It induces an isomorphism of complexes. That is, the mapping cone $M(i_B)$ is exact. Therefore

$$\operatorname{Hom}_B(M(i_B), E^\cdot(B)) = M(\operatorname{Hom}_B(i_B, E^\cdot(B)))$$

is also an exact complex. Hence

$$\operatorname{Hom}_B(i_B, E^\cdot(B)) : \operatorname{Hom}_B(E^\cdot(B), E^\cdot(B)) \to E^\cdot(B)$$

induces an isomorphism in cohomology. Now it is easy to check that the composition of the homomorphisms

$$B \to \operatorname{Hom}_B(E^\cdot(B), E^\cdot(B)) \to E^\cdot(B)$$

is just $i_B$. Therefore $j_B$ induces an isomorphism in cohomology. Moreover

$$\operatorname{Hom}_B(E^\cdot(B), E^\cdot(B))$$

is a complex of flat $B$-modules. Therefore the natural map $j_B$ induces a homomorphism of complexes $M \otimes_B B \to M \otimes \operatorname{Hom}_B(E^{\cdot}(B), E^{\cdot}(B))$ which induces – by the same argument as above – an isomorphism in cohomology. $\square$

By his recent result on Macaulayfications, see [25], T. Kawasaki proved the converse of 1.11, namely, that $A$ is the quotient of a Gorenstein ring provided $A$ posseses a dualizing complex.

In the following there is a characterization when a certain complex is a dualizing complex. To this end recall the following induction procedure well-suited to homological arguments.

**Proposition 1.12.** *Let $\mathfrak{P}$ denote a property of finitely generated $A$-modules, where $(A, \mathfrak{m}, k)$ denotes a local ring with residue field $k$. Suppose that $\mathfrak{P}$ satisfies the following properties:*

a) *The residue field $k$ has $\mathfrak{P}$.*
b) *If $0 \to M' \to M \to M'' \to 0$ denotes a short exact sequence of finitely generated $A$-modules such that $M'$ and $M''$ have $\mathfrak{P}$, then so does $M$.*
c) *If $x$ is an $M$-regular element such that $M/xM$ has $\mathfrak{P}$, then so does $M$.*

*Then any finitely generated $A$-module $M$ has $\mathfrak{P}$.*

The proof is easy. For the details see L. L. Avramov's notes [3, Proposition 0.0.9] in this volume. In the following we will use this arguments in order to sketch the characterization of dualizing complexes.

**Theorem 1.13.** *Let $D$ denote a bounded complex of injective $A$-modules. Assume that $D$ has finitely generated cohomology modules. Then $D$ is a dualizing complex if and only if*

$$H^n(\operatorname{Hom}_A(k, D)) \simeq \begin{cases} 0 \textit{ for} & n \neq t \\ k \textit{ for} & n = t \end{cases}$$

*for a certain integer $t \in \mathbb{Z}$.*

*Proof.* Suppose that $D$ is a dualizing complex. Then – by definition – the natural homomorphism

$$k \to \operatorname{Hom}_A(\operatorname{Hom}_A(k, D), D))$$

induces an isomorphism in cohomology. Furthermore $\operatorname{Hom}_A(k, D)$ is a complex consisting of $k$-vector spaces and whose cohomology modules are finite dimensional $k$-vector spaces. For $i \in \mathbb{Z}$ let $H^i = H^i(\operatorname{Hom}_A(k, D))$ and $h_i = \dim_k H_i$. Then it is easy to see that that there is an isomorphism of complexes $H^{\cdot} \xrightarrow{\sim} \operatorname{Hom}_A(k, D)$, where $H^{\cdot}$ denotes the complex consisting of $H^i$ and the zero homomorphisms as boundary maps. Then for $n \in \mathbb{Z}$ it follows that

$$\dim_k H^n(\operatorname{Hom}_A(\operatorname{Hom}_A(k, D), D)) = \sum_{i \in \mathbb{Z}} h_i h_{i+n}.$$

As easily seen this implies the existence of an integer $t \in \mathbb{Z}$ such that $h_t = 1$ and $h_i = 0$ for any $i \neq t$.

In order to prove the converse one has to show that the natural homomorphism

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D), D)$$

induces an isomorphism in cohomology for any finitely generated $A$-module $M$. To this end we proceed by 1.12. By the assumption it follows immediately that a) is true. In order to prove b) recall that $\operatorname{Hom}_A(\operatorname{Hom}_A(\cdot, D), D)$ transforms short exact sequences into short exact sequences of complexes. This holds because $D$ is a bounded complex of injective $A$-modules. Finally c) is true because the cohomology modules of $\operatorname{Hom}_A(\operatorname{Hom}_A(M, D), D)$ are finitely generated $A$-modules. Then one might apply Nakayama's Lemma. □

In the case the integer $t$ in 1.13 is equal to zero call $D$ a normalized dualizing complex.

It is noteworthy to say that 1.12 does not apply for the proof of 1.6. In general it will be not true that the complex $\operatorname{Hom}(M, D^{\cdot}_{\underline{x}})$ has finitely generated cohomology. So the Nakayama Lemma does not apply in proving condition c) in 1.12.

Under the previous assumptions on $A$ and $B$ with $r = \dim B$ let $D(M)$ denote the complex $\operatorname{Hom}_A(M, D_A)$, where $M$ denotes a finitely generated $A$-module $M$. Then there is an isomorphism $D(M) \xrightarrow{\sim} \operatorname{Hom}_B(M, E^{\cdot}(M))$. Therefore

$$H^n(D(M)) \simeq \operatorname{Ext}^n_B(M, B) \text{ for all } n \in \mathbb{Z}.$$

This implies $H^{r-d}(D(M)) \simeq K_M$ and $H^{r-n}(D(M)) \simeq K^n_M$ for all $n \neq d = \dim M$. Because of $D(M)^n = 0$ for all $n < r - d$ there is a natural homomorphism of complexes

$$i_M : K_M[d - r] \to D(M),$$

where $K_M$ is considered as a complex concentrated in homological degree zero. So the mapping cone $M(i_M)$ provides a short exact sequence of complexes

$$0 \to D(M)[-1] \to M(i_M) \to K_M[d - r] \to 0.$$

Therefore we see that $H^{r-n+1}(M(i_M)) \simeq K^n_M$ for all $0 \leq n < \dim M$ and $H^{r-n+1}(M(i_M)) = 0$ for all $n < 0$ and all $n \geq \dim M$. By applying the functor $D(\cdot) := \operatorname{Hom}_A(\cdot, D_A)$ it induces a short exact sequence of complexes

$$0 \to D(K_M)[r - d] \to D(M(i_M)) \to D^2(M)[1] \to 0.$$

Recall that $D_A$ is a complex consisting of injective $A$-modules. By 1.11 and the definition of $K_{K_M}$ this yields an exact sequence

$$0 \to H^{-1}(D(M(i_M))) \to M \xrightarrow{\tau_M} K_{K_M} \to H^0(D(M(i_M))) \to 0$$

and isomorphisms $H^n(D(M(i_M))) \simeq K^{d-n}_{K_M}$ for all $n \geq 1$.

Note that in the particular case of $M = A$ the homomorphism $\tau_A$ coincides with the natural map

$$A \to \operatorname{Hom}_A(K_A, K_A),\; a \mapsto f_a,$$

of the ring into the endomorphism ring of its canonical module. Here $f_a$ denotes the multiplication map by $a$.

**Theorem 1.14.** *Let $M$ denote a finitely generated, equidimensional $A$-module with $d = \dim M$, where $A$ is a factor ring of a Gorenstein ring. Then for an integer $k \geq 1$ the following statements are equivalent:*

(i) *$M$ satisfies condition $S_k$.*

(ii) *The natural map $\tau_M : M \to K_{K_M}$ is bijective (resp. injective for $k = 1$) and $H^n_{\mathfrak{m}}(K_M) = 0$ for all $d - k + 2 \leq n < d$.*

*Proof.* First recall that $H^n(D(M(i_M))) \simeq K^{d-n}_{K_M}$ for all $n \geq 1$. By the local duality it follows that $T(K^n_{K_M}) \simeq H^n_{\mathfrak{m}}(K_M)$ for all $n \in \mathbb{Z}$. By virtue of the short exact sequence and the above isomorphisms the statement in (ii) is equivalent to

(iii) $\quad H^n(D(M(i_M))) = 0$ *for all* $-1 \leq n < k - 1$.

Next show that (i) $\Rightarrow$ (iii). First recall that $D(M(i_M)) \xrightarrow{\sim} \operatorname{Hom}_B(M(i_M), E^{\cdot}(B))$. Then there is the following spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_B(H^{-j}(M(i_M)), B) \Rightarrow E^n = H^n(D(M(i_M)))$$

in order to compute $H^n(D(M(i_M)))$. Moreover it follows that

$$H^{-j}(M(i_M)) \simeq \begin{cases} K_M^{r+j+1} & \text{for } 0 \leq r + j + 1 < \dim M \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

By the assumptions and 1.9 it implies $\dim K_M^{r+j+1} \leq r + j + 1 - k$ for all $j \in \mathbb{Z}$. As a consequence of 1.8 it turns out that $E_2^{i,j} = 0$ for all $i, j \in \mathbb{Z}$ satisfying $i + j < k - 1$. That is the spectral sequence proves the condition (iii).

In order to prove the reverse conclusion first note that $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$ for all prime ideals $\mathfrak{p} \in \operatorname{Supp} M$ since $M$ is equi-dimensional. Then by 1.9 it follows easily

$$D(M(i_M)) \otimes_A A_{\mathfrak{p}} \xrightarrow{\sim} D(M(i_{M_{\mathfrak{p}}})).$$

This means that the claim is a local question. So by induction we have to show that $\operatorname{depth} M \geq k$. By induction hypothesis we know that $\dim K_M^j \leq j - k$ for all $j > k$ and $\dim K_M^j \leq 0$ for all $0 \leq j \leq k$. Then the above spectral sequences degenerates partially to the isomorphisms $\operatorname{Ext}^r_B(K_M^{j+1}, B) \simeq H^j(D(M(i_M)))$ for all $j < k - 1$. Recall that $\operatorname{Ext}^r_B(N, B)$ is the onliest possible non-vanishing Ext module for $N$ a $B$-module of finite length. By the local duality this implies $H^{j+1}_{\mathfrak{m}}(M) = 0$ for all $j < k - 1$ and $\operatorname{depth}_A M \geq k$, as required. $\square$

It turns out that for a module $M$ satisfying $S_2$ the natural map $\tau_M : M \to K_{K_M}$ is an isomorphism. In the case of the canonical module $K_A$ this means that the endomorphism ring of $K_A$ is isomorphic to $A$ if and only if $A$ satisfies $S_2$. The previous result has a dual statement which characterizes the vanishing of the local cohomology modules below the dimension of the module.

**Corollary 1.15.** *With the notation of 1.14 suppose that the $A$-module $M$ satisfies the condition $S_2$. For an integer $k \geq 2$ the following conditions are equivalent:*

(i) $K_M$ *satifies condition* $S_k$.
(ii) $H^n_{\mathfrak{m}}(M) = 0$ *for all* $d - k + 2 \leq n < d$.

*Proof.* This is just a consequence of 1.14 and the remark that $\tau_M : M \to K_{K_M}$ is an isomorphism. ☐

There are several further applications of these vanishing results in the case $A$ is a quasi-Gorenstein ring or in liaison. For the details compare [41]. We conclude with one of them, a Cohen-Macaulay characterization for a quasi-Gorenstein ring. To this end let us call a local ring $(A, \mathfrak{m})$ that is a quotient of a Gorenstein ring $B$ a quasi-Gorentein ring, provided $K_A \simeq A$. Note that a Cohen-Macaulay quasi-Gorenstein ring is a Gorenstein ring. A local factorial ring that is a quotient of a Gorenstein ring is a quasi-Gorenstein ring, see [32].

**Theorem 1.16.** *Let $(A, \mathfrak{m})$ denote a quasi-Gorenstein ring such that*

$$\operatorname{depth} A_{\mathfrak{p}} \geq \min\{\dim A, \frac{1}{2} \dim A_{\mathfrak{p}} + 1\} \text{ for all } \mathfrak{p} \in \operatorname{Spec} A.$$

*Then $A$ is a Gorenstein ring.*

*Proof.* Let $d = \dim A$. It is enough to show that $A$ is a Cohen-Macaulay ring. This is true for $d \leq 3$ by the assumption. By induction $A_{\mathfrak{p}}$ is a Cohen-Macaulay ring for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$. Therefore by the assumption the local ring $(A, \mathfrak{m})$ satisfies the condition $S_k$ for $k \geq \frac{1}{2} \dim A + 1$. Because of $K_A \simeq A$ the result 1.15 implies that $H^n_{\mathfrak{m}}(A) = 0$ for all $n < \dim A$, which proves the Cohen-Macaulayness of $A$. ☐

The previous result says something about the difficulty to construct non-Cohen-Macaulay factorial domains. If such a ring is 'half way' Cohen-Macaulay it is a Cohen-Macaulay ring. Originally this result was proved by R. Hartshorne and A. Ogus, see [17].

1.3. **Cohomological Annihilators.** Vanishing results on local cohomology modules provide strong information. More subtle information comes from consideration of their annihilators. This will be sampled in this subsection.

To this end we have to use a certain generalization of the notion of a regular sequence. First let us summarize basic facts about filter regular sequences. Let $M$ denote a finitely generated $A$-module over $(A, \mathfrak{m})$, a local Noetherian ring.

A system of elements $\underline{x} = x_1, \ldots, x_r \subseteq \mathfrak{m}$ is called a filter regular sequence of $M$ (or $M$-filter regular sequence), if

$$x_i \notin \mathfrak{p} \quad \text{for all } \mathfrak{p} \in (\operatorname{Ass} M/(x_1, \ldots, x_{i-1})M) \setminus \{\mathfrak{m}\}$$

for all $i = 1, \ldots, r$. This is equivalent to saying that the $A$-modules

$$(x_1, \ldots, x_{i-1})M : x_i/(x_1, \ldots, x_{i-1})M, \quad i = 1, \ldots, r,$$

are of finite length. Moreover $\underline{x}$ is an $M$-filter regular sequence if and only if $\{\frac{x_1}{1}, \ldots, \frac{x_i}{1}\} \in A_\mathfrak{p}$ is an $M_\mathfrak{p}$-regular sequence for all $\mathfrak{p} \in (V(x_1, \ldots, x_i) \cap \operatorname{Supp} M) \setminus \{\mathfrak{m}\}$ and $i = 1, \ldots, r$.

**Lemma 1.17.** *Let $M$ denote a finitely generated $A$-module. Suppose that $\underline{x} = x_1, \ldots, x_r$ denotes an $M$-filter regular sequence.*

a) *$H^i(\underline{x}; M)$ is an $A$-module of finite length for all $i < r$.*
b) *$H_i(\underline{x}; M)$ is an $A$-module of finite length for all $i > 0$.*
c) *$\operatorname{Supp} H^i_\mathfrak{c}(M) \subseteq V(\mathfrak{m})$ for all $0 \leq i < r$, where $\mathfrak{c} = (x_1, \ldots, x_r)A$.*

*Proof.* Because of the self duality of the Koszul complexes it will be enough to prove one of the first two statements. Now note that

$$\operatorname{Supp} H^i(\underline{x}; M) \subseteq V(\underline{x}) \cap \operatorname{Supp} M, \quad i \in \mathbb{Z}.$$

On the other hand $\underline{x}$ is an $M$-regular sequence if and only if $H^i(\underline{x}; M) = 0$ for all $i < r$. Then the result a) follows by a localization argument of the Koszul complexes. In order to prove c) note that $\operatorname{Supp} H^i_\mathfrak{c}(M) \subseteq V(\underline{x}) \cap \operatorname{Supp} M$. Let $\mathfrak{p}$ be a non-maximal prime ideal in $V(\underline{x}) \cap \operatorname{Supp} M$. By a localization argument it follows that

$$H^i_\mathfrak{c}(M) \otimes_A A_\mathfrak{p} \simeq H^i_{\mathfrak{c}A_\mathfrak{p}}(M_\mathfrak{p}) = 0 \quad \text{for } i < r,$$

since $\{\frac{x_1}{1}, \ldots, \frac{x_r}{1}\}$ is an $M_\mathfrak{p}$-regular sequence. □

Let $M$ denote a finitely generated $A$-module. Let $\mathfrak{a}$ denote an ideal of $(A, \mathfrak{m})$. The vanishing resp. non-vanishing of the local cohomology modules $H^n_\mathfrak{a}(M)$ provides useful local information on $M$. For a more subtle consideration the annihilators of $H^n_\mathfrak{a}(M)$ are of some interest. For a finitely generated $A$-module $M$ let

$$\mathfrak{a}_n(M) := \operatorname{Ann}_A H^n_\mathfrak{a}(M), \ n \in \mathbb{Z},$$

denote the $n$-th cohomological annihilator of $M$ with respect to $\mathfrak{a}$.

Now we relate the cohomological annihilators of $M$ to those of $M$ modulo a bunch of generic hyperplane sections.

**Theorem 1.18.** *Let $\underline{x} = x_1, \dots, x_r$ denote an $M$-filter regular sequence. Then*
$$\mathfrak{a}_n(M) \cdot \ldots \cdot \mathfrak{a}_{n+r}(M) \subseteq \mathfrak{a}_n(M/\underline{x}M)$$
*for all integers $n$.*

*Proof.* Let $K^{\cdot} := K^{\cdot}_{\underline{y}}$ denote the Čech complex of $A$ with respect to a system of generators $\underline{y} = y_1, \dots, y_s$ of the ideal $\mathfrak{a}$. Let $K^{\cdot}(\underline{x}; A) \otimes_A M$ be the Koszul co-complex of $M$ with respect to $\underline{x}$. Put
$$C^{\cdot} := (K^{\cdot} \otimes_A M) \otimes_A K^{\cdot}(\underline{x}; A) \simeq K^{\cdot} \otimes_A K^{\cdot}(\underline{x}; M).$$
There are two spectral sequences for computing the cohomology of $C^{\cdot}$. First consider
$$E_2^{ij} = H^i(K^{\cdot} \otimes_A H^j(\underline{x}; M)) \Rightarrow E^{i+j} = H^{i+j}(C^{\cdot}).$$
Note that $H^i(K^{\cdot} \otimes_A N) \simeq H^i_{\mathfrak{a}}(N)$, $i \in \mathbb{Z}$, for a finitely generated $A$-module $N$, see 1.3. Therefore
$$E_2^{ij} \simeq H^i_{\mathfrak{a}}(H^j(\underline{x}; M)) \quad \text{for all } i, j \in \mathbb{Z}.$$
By 1.17 the $A$-modules $H^i(\underline{x}; M)$, $i < r$, are of finite length. So there are the following isomorphisms
$$E_2^{ij} \simeq \begin{cases} 0 & \text{for } i \neq 0 \quad \text{and } j \neq r, \\ H^j(\underline{x}; M) & \text{for } i = 0 \quad \text{and } j \neq r, \\ H^i_{\mathfrak{a}}(M/\underline{x}M) & \text{for } j = r. \end{cases}$$
To this end recall that $H^r(\underline{x}; M) \simeq M/\underline{x}M$. By virtue of the spectral sequence it turns out that
$$E_\infty^{ij} = 0 \quad \text{for all } i \neq 0, \quad j \neq r.$$
Because of the subsequent stages of the spectral sequence
$$E_k^{i-k, r+k-1} \to E_k^{ir} \to E_k^{i+k, r-k+1}$$
and $E_k^{i-k,r+k-1} = E_k^{i+k,r-k+1} = 0$ for all $k \geq 2$ it yields that $E_\infty^{ir} \simeq H^i_{\mathfrak{a}}(M/\underline{x}M)$. By a similar consideration we obtain that $E_\infty^{0j} \simeq H^j(\underline{x}; M)$ for all $j \neq r$. Therefore there are the following isomorphisms
$$H^i(C^{\cdot}) \simeq \begin{cases} H^i(\underline{x}; M) & \text{for } 0 \leq i < r, \\ H^{i-r}_{\mathfrak{a}}(M/\underline{x}M) & \text{for } r \leq i \leq d, \\ 0 & \text{otherwise,} \end{cases}$$
where $d = \dim M$. On the other hand there is the spectral sequence
$$'E_2^{ij} = H^j(K^{\cdot}(\underline{x}; A) \otimes_A H^i_{\mathfrak{a}}(M)) \Rightarrow {'E}^{i+j} = H^{i+j}(C^{\cdot}).$$
Because of $'E_2^{ij} = H^j(\underline{x}; H^i_{\mathfrak{a}}(M))$ it follows that $'E_2^{ij} = 0$ for all $j < 0$ and $j > r$. By the construction of the Koszul complex $'E_2^{ij}$ is a subquotient of the direct sum of copies of $H^i_{\mathfrak{a}}(M)$. Therefore $\mathfrak{a}_i(M)('E_2^{ij}) = 0$ for all $i, j \in \mathbb{Z}$. Whence it implies

that $\mathfrak{a}_i(M)('E_\infty^{ij}) = 0$ for all $i, j \in \mathbb{Z}$. By view of the filtration of $H^{i+j}(C^{\cdot})$ defined by $'E_\infty^{ij}$ it follows that

$$\begin{aligned} \mathfrak{a}_0(M) \cdot \ldots \cdot \mathfrak{a}_i(M) H^i(C^{\cdot}) &= 0 \quad \text{for } 0 \le i < r \text{ and} \\ \mathfrak{a}_{i-r}(M) \cdot \ldots \cdot \mathfrak{a}_i(M) H^i(C^{\cdot}) &= 0 \quad \text{for } r \le i \le d. \end{aligned}$$

Hence, the above computation of $H^i(C^{\cdot})$ proves the claim. □

For a filter regular sequence $\underline{x} = x_1, \ldots, x_r$ the proof of Theorem 3.3 provides that $\mathfrak{a}_0(M) \cdot \ldots \cdot \mathfrak{a}_i(M) H^i(x_1, \ldots, x_r; M) = 0$ for all $i < r$. Because of the finite length of $H^i(\underline{x}; M)$ for all $i < r$ this is a particular case of the results shown in [41]. Moreover the notion of $M$-filter regular sequences provides an interesting expression of the local cohomology modules of $M$.

**Lemma 1.19.** *Let $\underline{x} = x_1, \ldots, x_r$ be an $M$-filter regular sequence contained in $\mathfrak{a}$. Put $\mathfrak{c} = (x_1, \ldots, x_r)A$. Then there are the following isomorphisms*

$$H^i_{\mathfrak{a}}(M) \simeq \begin{cases} H^i_{\mathfrak{c}}(M) & \text{for } 0 \le i < r, \\ H^{i-r}_{\mathfrak{a}}(H^r_{\mathfrak{c}}(M)) & \text{for } r \le i \le d, \end{cases}$$

*where $d = \dim_A M$.*

*Proof.* Consider the spectral sequence

$$E_2^{ij} = H^i_{\mathfrak{a}}(H^j_{\mathfrak{c}}(M)) \Rightarrow E^{i+j} = H^{i+j}_{\mathfrak{a}}(M).$$

By 1.17 we have that $\operatorname{Supp} H^j_{\mathfrak{c}}(M) \subseteq V(\mathfrak{m})$ for all $j < r$. Whence $E_2^{ij} = 0$ for all $i \neq 0$ and $j \neq r$. Furthermore, $E_2^{0j} = H^j_{\mathfrak{c}}(M)$ for $j \neq r$ and $E_2^{ir} = H^i_{\mathfrak{a}}(H^r_{\mathfrak{c}}(M))$. An argument similar to that of the proof given in 1.18 yields that

$$E_\infty^{0j} \simeq H^j_{\mathfrak{c}}(M) \quad \text{and} \quad E_\infty^{ir} \simeq H^i_{\mathfrak{a}}(H^r_{\mathfrak{c}}(M)).$$

Because of $E_\infty^{0j} = 0$ for $j > r$ the spectral sequence proves the claim. □

Let $\underline{x} = x_1, \ldots, x_r$ be a system of elements of $A$. For the following results put $\underline{x}^{(k)} = x_1^k, \ldots, x_r^k$ for an integer $k \in \mathbb{N}$.

**Corollary 1.20.** *Let $\underline{x} = x_1, \ldots, x_r$ be an $M$-filter regular sequence contained in $\mathfrak{a}$. The multiplication by $x_1 \cdots x_r$ induces a direct system $\{H^i_{\mathfrak{a}}(M/\underline{x}^{(k)}M)\}_{k \in \mathbb{N}}$, such that*

$$H^{i+r}_{\mathfrak{a}}(M) \simeq \varinjlim H^i_{\mathfrak{a}}(M/\underline{x}^{(k)}M)$$

*for all $i \ge 0$.*

*Proof.* There is a direct system $\{M/\underline{x}^{(k)}M\}_{k \in \mathbb{N}}$ with homomorphisms induced by the multiplication by $x_1 \cdots x_r$. By [13] there is an isomorphism

$$H^r_{\mathfrak{c}}(M) \simeq \varinjlim M/\underline{x}^{(k)}M.$$

Then the claim follows by 1.19 since the local cohomology commutes with direct limits. $\square$

In order to produce an 'upper' approximation of $\mathfrak{a}_i(M/\underline{x}M)$, $\underline{x} = x_1, \ldots, x_r$, an $M$-filter regular sequence, a few preliminaries are necessary. For a given $i$ and $j = 0, 1, \ldots, r$ set

$$\mathfrak{a}_{ij}(\underline{x}; M) = \bigcap_{k_1,\ldots,k_j \geq 1} \mathfrak{a}_i(M/(x_1^{k_1}, \ldots, x_j^{k_j})M).$$

Furthermore define $\mathfrak{a}_i(\underline{x}; M) = \bigcap_{j=0}^{r} \mathfrak{a}_{ij}(\underline{x}; M)$. The next result relates the cohomological annihilators of $M$ to those of $M/\underline{x}M$.

**Corollary 1.21.** *Let $\underline{x} = x_1, \ldots, x_r$ be an $M$-filter regular sequence contained in $\mathfrak{a}$. Then*

$$\mathfrak{a}_i(M) \cdot \ldots \cdot \mathfrak{a}_{i+r}(M) \subseteq \mathfrak{a}_i(\underline{x}; M) \subseteq \mathfrak{a}_i(M) \cap \ldots \cap \mathfrak{a}_{i+r}(M)$$

*for all $0 \leq i \leq d - r$. In particular, $\mathfrak{a}_i(\underline{x}; M)$ and $\mathfrak{a}_i(M) \cap \ldots \cap \mathfrak{a}_{i+r}(M)$ have the same radical.*

*Proof.* By 1.18 it follows that $\mathfrak{a}_i(M) \cdot \ldots \cdot \mathfrak{a}_{i+j}(M) \subseteq \mathfrak{a}_{ij}(\underline{x}; M)$ for $j = 0, 1, \ldots, r$. Recall that $x_1^{k_1}, \ldots, x_j^{k_j}$ forms an $M$-filter regular sequence, provided that $\underline{x} = x_1, \ldots, x_r$ is an $M$-filter regular sequence. Whence the first inclusion is true. Moreover, by 1.20 it yields that

$$\mathfrak{a}_i(\underline{x}; M) \subseteq \mathfrak{a}_{ij}(\underline{x}; M) \subseteq \mathfrak{a}_{i+j}(M)$$

for all $j = 0, 1, \ldots, r$. This proves the second containment relation. $\square$

The results of this section generalize those for the cohomological annihilators $\mathfrak{m}_n(M)$ of $H^n_{\mathfrak{m}}(M)$, $n \in \mathbb{Z}$, investigated in [41].

## 2. A Few Applications of Local Cohomology

2.1. **On Ideal Topologies.** Let $S$ denote a multiplicatively closed set of a Noetherian ring $A$. For an ideal $\mathfrak{a}$ of $A$ put $\mathfrak{a}_S = \mathfrak{a}A_S \cap A$. For an integer $n \in \mathbb{N}$ let $\mathfrak{a}_S^{(n)} = \mathfrak{a}^n A_S \cap A$ denote the $n$-th symbolic power of $\mathfrak{a}$ with respect to $S$. Note that this generalizes the notion of the $n$-th symbolic power $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$ of a prime ideal $\mathfrak{p}$ of $A$. The ideal $\mathfrak{a}_S$ is the so-called $S$-component of $\mathfrak{a}$, i.e.

$$\mathfrak{a}_S = \{r \in A \mid rs \in \mathfrak{a} \text{ for some } s \in S\}.$$

So the primary decomposition of $\mathfrak{a}_S$ consists of the intersection of all primary components of $\mathfrak{a}$ that do not meet $S$. In other words

$$\operatorname{Ass}_A A/\mathfrak{a}_S = \{\mathfrak{p} \in \operatorname{Ass}_A A/\mathfrak{a} \mid \mathfrak{p} \cap S = \emptyset\}.$$

Moreover it is easily seen that
$$\operatorname{Ass}_A \mathfrak{a}_S/\mathfrak{a} = \{\mathfrak{p} \in \operatorname{Ass}_A A/\mathfrak{a} \mid \mathfrak{p} \cap S \neq \emptyset\}.$$
However $\operatorname{Supp}_A \mathfrak{a}_S/\mathfrak{a} \subseteq V(\mathfrak{b})$, where $\mathfrak{b} = \prod_{\mathfrak{p}\in\operatorname{Ass}(\mathfrak{a}_S/\mathfrak{a})} \mathfrak{p}$. Whence it turns out that $\mathfrak{a}_S = \mathfrak{a} :_A \langle \mathfrak{b} \rangle$, where the last colon ideal denotes the stable value of the ascending chain of ideals
$$\mathfrak{a} \subseteq \mathfrak{a} :_A \mathfrak{b} \subseteq \mathfrak{a} :_A \mathfrak{b}^2 \subseteq \ldots .$$
Obviously $\mathfrak{a} :_A \langle \mathfrak{b} \rangle = \mathfrak{a} :_A \langle \mathfrak{b}' \rangle$ for two ideals $\mathfrak{b}, \mathfrak{b}'$ with the same radical. On the other hand for two ideals $\mathfrak{a}, \mathfrak{b}$ it follows that
$$\mathfrak{a} :_A \langle \mathfrak{b} \rangle = \mathfrak{a}_S, \quad \text{where} \quad S = \cap_{\mathfrak{p}\in\operatorname{Ass} A/\mathfrak{a}\setminus V(\mathfrak{b})} A \setminus \mathfrak{p}.$$
There is a deep interest in comparing the topology defined by $\{\mathfrak{a}_S^{(n)}\}_{n\in\mathbb{N}}$ with the $\mathfrak{a}$-adic topology. To this end we shall use the following variation of Chevalley's theorem, see [35].

**Theorem 2.1.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(A, \mathfrak{m})$. Let $\{\mathfrak{b}_n\}_{n\in\mathbb{N}}$ denote a descending sequence of ideals. Suppose that the following conditions are satisfied:*

a) *$A$ is $\mathfrak{a}$-adically complete,*
b) *$\cap_{n\in\mathbb{N}} \mathfrak{b}_n = (0)$, i.e. the filtration is separated, and*
c) *for all $m \in \mathbb{N}$ the family of ideals $\{\mathfrak{b}_n A/\mathfrak{a}^m\}_{n\in\mathbb{N}}$ satisfies the descending chain condition.*

*Then for any $m \in \mathbb{N}$ there exists an integer $n = n(m)$ such $\mathfrak{b}_n \subset \mathfrak{a}^m$.*

*Proof.* The assumption in c) guarantees that for any given $m \in \mathbb{N}$ there is an integer $n = n(m)$ such that
$$\mathfrak{b}_n + \mathfrak{a}^m = \mathfrak{b}_{n+k} + \mathfrak{a}^m \text{ for all } k \geq 1.$$
Call the ideal at the stable value $\mathfrak{c}_m$. Now suppose the conclusion is not true, i.e. $\mathfrak{b}_n \not\subseteq \mathfrak{a}^m$ for all $n \in \mathbb{N}$ and a fixed $m \in \mathbb{N}$. Therefore $\mathfrak{c}_m \neq \mathfrak{a}^m$. Moreover it is easily seen that $\mathfrak{c}_{m+1} + \mathfrak{a}^m = \mathfrak{c}_m$. Now construct inductively a series $(x_m)_{m\in\mathbb{N}}$ satisfying the following properties
$$x_m \in \mathfrak{c}_m \setminus \mathfrak{a}^m \text{ and } x_{m+1} \equiv x_m \mod \mathfrak{a}^m.$$
Therefore $(x_m)_{m\in\mathbb{N}}$ is a convergent series with a limit $0 \neq x \in A$. This follows since $A$ is $\mathfrak{a}$-adically complete by a). By definition that means for any $m \in \mathbb{N}$ there exists an $l \in \mathbb{N}$ such that $x - x_n \in \mathfrak{a}^m$ for all $n \geq l = l(m)$. Because of $x_n \in \mathfrak{c}_n$ this provides that $x \in \cap_{m\in\mathbb{N}} \cap_{n\in\mathbb{N}} (\mathfrak{c}_n + \mathfrak{a}^m)$. By Krull's Intersection Theorem and assumption b) it follows $x = 0$, a contradiction. $\square$

As a first application compare the $\mathfrak{a}$-adic topology with the topology derived by cutting the $\mathfrak{m}$-torsion of the powers of $\mathfrak{a}$.

**Corollary 2.2.** *For an ideal $\mathfrak{a}$ of a local ring $(A, \mathfrak{m})$ the following conditions are equivalent:*

(i) $\{\mathfrak{a}^n :_A \langle \mathfrak{m} \rangle\}_{n\in\mathbb{N}}$ *is equivalent to the $\mathfrak{a}$-adic topology.*

(ii) $\cap_{n\in\mathbb{N}}(\mathfrak{a}^n \widehat{A} : \langle \mathfrak{m}\widehat{A} \rangle) = 0$, *where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of $A$.*

(iii) $\operatorname{height}(\mathfrak{a}\widehat{A} + \mathfrak{p}/\mathfrak{p}) < \dim \widehat{A}/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$.

*Proof.* Without loss of generality we may assume that $A$ is a complete local ring. The conclusion (i) $\Rightarrow$ (ii) is obviously true by Krull's Intersection Theorem. Let us prove (ii) $\Rightarrow$ (iii). Suppose there is a $\mathfrak{p} \in \operatorname{Ass} A$ such that $\mathfrak{a} + \mathfrak{p}$ is $\mathfrak{m}$-primary. Then

$$0 \neq 0 : \langle \mathfrak{p} \rangle \subseteq \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{p} \rangle) = \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{p} + \mathfrak{a} \rangle) = \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle),$$

a contradiction. Finally we prove the implication (iii) $\Rightarrow$ (i).

First note that because $A$ is complete it is also $\mathfrak{a}$-adically complete. Moreover for a given $m \in \mathbb{N}$ the sequence $\{(\mathfrak{a}^n : \langle \mathfrak{m} \rangle)A/\mathfrak{a}^m\}_{n\in\mathbb{N}}$ satisfies the descending chain condition. Note that $((\mathfrak{a}^n : \langle \mathfrak{m} \rangle) + \mathfrak{a}^m)/\mathfrak{a}^m$ is a module of finite length for all large $n \in \mathbb{N}$. Suppose that (i) is not true. By virtue of 2.1 this means that $0 \neq \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle)$, since the conditions a) and c) are satisfied.

Now choose

$$\mathfrak{p} \in \operatorname{Ass}_A(\cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle))$$

an associated prime ideal. Then $\mathfrak{p} = 0 :_A x$ for some $0 \neq x \in \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle)$. Therefore $\mathfrak{p} \in \operatorname{Ass} A$.

By the Artin-Rees Lemma there exists a $k \in \mathbb{N}$ such that $\mathfrak{a}^k \cap xA \subseteq x\mathfrak{a}$. By the choice of $x$ there is an integer $l \in \mathbb{N}$ such that $\mathfrak{m}^l x \subseteq \mathfrak{a}^k$. Therefore

$$\mathfrak{m}^l x \subseteq \mathfrak{a}^k \cap xA \subseteq x\mathfrak{a},$$

which implies $\mathfrak{m}^l \subseteq \mathfrak{a} + \mathfrak{p}$, in contradiction to assumption (iii). $\square$

A remarkle improvement of 2.2 was shown by I. Swanson, see [47]. Under the equivalent conditions of 2.2 she proved the existence of a $k \in \mathbb{N}$ such that $\mathfrak{a}^{nk} :_A \langle \mathfrak{m} \rangle \subseteq \mathfrak{a}^n$ for all $n \in \mathbb{N}$.

In the following let us describe the obstruction for the equivalence of both of the topologies considered in 2.2. To this end let $u(\mathfrak{a})$ denote the intersection of those primary components $\mathfrak{q}$ of 0 in $A$ such that the associated prime ideal $\mathfrak{p}$ satisfies $\dim A/(\mathfrak{a} + \mathfrak{p}) > 0$.

**Proposition 2.3.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(A, \mathfrak{m})$. Then it follows that $u(\mathfrak{a}) = \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle)$.*

*Proof.* Let $x \in \cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle)$ be an arbitrary element. Then it is esily seen that $\frac{x}{1} \in \cap_{n\in\mathbb{N}}\mathfrak{a}^n A_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \in V(\mathfrak{a}) \setminus \{\mathfrak{m}\}$. That is $x \in 0_{\mathfrak{p}}$ for every $\mathfrak{p} \in V(\mathfrak{a}) \setminus \{\mathfrak{m}\}$. By taking the intersection over all those prime ideals it follows $x \in u(\mathfrak{a})$. But this means $\cap_{n\in\mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{m} \rangle) \subseteq u(\mathfrak{a})$.

In order to prove the converse containment relation let $\mathfrak{c} = \prod \mathfrak{p}$, where the product is taken over all prime ideals $\mathfrak{p} \in \operatorname{Ass} A$ such that $\dim A/(\mathfrak{a} + \mathfrak{p}) = 0$. Then $u(\mathfrak{a}) = 0 :_A \langle \mathfrak{c} \rangle$ and

$$0 :_A \langle \mathfrak{c} \rangle \subseteq \cap_{n \in \mathbb{N}} (\mathfrak{a}^n : \langle \mathfrak{c} \rangle) = \cap_{n \in \mathbb{N}} (\mathfrak{a}^n : \langle \mathfrak{c} + \mathfrak{a} \rangle) = \cap_{n \in \mathbb{N}} (\mathfrak{a}^n : \langle \mathfrak{m} \rangle)$$

because of $\operatorname{Rad}(\mathfrak{c} + \mathfrak{a}) = \mathfrak{m}$, as is easily seen. $\square$

Now consider the case of a principal ideal, important for the applications in the following.

**Corollary 2.4.** *Let $\mathfrak{a}$ denote an ideal of a commutative Noetherian ring $A$. For a regular element $x \in \mathfrak{a}$ the following conditions are equivalent:*

(i) *$\{x^n A :_A \langle \mathfrak{a} \rangle\}_{n \in \mathbb{N}}$ is equivalent to the $xA$-adic topology.*
(ii) *$\dim \widehat{A_P}/\mathfrak{p} > 1$ for all $P \in \operatorname{Ass} A/xA \cap V(\mathfrak{a})$ and all $\mathfrak{p} \in \operatorname{Ass} \widehat{A_P}$.*

*Proof.* First prove the implication (i) $\Rightarrow$ (ii). Suppose that there are prime ideals $P \in \operatorname{Ass} A/xA \cap V(\mathfrak{a})$ and $\mathfrak{p} \in \operatorname{Ass} \widehat{A_P}$ such that $\dim \widehat{A_P}/\mathfrak{p} = 1$. Because $x$ is an $\widehat{A_P}$-regular element this means that $\dim \widehat{A_P}/(x\widehat{A_P} + \mathfrak{p}) = 0$. Note that $\frac{x}{1}$ is not a unit in $\widehat{A_P}$. Now replace $\widehat{A_P}$ by $A$. Then by 2.2 there is an $n \in \mathbb{N}$ such that $x^m A :_A \langle \mathfrak{m} \rangle \not\subseteq x^n A$ for all $m \geq n$. Therefore $x^m A :_A \langle \mathfrak{a} \rangle \not\subseteq x^n A$ for all $m \geq n$ since $x^m A :_A \langle \mathfrak{m} \rangle \subseteq x^m A :_A \langle \mathfrak{a} \rangle$. This contradicts the assumption in (i).

In order to prove that (ii) $\Rightarrow$(i) consider the ideals

$$E_{m,n} = (x^m A :_A \langle \mathfrak{a} \rangle + x^n A)/x^n A \subseteq A/x^n A$$

for a given $n$ and all $m \geq n$. Obviously $\operatorname{Ass}_A E_{m,n} \subseteq \operatorname{Ass}_A A/xA \cap V(\mathfrak{a})$. Moreover $E_{m+1,n} \subseteq E_{m,n}$. That means, for a fixed $n \in \mathbb{N}$ the set $\operatorname{Ass}_A E_{m,n}$ becomes an eventually stable set of prime ideals, say $X_n$. The claim will follow provided $X_n = \emptyset$. Suppose that $X_n \neq \emptyset$. By a localization argument and changing notation one might assume that $X_n = \{\mathfrak{m}\}$, the maximal ideal of a local ring $(A, \mathfrak{m})$. Therefore $\operatorname{Supp} E_{m,n} = V(\mathfrak{m})$ for any fixed $n \in \mathbb{N}$ and all large $m$. Whence

$$x^m A :_A \langle \mathfrak{a} \rangle \subseteq x^n A :_A \langle \mathfrak{m} \rangle$$

for a given $n$ and all large $m$. By assumption (ii) and 2.2 it follows that for a given $k$ there is an integer $n$ such that $x^n A :_A \langle \mathfrak{m} \rangle \subseteq x^k A$. Therefore $X_n = \emptyset$, a contradiction to the choice of $X_n$. $\square$

While condition (ii) looks rather technical one should try to simplify it under reasonable conditions on $A$. Say a local ring $(A, \mathfrak{m})$ satisfies condition (C) provided

$$\dim A/P = \dim \hat{A}/\mathfrak{p} \text{ for all } P \in \operatorname{Ass} A \text{ and all } \mathfrak{p} \in \operatorname{Ass} \widehat{A}/P\widehat{A}.$$

This is equivalent to saying that $A/P$ is an unmixed local ring for any prime ideal $P \in \operatorname{Ass} A$.

Say that a commutative Noetherian $A$ satisfies locally condition (C) provided any localization $A_{\mathfrak{p}}, \mathfrak{p} \in \operatorname{Spec} A$, satisfies condition (C). Let $A$ be an unmixed ring resp. a factor ring of a Cohen-Macaulay ring. Then it follows that $A$ satisfies locally (C), see [35]. In particular, by Cohen's Structure Theorem it turns out that a complete local ring satisfies locally condition (C).

**Proposition 2.5.** *Let* $(A, \mathfrak{m})$ *denote a local ring satisfying condition* (C)*. Then it satisfies also locally* (C).

*Proof.* By definition condition (C) implies that, for $P \in \operatorname{Ass} A$, $A/P$ is unmixed, i.e., $\dim \widehat{A}/P = \dim \widehat{A}/\mathfrak{p}\widehat{A}$ for all $\mathfrak{p} \in \operatorname{Ass} \widehat{A}/P\widehat{A}$. Now unmixedness localizes, i.e. for any $Q \in V(P)$ the local ring $A_Q/PA_Q$ is again unmixed, see [33]. Therefore $A_Q$ satisfies condition (C) for any $Q \in \operatorname{Spec} A$. Recall that a for prime ideal $Q \in V(P)$ we have $\mathfrak{p} \in \operatorname{Ass} A$ and $\mathfrak{p} \subseteq Q$ if and only if $\mathfrak{p}A_Q \in \operatorname{Ass} A_Q$. □

Now use the results about condition (C) in order to simplify the result in 2.4.

**Corollary 2.6.** *Let $A$ denote a commutative Noetherian ring satisfying locally the condition* (C)*. Let $x \in \mathfrak{a}$ be a regular element. Then $\{x^n A :_A \langle \mathfrak{a} \rangle\}_{n \in \mathbb{N}}$ is equivalent to the $xA$-adic topology if and only if* $\operatorname{height}(\mathfrak{a}+\mathfrak{p}/\mathfrak{p}) > 1$ *for all* $\mathfrak{p} \in \operatorname{Ass} A$*. In particular, in the case of a local ring $(A, \mathfrak{m})$ this holds if and only if* $\operatorname{height}(\mathfrak{a}\widehat{A} + \mathfrak{p}/\mathfrak{p}) > 1$ *for all* $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$.

*Proof.* Let $\operatorname{height}(\mathfrak{a} + \mathfrak{p}/\mathfrak{p}) > 1$ for all primes $\mathfrak{p} \in \operatorname{Ass} A$. Then $\dim A_P/\mathfrak{p} > 1$ for all prime ideals $P \in \operatorname{Ass} A/xA \cap V(\mathfrak{a})$ and $\mathfrak{p} \in \operatorname{Ass} A_P$. Because of condition (C) for $A_P$ it implies that $\dim \widehat{A_P}/\mathfrak{q} > 1$ for all $\mathfrak{q} \in \operatorname{Ass} \widehat{A_P}/\mathfrak{p}\widehat{A_P}$. But now

$$\operatorname{Ass} \widehat{A_P} = \cup_{\mathfrak{p} \in \operatorname{Ass} A_P} \operatorname{Ass} \widehat{A_P}/\mathfrak{p}\widehat{A_P},$$

see [28, Theorem 23.2], which proves the first part of the result in view of 2.4.

Conversely suppose the equivalence of the ideal topologies and let $\operatorname{height}(\mathfrak{a} + \mathfrak{p}/\mathfrak{p}) = 1$ for some $\mathfrak{p} \in \operatorname{Ass} A$. Then there is a prime ideal $P \in V(\mathfrak{a})$ such that $\dim A_P/\mathfrak{p}A_P = 1$. Because $\frac{x}{1}$ is an $A_P$-regular element it follows that $P \in \operatorname{Ass} A/xA \cap V(\mathfrak{a})$. So condition (C) provides a contradiction by 2.4. □

2.2. **On Ideal Transforms.** In this subsection let us discuss the behaviour of certain intermediate rings lying between a commutative Noetherian ring and its full ring of quotients. To this end let $x \in A$ be a non-zero divisor and $A \subseteq B \subseteq A_x$ an intermediate ring.

**Lemma 2.7.** *For an intermediate ring $A \subseteq B \subseteq A_x$ the following conditions are equivalent:*

(i) *$B$ is a finitely generated $A$-module.*
(ii) *There is a $k \in \mathbb{N}$ such that $x^k B \subseteq A$.*

(iii) *There is a* $k \in \mathbb{N}$ *such that* $x^{k+1}B \cap A \subseteq xA$.
(iv) $\{x^n B \cap A\}_{n \in \mathbb{N}}$ *is equivalent to the* $xA$*-adic topology.*
(v) *There is a* $k \in \mathbb{N}$ *such that* $x^{n+k}B \cap A = x^n(x^k B \cap A)$ *for all* $n \geq 1$.

*Proof.* The implication (i) $\Rightarrow$ (v) is a consequence of the Artin-Rees lemma. The implications (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are easy to see. Finally we have (ii) $\Rightarrow$ (i) since $B$ is as an $A$-submodule of the finitely generated $A$-module $\frac{1}{x^k}A$ finitely generated. $\square$

In the following there is some need for the ideal transform. To this end let $\mathfrak{a}$ denote a regular ideal of a commutative Noetherian ring $A$. Define

$$T_{\mathfrak{a}}(A) = \{q \in Q(A) \mid \operatorname{Supp}(Aq + A/A) \subseteq V(\mathfrak{a})\},$$

where $Q(A)$ denotes the full ring of quotients of $A$. It follows that

$$T_{\mathfrak{a}}(A) = \{q \in Q(A) \mid \mathfrak{a}^n q \subseteq A \text{ for some } n \in \mathbb{N}\},$$

Note that $A \subseteq T_{\mathfrak{a}}(A) \subseteq A_x$, where $x \in \mathfrak{a}$ is a non-zero divisor. Moreover, suppose that $\mathfrak{a} = (a_1, \ldots, a_s)A$, where each of the $a_i$'s is a non-zero divisor. Then $T_{\mathfrak{a}}(A) = \cap_{i=1}^s A_{a_i}$ as is easily seen. Moreover one might define $T_{\mathfrak{a}}(B)$ for an arbitrary intermediate ring $A \subseteq B \subseteq Q(A)$ in a corresponding way.

Ideal transforms were first studied by M. Nagata in connection with Hilbert's 14th problem, see [34]. It is of some interest to describe when $T_{\mathfrak{a}}(A)$ is an $A$-algebra of finite type. As a first step towards this direction consider when it is a finitely generated $A$-module.

**Lemma 2.8.** *Let* $\mathfrak{a}$ *denote a regular ideal of a commutative ring* $A$. *Let* $x \in \mathfrak{a}$ *be a non-zero divisor. Then the following conditions are satisfied:*

a) $T_{\mathfrak{a}}(T_{\mathfrak{a}}(A)) = T_{\mathfrak{a}}(A)$,
b) $\operatorname{Ass}_A T_{\mathfrak{a}}(A)/A = \operatorname{Ass} A/xA \cap V(\mathfrak{a})$, *and*
c) $rT_{\mathfrak{a}}(A) \cap A = rA : \langle \mathfrak{a} \rangle$ *for any regular element* $r \in A$.

*Proof.* The first claim is obvious by definition. In order to prove b) let $\mathfrak{p} = A : q$ for some $q = \frac{s}{x^n} \in T_{\mathfrak{a}}(A) \setminus A$. Then $\mathfrak{p} = x^n A :_A s$ and $s \notin x^n A$. That is $\mathfrak{p} \in \operatorname{Ass} A/xA$. Furthermore $\mathfrak{a} \subseteq \mathfrak{p}$ since $\mathfrak{a}^k q \subseteq A$ and $\mathfrak{a}^k \subseteq A : q = \mathfrak{p}$, for some $k \in \mathbb{N}$. The reverse conclusion follows by similar arguments.

In order to prove c) first note that $rT_{\mathfrak{a}}(A) \cap A \subseteq rA :_A \langle \mathfrak{a} \rangle$ as easily seen. For the reverse inclusion note that $rT_{\mathfrak{a}}(A) : \langle \mathfrak{a} \rangle = rT_{\mathfrak{a}}(A)$. $\square$

As a consequence of 2.8 it follows that $\operatorname{Ass}_A T_{\mathfrak{a}}(A)/A = \operatorname{Ass} \operatorname{Ext}_A^1(A/\mathfrak{a}, A)$. Therefore $T_{\mathfrak{a}}(A) = A$ if and only if $\operatorname{grade} \mathfrak{a} > 1$. There is a relation of ideal transforms to a more functorial construction. First note that $\operatorname{Hom}_A(\mathfrak{a}, A) \simeq A :_{Q(A)} \mathfrak{a}$ for a regular ideal $\mathfrak{a}$ in $A$. Therefore $\varinjlim \operatorname{Hom}_A(\mathfrak{a}^n, A) \simeq \cup_{n\in\mathbb{N}}(A :_{Q(A)} \mathfrak{a}^n) = T_{\mathfrak{a}}(A)$. This yields a short exact sequence

$$0 \to A \to T_{\mathfrak{a}}(A) \to H^1_{\mathfrak{a}}(A) \to 0,$$

where the monomorphism is just the inclusion map. Therefore $T_{\mathfrak{a}}(A)/A \simeq H^1_{\mathfrak{a}}(A)$. So the ideal transform enables another approach to $H^1_{\mathfrak{a}}(A)$.

It is of a particular interest when $T_{\mathfrak{a}}(A)$ or - equivalently $H^1_{\mathfrak{a}}(A)$ - is a finitely generated $A$-module. In the following there is a generalization of A. Grothendieck's finiteness result, see [13].

**Theorem 2.9.** (Grothendieck's Finiteness Result) *Let $\mathfrak{a}$ denote a regular ideal of a commutative Noetherian ring $A$. Then the following conditions are equivalent:*

(i) *$T_{\mathfrak{a}}(A)$ is a finitely generated $A$-module.*
(ii) $\dim \widehat{A_P}/\mathfrak{p} > 1$ *for all* $P \in \operatorname{Ass}_A \operatorname{Ext}^1_A(A/\mathfrak{a}, A)$ *and all* $\mathfrak{p} \in \operatorname{Ass} \widehat{A_P}$.

*Proof.* By 2.7 and 2.8 the statement in condition (i) is equivalent to the fact that $\{x^n A :_A \langle \mathfrak{a} \rangle\}_{n\in\mathbb{N}}$ is equivalent to the $xA$-adic topology for a non-zero divisor $x \in \mathfrak{a}$. Note that $T_{\mathfrak{a}}(A) \subseteq A_x$. By 2.4 this proves the statement because of

$$\operatorname{Ass} A/xA \cap V(\mathfrak{a}) = \operatorname{Ass} \operatorname{Ext}^1_A(A/\mathfrak{a}, A)$$

as mentioned above. □

Under the additional assumption of condition (C) on $A$ there is a further simplification of the finiteness result.

**Corollary 2.10.** a) *Suppose that $A$ is a factor ring of a Cohen-Macaulay ring. Then $T_{\mathfrak{a}}(A)$ is a finitely generated $A$-module if and only if* $\operatorname{height}(\mathfrak{a} + \mathfrak{p}/\mathfrak{p}) > 1$ *for all $\mathfrak{p} \in \operatorname{Ass} A$. In particular $T_{\mathfrak{a}}(A)$ is a finitely generated $A$-module if and only if $T_{\mathfrak{a}+\mathfrak{p}/\mathfrak{p}}(A/\mathfrak{p})$ is a finitely generated $A/\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Ass} A$.*
b) *Suppose that $(A, \mathfrak{m})$ is a local ring. Then $T_{\mathfrak{a}}(A)$ is a finitely generated $A$-module if and only if* $\operatorname{height}(\mathfrak{a}\widehat{A} + \mathfrak{p}/\mathfrak{p}) > 1$ *for all $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$.*

*Proof.* It is a consequence of 2.9 with the aid of 2.6. □

In the case of a local ring $(A, \mathfrak{m})$ which is a factor ring of a Cohen-Macaulay ring the finiteness of $H^1_{\mathfrak{m}}(A)$ is therefore equivalent to $\dim A/\mathfrak{p} > 1$ for all prime ideals $\mathfrak{p} \in \operatorname{Ass} A$.

A more difficult problem is a characterization of when the ideal transform $T_{\mathfrak{a}}(A)$ is an $A$-algebra of finite type. This does not hold even in the case of a polynomial ring over a field as shown by M. Nagata, see [34], in the context of Hilbert's 14th problem.

2.3. **Asymptotic Prime Divisors.** In the following we apply some of the previous considerations to the study of asymptotic prime ideals. To this end there is a short excursion about graded algebras.

For a commutative Noetherian ring $A$ let $F = \{\mathfrak{a}_n\}_{n\in\mathbb{Z}}$ denote a filtration of ideals, i.e. a family of ideals satisfying the following conditions:

a) $\mathfrak{a}_n = A$ for all $n \leq 0$,
b) $\mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n$ for all $n \in \mathbb{Z}$, and
c) $\mathfrak{a}_n\mathfrak{a}_m \subseteq \mathfrak{a}_{n+m}$ for all $n, m \in \mathbb{Z}$.

Then one may form $R(F)$, the Rees ring associated to $F$, i.e. $R(F) = \oplus_{n\in\mathbb{Z}}\mathfrak{a}_n t^n \subseteq A[t, t^{-1}]$, where $t$ denotes an indeterminate. Let $\mathfrak{a} = (a_1, \ldots, a_s)A$ denote an ideal of $A$. Then $F$ is called an $\mathfrak{a}$-admissible filtration, whenever $\mathfrak{a}^n \subseteq \mathfrak{a}_n$ for all $n \in \mathbb{Z}$. For an $\mathfrak{a}$-admissible filtration it is easily seen that $R(F)$ is an $R(\mathfrak{a})$-module, where

$$R(\mathfrak{a}) = \oplus_{n\in\mathbb{Z}}\mathfrak{a}^n t^n = A[a_1t, \ldots, a_st, t^{-1}]$$

denotes the (extended) Rees ring of $A$ with respect to $\mathfrak{a}$. Note that $R(\mathfrak{a})$ is the Rees ring associated to the $\mathfrak{a}$-adic filtration $F = \{\mathfrak{a}^n\}_{n\in\mathbb{Z}}$.

There are several possibilities to associate an $\mathfrak{a}$-admissible filtration $F$ to a given ideal $\mathfrak{a}$. One of these is defined for a multiplicatively closed subset $S$ of $A$. Let $\mathfrak{a}_S^{(n)}, n \in \mathbb{N}$, denote the $n$-th symbolic power of $\mathfrak{a}$ with respect to $S$. Then $F = \{\mathfrak{a}_S^{(n)}\}_{n\in\mathbb{Z}}$ forms an $\mathfrak{a}$-admissible filtration. The corresponding Rees ring $R_S(\mathfrak{a}) := R(F)$ is called the symbolic Rees ring of $\mathfrak{a}$ with respect to $S$. In the case of $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of $A$ write $R_S(\mathfrak{p})$ instead of $R_{A\setminus\mathfrak{p}}(\mathfrak{p})$.

Let $F$ denote an $\mathfrak{a}$-admissible filtration. It follows that $R(F)$ is a finitely generated $R(\mathfrak{a})$-module if and only if there is an integer $k \in \mathbb{N}$ such that $\mathfrak{a}_{n+k} = \mathfrak{a}^n\mathfrak{a}_k$ for all $n \in \mathbb{N}$. Equivalently this holds if and only if $\mathfrak{a}_{n+k} \subseteq \mathfrak{a}^n$ for all $n \in \mathbb{N}$ and a certain integer $k \geq 0$. This behaviour sometimes is called linear equivalence of $F$ to the $\mathfrak{a}$-adic topology.

For an integer $k \in \mathbb{N}$ let $F_k = \{\mathfrak{a}_{nk}\}_{n\in\mathbb{Z}}$. Then $R(F_k) \simeq R^{(k)}(F)$, where $R^{(k)}(F) = \oplus_{n\in\mathbb{Z}}\mathfrak{a}_{nk}t^{nk}$ denotes the $k$-th Veronesean subring of $R(F)$. Before we continue with the study of ideal transforms there is a characterization of when $R(F)$ is an $A$-algebra of finite type.

**Proposition 2.11.** *Let $F = \{\mathfrak{a}_n\}_{n\in\mathbb{Z}}$ denote a filtration of ideals. Then the following conditions are equivalent:*

(i) *$R(F)$ is an $A$-algebra of finite type.*
(ii) *There is a $k \in \mathbb{N}$ such that $R(F_k)$ is an $A$-algebra of finite type.*
(iii) *There is a $k \in \mathbb{N}$ such that $R(F_k)$ is a finitely generated $R(\mathfrak{a}_k)$-module.*
(iv) *There is a $k \in \mathbb{N}$ such that $\mathfrak{a}_{nk} = (\mathfrak{a}_k)^n$ for all $n \geq k$.*
(v) *There is a $k \in \mathbb{N}$ such that $\mathfrak{a}_{n+k} = \mathfrak{a}_n\mathfrak{a}_k$ for all $n \geq k$.*

*Proof.* First show (i) $\Rightarrow$ (v). By the assumption there is an $r \in \mathbb{N}$ such that $R(F) = A[\mathfrak{a}_1 t, \ldots, \mathfrak{a}_r t^r]$. Put $l = r!$ and $k = rl$. Then it follows that $\mathfrak{a}_n = \sum \mathfrak{a}_1^{n_1} \cdots \mathfrak{a}_r^{n_r}$, where the sum is taken over all $n_1, \ldots, n_r$ such that $\sum_{i=1}^{r} i n_i \geq n$. For $n \geq k$ it is easy to see that there is an integer $1 \leq i \leq r$ such that $n_i \geq \frac{l}{i}$. Whence $\mathfrak{a}_n \subseteq \mathfrak{a}_{n-l}\mathfrak{a}_l$ for any $n \geq k$. That means $\mathfrak{a}_{n+k} = \mathfrak{a}_n \mathfrak{a}_k$ for any $n \geq k$ as is easily seen.

While the implication (v) $\Rightarrow$ (iv) holds trivially the implication (iv) $\Rightarrow$ (iii) is a consequence of the Artin-Rees lemma. In order to show (iii) $\Rightarrow$ (ii) note that $R(\mathfrak{a}_k)$ is an $A$-algebra of finite type.

Finally show (ii) $\Rightarrow$ (i). For $0 \leq i < k$ it follows that $\mathfrak{A}_i = \oplus_{n \in \mathbb{Z}} \mathfrak{a}_{nk+i} t^{nk}$ is an ideal of $R^{(k)}(F)$, and $R^{(k)}(F)$ is isomorphic to $R(F_k)$. So $\mathfrak{A}_i, 0 \leq i < r$, is a finitely generated $R^{(k)}(F)$-module. Because of $R(F) = \oplus_{i=0}^{k-1} \mathfrak{A}_i t^i$ it turns out that $R(F)$ is a finitely generated $R^{(k)}(F)$-module. This proves that $R(F)$ is an $A$-algebra of finite type. □

The implication (i) $\Rightarrow$ (v) was shown by D. Rees, see [37]. In the case of a local ring (ii) $\Rightarrow$ (i) was proved by a different argument in [43].

Before we shall continue with the study of certain ideal transforms consider two applications of the Artin-Rees Lemma. They will be useful in the study of the Ratliff-Rush closure of an ideal.

**Proposition 2.12.** *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}_1, \ldots, \mathfrak{b}_t$, $t \in \mathbb{N}$, denote ideals of a commutative Noetherian ring $A$.*

a) *There is a $k \in \mathbb{N}$ such that*

$$\cap_{i=1}^{t}(\mathfrak{a}^{n+k} + \mathfrak{b}_i) = \mathfrak{a}^n(\cap_{i=1}^{t}(\mathfrak{a}^k + \mathfrak{b}_i)) + \cap_{i=1}^{t}\mathfrak{b}_i \text{ for all } n \geq 1.$$

b) *There is a $k \in \mathbb{N}$ such that*

$$\mathfrak{a}^{n+k} :_A \mathfrak{b} = \mathfrak{a}^n(\mathfrak{a}^k :_A \mathfrak{b}) + 0 :_A \mathfrak{b} \text{ for all } n \geq 1.$$

*Proof.* In order to prove a) consider the natural injective homomorphism of finitely generated $A$-modules

$$A/\cap_{i=1}^{t} \mathfrak{b}_i \to \oplus_{i=1}^{t} A/\mathfrak{b}_i,\ a + \cap_{i=1}^{t}\mathfrak{b}_i \mapsto (a + \mathfrak{b}_1, \ldots, a + \mathfrak{b}_t).$$

Then the Artin-Rees Lemma provides the existence of an integer $k \in \mathbb{N}$ such that

$$\mathfrak{a}^{n+k}(\oplus_{i=1}^{t} A/\mathfrak{b}_i) \cap (A/\cap_{i=1}^{t} \mathfrak{b}_i) = \mathfrak{a}^n(\mathfrak{a}^k(\oplus_{i=1}^{t} A/\mathfrak{b}_i) \cap (A/\cap_{i=1}^{t} \mathfrak{b}_i))$$

for all $n \in \mathbb{N}$. In fact this proves the statement a).

For the proof of b) let $\mathfrak{b} = (b_1, \ldots, b_s)A$. Then by the Artin-Rees Lemma there is a $c \in \mathbb{N}$ such that

$$\mathfrak{a}^{n+c} :_A b_i \subseteq \mathfrak{a}^n + (0 :_A b_i)$$

for all $n \in \mathbb{N}$ and $i = 1, \ldots, s$. Because of the statement in a) there exists a $d \in \mathbb{N}$ such that

$$\cap_{i=1}^{s}(\mathfrak{a}^{n+d} + 0 :_A b_i) \subseteq \mathfrak{a}^n + \cap_{i=1}^{s}(0 :_A b_i) = \mathfrak{a}^n + (0 :_A \mathfrak{b}).$$

But now we have that $\cap_{i=1}^{s}(\mathfrak{a}^n :_A b_i) = \mathfrak{a}^n :_A \mathfrak{b}$ for all $n \in \mathbb{N}$. So finally there exists a $k \in \mathbb{N}$ such that $\mathfrak{a}^{n+k} :_A \mathfrak{b} \subseteq \mathfrak{a}^n + (0 :_A \mathfrak{b})$ for all $n \in \mathbb{N}$. By passing to $A/0 :_A \mathfrak{b}$ the Artin-Rees Lemma proves the claim in b). ☐

As a first sample of ideal transforms consider $T_{(\mathfrak{a}t,t^{-1})}(R(\mathfrak{a}))$. But now we have that $T_{(\mathfrak{a}t,t^{-1})}(R(\mathfrak{a})) \subseteq A[t, t^{-1}]$. So it is an easy exercise to prove that the $n$-th graded piece of the ideal transform is given by

$$T_{(\mathfrak{a}t,t^{-1})}(R(\mathfrak{a}))_n = \begin{cases} A & \text{for } n \leq 0, \\ (\mathfrak{a}^n)^* & \text{for } n > 0, \end{cases}$$

where $(\mathfrak{a}^n)^* = \cup_{m\in\mathbb{N}}(\mathfrak{a}^{n+m} : \mathfrak{a}^m)$ denotes the Ratliff-Rush closure of $\mathfrak{a}^n$. In the following put $R^*(\mathfrak{a}) = \oplus_{n\in\mathbb{Z}}(\mathfrak{a}^n)^* t^n$. A few basic results of the Ratliff-Rush closure are listed in the following result.

**Lemma 2.13.** *Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring $A$.*

a) *There is an integer $k \in \mathbb{N}$ such that $(\mathfrak{a}^n)^* = \mathfrak{a}^n + 0 :_A \langle\mathfrak{a}\rangle$ for all $n \geq k$. In particular $(\mathfrak{a}^n)^* = \mathfrak{a}^n$ for all $n \geq k$ provided $\mathfrak{a}$ is a regular ideal.*
b) *$(\mathfrak{a}^{n+1})^* :_A \mathfrak{a} = (\mathfrak{a}^n)^*$ for all $n \in \mathbb{N}$.*
c) *$T_{(\mathfrak{a}t,t^{-1})}(R(\mathfrak{a}))$ is a finitely generated $R(\mathfrak{a})$-module if and only if $\mathfrak{a}$ is a regular ideal.*

*Proof.* Fix an integer $n \in \mathbb{N}$. Then for a sufficiently large integer $m$ it follows that $0 :_A \mathfrak{a}^m = 0 :_A \langle\mathfrak{a}\rangle$ and $(\mathfrak{a}^n)^* = (\mathfrak{a}^{n+m} + 0 :_A \langle\mathfrak{a}\rangle) :_A \mathfrak{a}^m$. Therefore, by passing to $A/0 :_A \langle\mathfrak{a}\rangle$ we may assume that $\mathfrak{a}$ is a regular ideal in order to prove a). Then by 2.12 it follows that

$$\oplus_{n\in\mathbb{Z}}(\mathfrak{a}^{n+1} :_A \mathfrak{a})t^n$$

is a finitely generated $R(\mathfrak{a})$-module. Therefore the Artin-Rees Lemma provides the existence of an integer $k \in \mathbb{N}$ such that $\mathfrak{a}^{n+k+1} :_A \mathfrak{a} = \mathfrak{a}^n(\mathfrak{a}^{k+1} :_A \mathfrak{a})$ for all $n \geq 1$. Therefore $\mathfrak{a}^{n+k+1} :_A \mathfrak{a} = \mathfrak{a}^{n+k}$ for all $n \geq 1$. This proves the claim in a).

The statement in b) follows easily by the definitions. Finally c) is a consequence of a) and the Artin-Rees Lemma. ☐

Next let $(A, \mathfrak{m})$ denote a local Noetherian ring. For an ideal $\mathfrak{a}$ of $A$ consider the ideal transform $T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$. It is easily seen that its $n$-th graded component has the following form

$$T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))_n = \begin{cases} A & \text{for } n \leq 0, \\ \mathfrak{a}^n :_A \langle\mathfrak{m}\rangle & \text{for } n > 0. \end{cases}$$

Therefore the finiteness of $T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$ yields some information about the existence of an integer $k \in \mathbb{N}$ such that $\mathfrak{a}^{n+k} : \langle \mathfrak{m} \rangle \subseteq \mathfrak{a}^n$ for all $n \geq 1$ as it is clear by the Artin-Rees Lemma. This is a sharpening of the problem on the equivalence of the topologies investigated at the beginning.

In the following let $l(\mathfrak{a})$ denote the analytic spread of $\mathfrak{a}$, i.e.

$$l(\mathfrak{a}) = \dim R(\mathfrak{a})/(\mathfrak{m}, t^{-1})R(\mathfrak{a}),$$

see D. G. Northcott and D. Rees [36] for basic results. Recall that

$$\operatorname{height} \mathfrak{a} \leq l(\mathfrak{a}) \leq \dim A.$$

Moreover, $l(\mathfrak{a}) = \dim gr_A(\mathfrak{a})/\mathfrak{m} gr_A(\mathfrak{a})$, where $gr_A(\mathfrak{a}) = \oplus_{n\in\mathbb{N}} \mathfrak{a}^n/\mathfrak{a}^{n+1}$ denotes the form ring with respect to $\mathfrak{a}$.

**Theorem 2.14.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(A, \mathfrak{m})$.*

a) *The ideal transform $T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$ is a finitely generated $R(\mathfrak{a})$-module if and only if*
$$l(\mathfrak{a}\widehat{A} + \mathfrak{p}/\mathfrak{p}) < \dim \widehat{A}/\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass} \widehat{A}.$$

b) *$T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$ is an $A$-algebra of finite type if and only if there is a $k \in \mathbb{N}$ such that*
$$l(\mathfrak{a}^k\widehat{A} : \langle \mathfrak{m}\widehat{A} \rangle + \mathfrak{p}/\mathfrak{p}) < \dim \widehat{A}/\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass} \widehat{A}.$$

*Proof.* At first prove a). As a consequence of the Artin-Rees Lemma the ideal transform $T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$ is finitely generated over $R(\mathfrak{a})$ if and only if the corresponding result holds for $\mathfrak{a}\widehat{A}$ in $(\widehat{A}, \widehat{\mathfrak{m}})$. Therefore, without loss of generality we may assume that $A$ is a complete local ring.

So we may assume that $R(\mathfrak{a})$ is the quotient of a Cohen-Macaulay ring. Furthermore there is a 1-to-1 correspondence between the associated prime ideals $\mathfrak{P}$ of $R(\mathfrak{a})$ and the associated prime ideals $\mathfrak{p}$ of $A$ given by

$$\mathfrak{P} \mapsto \mathfrak{p} = \mathfrak{P} \cap A \text{ resp. } \mathfrak{p} \mapsto \oplus_{n\in\mathbb{Z}} (\mathfrak{a}^n \cap \mathfrak{p}) t^n.$$

By virtue of 2.10 $T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$ is a finitely generated $R(\mathfrak{a})$-module if and only if $T_{(\mathfrak{m}/\mathfrak{p},t^{-1})}(R(\mathfrak{a} + \mathfrak{p}/\mathfrak{p}))$ is a finitely generated $R(\mathfrak{a} + \mathfrak{p}/\mathfrak{p}))$-module for all $\mathfrak{p} \in \operatorname{Ass} A$. That is, without loss of generality we may assume $(A, \mathfrak{m})$ a complete local domain after changing the notation. But under this assumption $T_{(\mathfrak{m},t^{-1})}(R(\mathfrak{a}))$ is a finitely generated $R(\mathfrak{a})$-module if and only if $\operatorname{height}(\mathfrak{m}, t^{-1})R(\mathfrak{a}) > 1$. Finally $A$ is a universally catenarian domain. Therefore it holds

$$\operatorname{height}(\mathfrak{m}, t^{-1})R(\mathfrak{a}) = \dim R(\mathfrak{a}) - \dim R(\mathfrak{a})/(\mathfrak{m}, t^{-1})R(\mathfrak{a}).$$

Because of $\dim R(\mathfrak{a}) = \dim A + 1$ and $\dim R(\mathfrak{a})/(\mathfrak{m}, t^{-1})R(\mathfrak{a}) = l(\mathfrak{a})$ this completes the proof.

With the aid of statement a) the conclusion in b) follows by virtue of 2.11 □

As above let $\mathfrak{a}$ denote an ideal of a commutative Noetherien ring $A$. Let $\mathrm{As}(\mathfrak{a})$ resp. $\mathrm{Bs}(\mathfrak{a})$ denote the ultimately constant values of $\operatorname{Ass} A/\mathfrak{a}^n$ resp. $\operatorname{Ass} \mathfrak{a}^n/\mathfrak{a}^{n+1}$ for all large $n \in \mathbb{N}$, as shown by M. Brodmann in [4], see also [42].

As it will be shown in the following the previous result 2.14 has to do with the property $\mathfrak{m} \in \mathrm{As}(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of a local ring $(A, \mathfrak{m})$. To this end we modify a result originally shown by L. Burch, see [8]. Further results in this direction were shown by C. Huneke, see [21].

**Theorem 2.15.** *Suppose that $(A, \mathfrak{m})$ denotes a local Noetherian ring. Then the following results are true:*

a) *If $\mathfrak{m} \notin \mathrm{Bs}(\mathfrak{a})$, then $l(\mathfrak{a}) < \dim A$.*
b) *The converse is true provided $A$ is a universally catenarian domain and $gr_A(\mathfrak{a})$ is unmixed.*

*Proof.* In order to show a) first note that the natural epimorphism

$$\phi_n : \mathfrak{a}^n/\mathfrak{a}^{n+1} \to \mathfrak{a}^n \bar{A}/\mathfrak{a}^{n+1}\bar{A} \text{ with } \bar{A} = A/0 :_A \langle \mathfrak{a} \rangle$$

is an isomorphism for all large $n \in \mathbb{N}$. This follows easily by the Artin-Rees Lemma. By passing to $\bar{A}$ one might assume that $\mathfrak{a}$ is a regular ideal. Then $\mathfrak{m} \notin \mathrm{As}(\mathfrak{a})$ because of $\mathrm{As}(\mathfrak{a}) = \mathrm{Bs}(\mathfrak{a})$ for the regular ideal $\mathfrak{a}$, see [30].

Next investigate the Noetherian ring $R^*(\mathfrak{a})$. Now we claim that

$$\operatorname{Ass} A/(\mathfrak{a}^n)^* \subseteq \operatorname{Ass} A/(\mathfrak{a}^{n+1})^* \text{ for all } n \in \mathbb{N}.$$

To this end note that $(\mathfrak{a}^{n+1})^* : \mathfrak{a} = (\mathfrak{a}^n)^*$ for all $n \in \mathbb{N}$, see 2.13. Let $\mathfrak{a} = (a_1, \ldots, a_s)A$. Then the natural homomorphism

$$A/(\mathfrak{a}^n)^* \to \oplus_{i=1}^s A/(\mathfrak{a}^{n+1})^*, \quad r + (\mathfrak{a}^n)^* \mapsto (ra_i + (\mathfrak{a}^{n+1})^*)$$

is injective for all $n \in \mathbb{N}$. Therefore $\operatorname{Ass} A/(\mathfrak{a}^n)^* \subseteq \operatorname{Ass} A/(\mathfrak{a}^{n+1})^*$, as required. Because of $(\mathfrak{a}^n)^* = \mathfrak{a}^n$ for all large $n$ it turns out that $\mathfrak{m} \notin \operatorname{Ass} A/(\mathfrak{a}^n)^*$ for $\underline{\text{all}}$ $n \in \mathbb{N}$. Because of $T_{(\mathfrak{m},t^{-1})}(R^*(\mathfrak{a})) = \oplus_{n \in \mathbb{Z}}((\mathfrak{a}^n)^* : \langle \mathfrak{m} \rangle)t^n$ it follows that $T_{(\mathfrak{m},t^{-1})}(R^*(\mathfrak{a})) = R^*(\mathfrak{a})$. By 2.8 this means that $\operatorname{grade}(\mathfrak{m}, t^{-1})R^*(\mathfrak{a}) > 1$. But now

$$1 < \operatorname{height}(\mathfrak{m}, t^{-1})R^*(\mathfrak{a}) \le \dim R^*(\mathfrak{a}) - \dim R^*(\mathfrak{a})/(\mathfrak{m}, t^{-1}).$$

Because $R^*(\mathfrak{a})$ is a finitely generated $R(\mathfrak{a})$-module it implies that

$$\dim R^*(\mathfrak{a}) = \dim A + 1 \text{ and } \dim R^*(\mathfrak{a})/(\mathfrak{m}, t^{-1})R^*(\mathfrak{a}) = l(\mathfrak{a}),$$

which finally proves the claim a).

In order to prove b) first note that $\operatorname{height}(\mathfrak{m}, t^{-1})R^*(\mathfrak{a}) = \dim R^* - l(\mathfrak{a})$ since $A$ is universally catenarian and $gr_A(\mathfrak{a})$ is unmixed. Since $(\mathfrak{a}^{n+1})^* :_A \mathfrak{a} = (\mathfrak{a}^n)^*$ for all $n \in \mathbb{N}$ there is no prime ideal $\mathfrak{P}$ of $R(\mathfrak{a})$ associated to $R^*(\mathfrak{a})/(t^{-1})R^*(\mathfrak{a})$ that contains

$(\mathfrak{a}t, t^{-1})R(\mathfrak{a})$. Because of $(\mathfrak{a}^n)^* = \mathfrak{a}^n + 0 :_A \langle \mathfrak{a} \rangle$ for all sufficiently large n, see 2.13, it is easy to see that the kernel and the cokernel of the natural graded homomorphism

$$R(\mathfrak{a})/(t^{-1})R(\mathfrak{a}) \to R(\mathfrak{a})^*/(t^{-1})R(\mathfrak{a})^*$$

are finitely generated $R(\mathfrak{a})$-modules whose support is contained in $V((\mathfrak{a}t, t^{-1})R(\mathfrak{a}))$. This implies

$$\operatorname{Ass} R^*(\mathfrak{a})/(t^{-1})R^*(\mathfrak{a}) = \{\mathfrak{P} \in \operatorname{Ass} R(\mathfrak{a})/(t^{-1})R(\mathfrak{a}) \mid \mathfrak{P} \not\supseteq (\mathfrak{a}t, t^{-1})R(\mathfrak{a})\}$$

as is easily seen. By the assumption it follows that $R^*(\mathfrak{a})/(t^{-1})R^*(\mathfrak{a})$ is unmixed. Therefore

$$\operatorname{grade}(\mathfrak{m}, t^{-1})R^*(\mathfrak{a}) > 1 \text{ and } T_{(\mathfrak{m},t^{-1})}(R^*(\mathfrak{a})) = R^*(\mathfrak{a}),$$

see 2.8. By definition this means $(\mathfrak{a}^n)^* : \langle \mathfrak{m} \rangle = (\mathfrak{a}^n)^*$ for all $n \in \mathbb{N}$. Because of $(\mathfrak{a}^n)^* = \mathfrak{a}^n$ for all large $n$ this proves the statement. □

A corresponding result is true for the integral closures $\overline{\mathfrak{a}^n}$ of $\mathfrak{a}^n$. To this end let $\overline{R}(\mathfrak{a})$ denote the integral closure of $R(\mathfrak{a})$ in $A[t, t^{-1}]$. Then

$$\overline{R}(\mathfrak{a})_n = \begin{cases} \overline{\mathfrak{a}^n} & \text{for } n > 0, \\ A & \text{for } n \leq 0, \end{cases}$$

where $\overline{\mathfrak{a}^n}$ denotes the integral closure of $\mathfrak{a}^n$, i.e. the ideal of all elements $x \in A$ satisfying an equation $x^m + a_1 x^{m-1} + \ldots + a_m = 0$, where $a_i \in (\mathfrak{a}^n)^i, i = 1, \ldots, m$. Note that $\operatorname{Ass} A/\overline{\mathfrak{a}^n}$ is an increasing sequence that becomes eventually stable for large $n$, as shown by L. J. Ratliff, see [40]. Call $\overline{\operatorname{As}}(\mathfrak{a})$ the stable value.

**Theorem 2.16.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(A, \mathfrak{m})$. Then the following conditions are true:*

a) *If $\mathfrak{m} \notin \overline{\operatorname{As}}(\mathfrak{a})$, then $l(\mathfrak{a}) < \dim A$.*
b) *The converse is true, provided $A$ is a universally catenarian domain.*

*Proof.* First note that $\overline{\mathfrak{a}^n} : \langle \mathfrak{m} \rangle = \overline{\mathfrak{a}^n}$ for all $n \in \mathbb{N}$ since $\operatorname{Ass} A/\overline{\mathfrak{a}^n}, n \in \mathbb{N}$, forms an increasing sequence. Hence it follows that $T_{(\mathfrak{m},t^{-1})}(\overline{R}(\mathfrak{a})) = \overline{R}(\mathfrak{a})$. By 2.8 it implies $\operatorname{height}(\mathfrak{m}, t^{-1})\overline{R}(\mathfrak{a}) > 1$. Therefore

$$1 < \operatorname{height}(\mathfrak{m}, t^{-1})R(\mathfrak{a}) \leq \dim R(\mathfrak{a}) - l(\mathfrak{a}),$$

which proves the claim.

In order to prove the converse first note that $\operatorname{height}(\mathfrak{m}, t^{-1})R(\mathfrak{a}) = \dim R(\mathfrak{a}) - l(\mathfrak{a})$, since $A$ is a universally catenarian domain. Therefore the assumption implies that

$$1 < \operatorname{height}(\mathfrak{m}, t^{-1})R(\mathfrak{a}) = \operatorname{height}(\mathfrak{m}, t^{-1})\overline{R}(\mathfrak{a}).$$

But now $\overline{R}(\mathfrak{a})$ is a Krull domain. Hence any associated prime ideal of the principal ideal $(t^{-1})\overline{R}(\mathfrak{a})$ is of height 1. Whence by 2.8 it follows that $T_{(\mathfrak{m},t^{-1})}(\overline{R}(\mathfrak{a})) = \overline{R}(\mathfrak{a})$. That is, $\overline{\mathfrak{a}^n} : \langle \mathfrak{m} \rangle = \overline{\mathfrak{a}^n}$ for all $n \in \mathbb{N}$, as required. □

The statements of 2.16 were shown by J. Lipman, see [27]. It extends in a straightforward way to an ideal $\mathfrak{a}$ of an arbitrary local ring $(A, \mathfrak{m})$. This was done by S. McAdam in [29], see also [42] for a different approach. In order to describe this result let $\operatorname{mAss} A$ denote the set of minimal prime ideals of $\operatorname{Ass} A$.

**Corollary 2.17.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(A, \mathfrak{a})$. Then $\mathfrak{m} \in \overline{\operatorname{As}}(\mathfrak{a})$ if and only if $l(\mathfrak{a}\widehat{A} + \mathfrak{p}/\mathfrak{p}) < \dim \widehat{A}/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{mAss} \widehat{A}$.*

*Proof.* First note that $\mathfrak{m} \in \operatorname{Ass} A/\overline{\mathfrak{a}}$ if and only if $\mathfrak{m}\widehat{A} \in \operatorname{Ass} \widehat{A}/\overline{\mathfrak{a}\widehat{A}}$, see [40]. Furthermore $\mathfrak{m} \in \operatorname{Ass} A/\overline{\mathfrak{a}}$ if and only if there is a minimal prime ideal $\mathfrak{p} \in \operatorname{mAss} A$ such that $\mathfrak{m}/\mathfrak{p} \in \operatorname{Ass}(A/\mathfrak{p})/(\overline{\mathfrak{a}A/\mathfrak{p}})$, see e.g. [42]. So the claim follows by 2.16 since the ring $\widehat{A}/\mathfrak{p}$ is – as a complete local domain – a universally catenarian domain. $\square$

Some of the previous ideas will be applied to the comparison of the ordinary powers of an ideal $\mathfrak{a}$ to the $S$-symbolic powers $\{\mathfrak{a}_S^{(n)}\}_{n\in\mathbb{N}}$ for a multiplicatively closed subset $S$ of the ring $A$. To this end use also the symbolic Rees ring $R_S(\mathfrak{a}) = \oplus_{n\in\mathbb{Z}}\mathfrak{a}_S^{(n)}t^n$ of $\mathfrak{a}$ with respect to $S$.

**Corollary 2.18.** *Let $S$ denote a multiplicatively closed subset of $A$. Let $\mathfrak{a}$ denote a regular ideal of $A$. Suppose that the following conditions are satisfied:*

a) *$A$ is a universally catenarian domain,*
b) *$\operatorname{depth} gr_A(\mathfrak{a})_{\mathfrak{P}} \geq \min\{1, \dim gr_A(\mathfrak{a})_{\mathfrak{P}}\}$ for all $\mathfrak{P} \not\supseteq gr_A(\mathfrak{a})_+$, and*
c) *$l(\mathfrak{a}A_{\mathfrak{p}}) < \dim A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{As}(\mathfrak{a})$ with $\mathfrak{p} \cap S \neq \emptyset$.*

*Then $\mathfrak{a}^n = \mathfrak{a}_S^{(n)}$ for all sufficiently large $n \in \mathbb{N}$.*

*Proof.* As shown in the proof of 2.15 the assumption b) implies that $R^*(\mathfrak{a})/(t^{-1})R^*(\mathfrak{a})$ is unmixed. Furthermore recall that

$$\operatorname{Ass} \mathfrak{a}_S^{(n)}/\mathfrak{a}^n = \{\mathfrak{p} \in \operatorname{Ass} A/\mathfrak{a}^n \mid \mathfrak{p} \cap S \neq \emptyset\}.$$

Therefore, for large $n$ the set $\operatorname{Ass} \mathfrak{a}_S^{(n)}/\mathfrak{a}^n$ will stabilize to a finite set, say $T(\mathfrak{a})$. The claim says that $T(\mathfrak{a}) = \emptyset$. Suppose that $T(\mathfrak{a}) \neq \emptyset$. Now recall that the claim is a local question. Hence without loss of generality we may assume that $(A, \mathfrak{m})$ is a local ring and $T(\mathfrak{a}) = \{\mathfrak{m}\}$. Whence $\mathfrak{a}_S^{(n)} = \mathfrak{a}^n : \langle \mathfrak{m} \rangle$ for all large $n \in \mathbb{N}$.

But now investigate $R^*(\mathfrak{a})$ and $T_{(\mathfrak{m},t^{-1})}(R^*(\mathfrak{a}))$. Since $A$ is universally catenarian and $R^*(\mathfrak{a})/(t^{-1})R^*(\mathfrak{a})$ is unmixed it follows by c) that

$$1 < \dim R^*(\mathfrak{a}) - l(\mathfrak{a}) = \operatorname{height}(\mathfrak{m}, t^{-1})R^*(\mathfrak{a}).$$

Therefore $T_{(\mathfrak{m},t^{-1})}(R^*(\mathfrak{a})) = R^*(\mathfrak{a})$ and $(\mathfrak{a}^n)^* : \langle \mathfrak{m} \rangle = (\mathfrak{a}^n)^*$ for <u>all</u> $n \in \mathbb{N}$. Moreover $(\mathfrak{a}^n)^* = \mathfrak{a}^n$ for all large $n$. Putting together all of these equalities it follows that $T(\mathfrak{a}) = \emptyset$, contracting the choice of $\mathfrak{m}$. $\square$

Suppose that condition b) in 2.18 holds for any homogeneous prime ideal. That means that $gr_A(\mathfrak{a})$ is unmixed with respect to the height. Then the conclusion of 2.18 holds for all $n \in \mathbb{N}$. This follows by a slight modification of the proof of 2.18. To this end one has to replace $R^*(\mathfrak{a})$ by $R(\mathfrak{a})$.

In order to conclude with this section let us relate the finiteness conditions of the symbolic Rees ring $R_S(\mathfrak{a})$ to the existence of an ideal $\mathfrak{b}$ whose $n$-th symbolic power with respect to $S$ coincides with its ordinary power for all large $n \in \mathbb{N}$.

**Theorem 2.19.** *Let $\mathfrak{a}$ resp. $S$ denote an ideal resp. a multiplicatively closed subset of a commutative Noetherian ring $A$.*

a) *$R_S(\mathfrak{a})$ is a finitely generated $R(\mathfrak{a})$-module if and only if $l(\widehat{\mathfrak{a}A_P} + \mathfrak{p}/\mathfrak{p}) < \dim \widehat{A_P}/\mathfrak{p}$ for all prime ideals $P \in \operatorname{As}(\mathfrak{a})$ such that $P \cap S \neq \emptyset$ and all $\mathfrak{p} \in \operatorname{Ass} \widehat{A_P}$.*

b) *$R_S(\mathfrak{a})$ is an $A$-algebra of finite type if and only if there is a $k \in \mathbb{N}$ such that $\mathfrak{b}^n = \mathfrak{b}_S^{(n)}$ for all large $n \in \mathbb{N}$, where $\mathfrak{b} = \mathfrak{a}_S^{(k)}$.*

*Proof.* Firstly show a). Suppose that $R_S(\mathfrak{a})$ is a finitely generated $R(\mathfrak{a})$-module. Let $P \in \operatorname{As}(\mathfrak{a})$ denote a prime ideal such that $P \cap S \neq \emptyset$. Then $\mathfrak{a}^n A_P : \langle PA_P \rangle \subseteq \mathfrak{a}_S^{(n)} A_P$ for all $n \in \mathbb{Z}$. Therefore $T_{(PA_P, t^{-1})}(R(\mathfrak{a}A_P))$ is – as a submodule of $R_{SA_P}(\mathfrak{a}A_P)$ – a finitely generated $R(\mathfrak{a}A_P)$-module. Therefore 2.14 proves the 'only if' part of the claim.

In order to prove the reverse implication note that by the Artin-Rees Lemma it will be enough to show that there is a $k \in \mathbb{N}$ such that the module

$$E_{k,n} = (\mathfrak{a}_S^{(n+k)} + \mathfrak{a}^n)/\mathfrak{a}^n$$

vanishes for all $n \geq 1$. The set of associated prime ideals of $E_{k,n}$ is contained in the finite set $X = \cup_{n \geq 1} \{P \in \operatorname{Ass} A/\mathfrak{a}^n \mid P \cap S \neq \emptyset\}$. Therefore the vanishing of $E_{k,n}$ is a local question for finitely many prime ideals in $X$. By induction it will be enough to prove the vanishing of $E_{k,n}$ at the localization with respect to a minimal prime ideal in $X$. By changing the notation let $(A, \mathfrak{m})$ denote the local ring at this localization. Because of the choice of $\mathfrak{m}$ it implies that $\mathfrak{a}_S^{(n)} = \mathfrak{a}^n :_A \langle \mathfrak{m} \rangle$ for all $n \in \mathbb{N}$. By 2.14 it follows that $T_{(\mathfrak{m}, t^{-1})}(R(\mathfrak{a}))$ is a finitely generated $R(\mathfrak{a})$-module. Therefore $\{\mathfrak{m}\} \not\in \operatorname{Ass} E_{k,n}$ for a certain $k \in \mathbb{N}$ and all $n \geq 1$, i.e. $E_{k,n} = (0)$, as required.

Finally show b). The claim is an easy consequence of 2.11. Recall that $\mathfrak{b}^n = \mathfrak{a}_S^{(nk)}$ if and only if $\mathfrak{b}^n = \mathfrak{b}_S^{(n)}$. □

2.4. **The Lichtenbaum-Hartshorne Vanishing Theorem.** The Lichtenbaum-Hartshorne vanishing theorem for local cohomology, see [16], characterizes the vanishing of $H_{\mathfrak{a}}^d(A)$ for an ideal $\mathfrak{a}$ in a $d$-dimensional local ring $(A, \mathfrak{m})$. Our proof yields an essential simplification by the use of ideal topologies.

For a finitely generated $d$-dimensional $A$-module $M$ let $(\operatorname{Ass} M)_d$ denote all the associated prime ideals of $M$ with $\dim A/\mathfrak{p} = d$. For an ideal $\mathfrak{a}$ of $A$ let $\mathfrak{u} = \mathfrak{u}(\mathfrak{a}\widehat{A})$ denote the intersection of those primary components $\mathfrak{q}$ of 0 in $\widehat{A}$ such that $\dim \widehat{A}/(\mathfrak{a}\widehat{A} + \mathfrak{p}) > 0$ for $\mathfrak{p} \in (\operatorname{Ass} \widehat{A})_d$, where $\mathfrak{p} = \operatorname{Rad} \mathfrak{q}$ and $d = \dim A$.

**Theorem 2.20.** *Let $\mathfrak{a}$ denote an ideal in a $d$-dimensional local ring $(A, \mathfrak{m})$. Then $H^d_{\mathfrak{a}}(A) \simeq \operatorname{Hom}_A(\mathfrak{u}, E)$, where $E$ denotes the injective hull of the residue field $A/\mathfrak{m}$. In particular $H^d_{\mathfrak{a}}(A)$ is an Artinian $A$-module and $H^d_{\mathfrak{a}}(A) = 0$ if and only if $\dim \widehat{A}/(\mathfrak{a}\widehat{A} + \mathfrak{p}) > 0$ for all $\mathfrak{p} \in (\operatorname{Ass} \widehat{A})_d$.*

*Proof.* As above let $T = \operatorname{Hom}_A(\cdot, E)$ denote the Matlis duality functor. Because of the following isomorphisms

$$T(H^d_{\mathfrak{a}}(A)) \simeq T(H^d_{\mathfrak{a}}(A) \otimes_A \widehat{A}) \simeq T(H^d_{\mathfrak{a}\widehat{A}}(\widehat{A}))$$

one may assume without loss of generality that $A$ is a complete local ring. So $A$ is the factor ring of a complete local Gorenstein ring $(B, \mathfrak{n})$ with $\dim A = \dim B = d$, say $A = B/\mathfrak{b}$. Replacing $\mathfrak{a}$ by its preimage in $B$ we have to consider $T(H^d_{\mathfrak{a}}(B/\mathfrak{b}))$. Let $\mathfrak{b}_d$ denote the intersection of all of the primary components $\mathfrak{q}$ of $\mathfrak{b}$ such that $\dim B/\mathfrak{p} = d$ for $\mathfrak{p}$ its associated prime ideal. Because of $\dim \mathfrak{b}_d/\mathfrak{b} < d$ the short exact sequence

$$0 \to \mathfrak{b}_d/\mathfrak{b} \to B/\mathfrak{b} \to B/\mathfrak{b}_d \to 0$$

implies that $H^d_{\mathfrak{a}}(B/\mathfrak{b}) \simeq H^d_{\mathfrak{a}}(B/\mathfrak{b}_d)$. Replacing $\mathfrak{b}_d$ by $\mathfrak{b}$ one may assume that $B/\mathfrak{b}$ is unmixed with respect to the dimension. There is an isomorphism

$$T(H^d_{\mathfrak{a}}(B) \otimes B/\mathfrak{b}) \simeq \operatorname{Hom}_B(B/\mathfrak{b}, T(H^d_{\mathfrak{a}}(B))).$$

Because the Hom-functor transforms direct into inverse limits it turns out that

$$T(H^d_{\mathfrak{a}}(B)) \simeq \varprojlim T(\operatorname{Ext}^d_B(B/\mathfrak{a}^n, B)) \simeq \varprojlim H^0_{\mathfrak{n}}(B/\mathfrak{a}^n),$$

as follows by the local duality, see 1.8. Because of $H^0_{\mathfrak{n}}(B/\mathfrak{a}^n) = \mathfrak{a}^n : \langle \mathfrak{n} \rangle/\mathfrak{a}^n$ and because $B$ is a complete local ring we see that $\varprojlim H^0_{\mathfrak{n}}(B/\mathfrak{a}^n) \simeq \cap_{n \in \mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{n} \rangle)$. But now the ideal $\cap_{n \in \mathbb{N}}(\mathfrak{a}^n : \langle \mathfrak{n} \rangle)$ is the ideal $\mathfrak{v}$ of $B$ that is the intersection of all primary components $\mathfrak{q}$ of 0 such that $\dim B/(\mathfrak{a} + \mathfrak{p}) > 0$ for $\mathfrak{p}$ the associated prime ideal of $\mathfrak{q}$, see 2.3. Therefore

$$T(H^d_{\mathfrak{a}}(B) \otimes_B B/\mathfrak{b}) \simeq \operatorname{Hom}_B(B/\mathfrak{b}, \mathfrak{v}) \simeq (0 :_B \mathfrak{b}) \cap \mathfrak{v}.$$

Furthermore it follows that $(0 :_B \mathfrak{b}) \cap \mathfrak{b} = 0$ since $\mathfrak{b}$ is an unmixed ideal in a Gorenstein ring $B$ with $\dim B = B/\mathfrak{b}$. Therefore $(0 :_B \mathfrak{b}) \cap \mathfrak{v} \simeq ((0 :_B \mathfrak{b}) \cap \mathfrak{v} + \mathfrak{b})/\mathfrak{b}$. Hence $(0 :_B \mathfrak{b}) \cap \mathfrak{v}$ is isomorphic to an ideal of $B/\mathfrak{b}$. Finally note that $\mathfrak{u} \simeq (0 :_B \mathfrak{b}) \cap \mathfrak{v}$ as follows by considering the set of associated prime ideals. Then the statement is a consequence of the Matlis duality. $\square$

The vanishing of the ideal $\mathfrak{u}$ is equivalent to the equivalence of certain ideal topologies, see 2.2. So there is another characterization of the vanishing of $H^d_{\mathfrak{a}}(A)$ for certain local rings.

**Corollary 2.21.** *Suppose that* $(A, \mathfrak{m})$ *denotes a formally equidimensional local ring. Then* $H^d_{\mathfrak{a}}(A) = 0$, $d = \dim A$, *if and only if the topology defined by* $\{\mathfrak{a}^n : \langle \mathfrak{m} \rangle\}_{n \in \mathbb{N}}$ *is equivalent to the* $\mathfrak{a}$*-adic topology.*

*Proof.* By virtue of 2.2 $\{\mathfrak{a}^n : \langle \mathfrak{m} \rangle\}_{n \in \mathbb{N}}$ is equivalent to the $\mathfrak{a}$-adic topology if and only if $\dim \widehat{A}/(\mathfrak{a}\widehat{A} + \mathfrak{p}) > 0$ for all $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$. But now $\dim A = \dim \widehat{A}/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$ by the assumption on $A$. So the claim follows by 2.20. $\square$

2.5. **Connectedness results.** Let $\mathfrak{a}, \mathfrak{b}$ denote two ideals of a commutative Noetherian ring $A$. Then there is a short exact sequence

$$0 \to A/\mathfrak{a} \cap \mathfrak{b} \xrightarrow{i} A/\mathfrak{a} \oplus A/\mathfrak{b} \xrightarrow{p} A/(\mathfrak{a} + \mathfrak{b}) \to 0,$$

where $i(a + \mathfrak{a} \cap \mathfrak{b}) = (a + \mathfrak{a}, -a + \mathfrak{b})$ and $p(a + \mathfrak{a}, b + \mathfrak{b}) = a + b + (\mathfrak{a} + \mathfrak{b})$ for $a, b \in A$. Because of the direct summand in the middle this sequence provides a helpful tool for connecting properties. This short exact sequence is an important ingredient for the next lemma.

In order to prove the connectedness theorem we need some preparations. A basic tool for this section will be the so-called Mayer-Vietoris sequence for local cohomology helpful also for different purposes.

**Lemma 2.22.** *Let* $\mathfrak{a}, \mathfrak{b}$ *denote two ideals of a commutative Noetherian ring* $A$*. Then there is a functorial long exact sequence*

$$\ldots \to H^n_{\mathfrak{a}+\mathfrak{b}}(M) \to H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) \to H^n_{\mathfrak{a}\cap\mathfrak{b}}(M) \to H^{n+1}_{\mathfrak{a}+\mathfrak{b}}(M) \to \ldots$$

*for any* $A$*-module* $M$*.*

*Proof.* Let $\mathfrak{c}$ denote an ideal of the ring $A$. Then first note that

$$\varinjlim \operatorname{Ext}^n(A/\mathfrak{c}^n, M) \simeq H^n_{\mathfrak{c}}(M)$$

for any $A$-module $M$ and all $n \in \mathbb{Z}$, see e.g. [14]. Now consider the short exact sequence at the beginning of this subsection for the ideals $\mathfrak{a}^n$ and $\mathfrak{b}^n$. Then take into account that the topologies defined by the families $\{\mathfrak{a}^n + \mathfrak{b}^n\}_{n \in \mathbb{N}}$ and $\{\mathfrak{a}^n \cap \mathfrak{b}^n\}_{n \in \mathbb{N}}$ are equivalent to the $(\mathfrak{a} + \mathfrak{b})$-adic and $\mathfrak{a} \cap \mathfrak{b}$-adic topology resp. Therefore the direct limit of the long exact Ext-sequence proves the claim. $\square$

Our first connectedness result is the following statement, a slight generalization of Hartshorne's connectedness result, see [15].

**Theorem 2.23.** *Let* $\mathfrak{c}$ *denote an ideal of a local ring* $(A, \mathfrak{m})$*. Suppose that* $\operatorname{grade} \mathfrak{c} > 1$*. Then the scheme* $\operatorname{Spec} A \setminus V(\mathfrak{c})$ *is connected.*

*Proof.* Because of grade $\mathfrak{c} > 1$ it follows that $H^i_{\mathfrak{c}}(A) = 0$ for $i = 0, 1$. Assume that $\operatorname{Spec} A \setminus V(\mathfrak{c})$ is not connected. Then there are non-nilpotent ideals $\mathfrak{a}, \mathfrak{b}$ satisfying the following properties:

1) $\mathfrak{a} \cap \mathfrak{b}$ is nilpotent,
2) $\operatorname{Spec} A \setminus V(\mathfrak{a})$ and $\operatorname{Spec} A \setminus V(\mathfrak{b})$ are disjoint and non-empty subsets of $\operatorname{Spec} A$.
3) $\operatorname{Spec} A \setminus V(\mathfrak{c}) = (\operatorname{Spec} A \setminus V(\mathfrak{a})) \cup (\operatorname{Spec} A \setminus V(\mathfrak{b}))$

Note that these conditions imply that $\operatorname{Rad}(\mathfrak{a} + \mathfrak{b}) = \operatorname{Rad} \mathfrak{c}$. Now consider the first part of the Mayer-Vietoris sequence

$$0 \to H^0_{\mathfrak{a}+\mathfrak{b}}(A) \to H^0_{\mathfrak{a}}(A) \oplus H^0_{\mathfrak{b}}(A) \to H^0_{\mathfrak{a}\cap\mathfrak{b}}(A) \to H^1_{\mathfrak{a}+\mathfrak{b}}(A).$$

Because of grade $\mathfrak{c} > 1$ and $\operatorname{Rad}(\mathfrak{a} + \mathfrak{b}) = \operatorname{Rad} \mathfrak{c}$ it turns out that $H^i_{\mathfrak{a}+\mathfrak{b}}(A) = 0$ for $i = 0, 1$. Moreover $\mathfrak{a} \cap \mathfrak{b}$ is nilpotent. Whence it yields that $H^0_{\mathfrak{a}\cap\mathfrak{b}}(A) = A$. So the Mayer-Vietoris sequence implies an isomorphism $H^0_{\mathfrak{a}}(A) \oplus H^0_{\mathfrak{b}}(A) \simeq A$. Since the ring $A$ – as a local ring – is indecomposable it follows either $H^0_{\mathfrak{a}}(A) = A$ and $H^0_{\mathfrak{b}}(A) = 0$ or $H^0_{\mathfrak{a}}(A) = 0$ and $H^0_{\mathfrak{b}}(A) = A$. But this means that $\mathfrak{a}$ resp. $\mathfrak{b}$ is a nilpotent ideal. Therefore we have a contradiction, so $\operatorname{Spec} A \setminus V(\mathfrak{c})$ is connected. $\square$

The author is grateful to Leif Melkersson for suggesting the above simplification of the original arguments.

Let $\mathfrak{a}$ denote the homogeneous ideal in $A = k[x_0, \ldots, x_3]$ describing the union of two disjoint lines in $\mathbb{P}^3_k$. Suppose that $\mathfrak{a}$ is up to the radical equal to an ideal $\mathfrak{c}$ generated by two elements. Then $\operatorname{Spec} A \setminus V(\mathfrak{a}) = \operatorname{Spec} A \setminus V(\mathfrak{c})$ is disconnected. Therefore grade $\mathfrak{c} \leq 1$, contradicting the fact that $\mathfrak{c}$ is an ideal of height 2 in a Cohen-Macaulay ring $A$. So $\mathfrak{a}$ is not set-theoretically a complete intersection. For further examples of this type see [15].

The previous result implies as a corollary a result on the length of chains of prime ideals in a catenarian local ring.

**Corollary 2.24.** *Let $(A, \mathfrak{m})$ denote a local Noetherian ring satisfying the condition $S_2$. Suppose that $A$ is catenarian. Then it is equidimensional, i.e. all of the minimal prime ideals have the same dimension.*

*Proof.* Let $\mathfrak{p}, \mathfrak{q}$ denote two minimal prime ideals of $\operatorname{Spec} A$. Then it is easily seen that there is a chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_r = \mathfrak{q}$$

such that $\operatorname{height}(\mathfrak{p}_i, \mathfrak{p}_{i+1}) = 1$ for all $i = 1, \ldots, r-1$. Hence by the catenarian condition

$$\dim A/\mathfrak{p}_i = 1 + \dim A/(\mathfrak{p}_i, \mathfrak{p}_{i+1}) = \dim A/\mathfrak{p}_{i+1}.$$

By iterating this $(r-1)$-times it follows that $\dim A/\mathfrak{p} = \dim A/\mathfrak{q}$, as required. $\square$

In order to prove the connectedness theorem inspired by G. Faltings we need a lemma first invented by M. Brodmann and J. Rung, see [5].

**Lemma 2.25.** *Let* $(A, \mathfrak{m})$ *denote an analytically irreducible domain with* $d = \dim A > 1$. *Suppose there are two ideals* $\mathfrak{b}, \mathfrak{c}$ *of* $A$ *such that* $\dim A/\mathfrak{b} > 0, \dim A/\mathfrak{c} > 0$, *and* $H^{d-1}_{\mathfrak{b}\cap\mathfrak{c}}(A) = 0$. *Then* $\dim A/\mathfrak{b} + \mathfrak{c} > 0$.

*Proof.* Suppose the contrary is true. Then $\mathfrak{b} + \mathfrak{c}$ is an $\mathfrak{m}$-primary ideal. The Mayer-Vietoris sequence provides an exact sequence

$$H^{d-1}_{\mathfrak{b}\cap\mathfrak{c}}(A) \to H^d_{\mathfrak{m}}(A) \to H^d_{\mathfrak{b}}(A) \oplus H^d_{\mathfrak{c}}(A) \to H^d_{\mathfrak{b}\cap\mathfrak{c}}(A).$$

Because $A$ is analytically irreducible and because of the vanishing of $H^{d-1}_{\mathfrak{b}\cap\mathfrak{c}}(A)$ the vanishing result 2.20 yields an isomorphism $H^d_{\mathfrak{m}}(A) \simeq H^d_{\mathfrak{b}}(A) \oplus H^d_{\mathfrak{c}}(A)$. Because of the non-vanishing of $H^d_{\mathfrak{m}}(A)$ this provides the non-vanishing of one of the direct summands, say $H^d_{\mathfrak{b}}(A)$. By 2.20 this means that $\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal, contradicting the assumption. ☐

This lemma is the main technical tool for the connectedness result given in the sequel.

**Theorem 2.26.** *Suppose* $\mathfrak{a}$ *denotes an ideal of an analytically irreducible domain* $(A, \mathfrak{m})$ *with* $d = \dim A > 1$. *Suppose that* $H^n_{\mathfrak{a}}(A) = 0$ *for* $n = d-1, d$. *Then* $(\operatorname{Spec} A/\mathfrak{a}) \setminus V(\mathfrak{m}/\mathfrak{a})$ *is connected.*

*Proof.* Suppose the contrary. Then there exist ideals $\mathfrak{b}, \mathfrak{c}$ of $A$ such that $\operatorname{Rad}(\mathfrak{b}\cap\mathfrak{c}) = \operatorname{Rad}\mathfrak{a}$ and $\mathfrak{b} + \mathfrak{c}$ is $\mathfrak{m}$-primary, but neither $\mathfrak{b}$ nor $\mathfrak{c}$ is an $\mathfrak{m}$-primary ideal. Because of the vanishing of $H^{d-1}_{\mathfrak{a}}(A)$ Lemma 2.25 provides a contradiction. ☐

The preceding result shows for instance the non-vanishing of $H^3_{\mathfrak{a}}(A)$ for the ideal $\mathfrak{a}$ of the union of two disjoint lines in $\mathbb{P}^3_k$ and $A = k[x_0, \ldots, x_3]$. This yields another proof that $\mathfrak{a}$ is not set-theoretically a complete intersection.

In the following let us generalize this connectedness result to the case of local rings that are not necessarily analytically unmixed. This was obtained by C. Huneke and M. Hochster, see [20], by a different argument.

**Theorem 2.27.** *Let* $(A, \mathfrak{m})$ *denote a d-dimensional local ring which is the quotient of a Gorenstein ring. Assume that* $A$ *satisfies the condition* $S_2$. *Suppose that* $\mathfrak{a}$ *denotes an ideal of* $A$ *such that* $H^n_{\mathfrak{a}}(A) = 0$ *for* $n = d-1, d$. *Then the scheme* $(\operatorname{Spec} A/\mathfrak{a}) \setminus V(\mathfrak{m}/\mathfrak{a})$ *is connected.*

*Proof.* As in the proof of 2.26 suppose the contrary. That is, there exist ideals $\mathfrak{b}, \mathfrak{c}$ of $A$ such that $\operatorname{Rad}(\mathfrak{b}\cap\mathfrak{c}) = \operatorname{Rad}\mathfrak{a}$ and $\mathfrak{b} + \mathfrak{c}$ is $\mathfrak{m}$-primary, but neither $\mathfrak{b}$ nor $\mathfrak{c}$ is an

$\mathfrak{m}$-primary ideal. By changing the notation let us assume that $A$ is a complete local ring. Then the Mayer-Vietoris sequence provides an isomorphism

$$H^d_{\mathfrak{m}}(A) \simeq H^d_{\mathfrak{b}}(A) \oplus H^d_{\mathfrak{c}}(A).$$

This implies that $K_A \simeq \mathfrak{u} \oplus \mathfrak{v}$, where $\mathfrak{u}$ resp. $\mathfrak{v}$ denotes the intersection of those primary components $\mathfrak{q}$ of the zero-ideal of $A$ such that $\dim A/\mathfrak{b} + \mathfrak{p} > 0$ resp. $\dim A/\mathfrak{c} + \mathfrak{p} > 0$ for $\mathfrak{p}$, the associated prime ideal of $\mathfrak{q}$, see 2.20. By the definitions, see Section 1.2, it follows now that $K_{K_A} \simeq K_{\mathfrak{u}} \oplus K_{\mathfrak{v}}$. Moreover since $A$ satisfies $S_2$ it is equidimensional by 2.22. Therefore $A \simeq K_{K_A}$, as turns out by 1.14.

By the Nakayama lemma one of the direct summands, say $K_{\mathfrak{v}}$, is zero, while for the second summand $A \simeq K_{\mathfrak{u}}$. By 1.9 it follows that $\operatorname{Ass} K_{\mathfrak{u}} = \operatorname{Ass} \mathfrak{u}$. Because of

$$\operatorname{Ass}_A \mathfrak{u} = \{\mathfrak{p} \in \operatorname{Ass} A \mid \dim A/\mathfrak{b} + \mathfrak{p} = 0\}$$

the equality $\operatorname{Ass} A = \operatorname{Ass} K_{\mathfrak{u}}$ implies that $\mathfrak{m} \subseteq \operatorname{Rad}(\mathfrak{b} + \mathfrak{p})$ for any associated prime ideal $\mathfrak{p}$ of $A$. As is easily seen it follows that $\mathfrak{m} \subseteq \operatorname{Rad}(\mathfrak{b})$, a contradiction. Therefore $(\operatorname{Spec} A/\mathfrak{a}) \setminus V(\mathfrak{m}/\mathfrak{a})$ is connected, as required. □

## 3. Local Cohomology and Syzygies

3.1. **Local cohomology and Tor's.** As above let $(A, \mathfrak{m})$ denote a local ring. Let $T(\cdot) = \operatorname{Hom}_A(\cdot, E)$ denote the Matlis duality functor, where $E = E_A(A/\mathfrak{m})$ is the injective hull of the residue field. In the following consider a length estimate for the length of $\operatorname{Tor}^A_n(M, N)$ resp. $\operatorname{Ext}^n_A(M, N)$ under the additional assumption that $M \otimes_A N$ is an $A$-module of finite length.

**Lemma 3.1.** *Let $M, N$ denote finitely generated $A$-modules such that $M \otimes_A N$ is an $A$-module of finite length. Then*

$$\operatorname{Ext}^i_A(M, H^j_{\mathfrak{m}}(N)) \text{ and } \operatorname{Tor}^A_i(M, H^j_{\mathfrak{m}}(N))$$

*are $A$-modules of finite length for all $i, j \in \mathbb{Z}$.*

*Proof.* Without loss of generality we may assume that $A$ is a complete local ring. Then $A$ is a quotient of a local Gorenstein ring $B$ with $\dim B = r$. By the local duality theorem, see 1.8, it turns out that $H^j_{\mathfrak{m}}(N) \simeq T(K^j_N)$ for all $j \in \mathbb{Z}$, where $K^j_N \simeq \operatorname{Ext}^{r-j}_B(N, B)$ with the natural $A$-module structure, see 1.8. Moreover there are natural isomorphisms

$$\operatorname{Ext}^i_A(M, H^j_{\mathfrak{m}}(N)) \simeq T(\operatorname{Tor}^A_i(M, K^j_N)) \text{ and } \operatorname{Tor}^A_i(M, H^j_{\mathfrak{m}}(N)) \simeq T(\operatorname{Ext}^i_A(M, K^j_N))$$

for all $i, j \in \mathbb{Z}$. But now $\operatorname{Supp} K^j_N \subseteq V(\operatorname{Ann}_A N)$ for all $j \in \mathbb{Z}$. Therefore

$$\operatorname{Supp} \operatorname{Tor}^A_i(M, K^j_N) \subseteq V(\operatorname{Ann}_A M, \operatorname{Ann}_A N).$$

By the assumption $M \otimes_A N$ is an $A$-module of finite length. That is,

$$V(\operatorname{Ann}_A M, \operatorname{Ann}_A N) \subseteq V(\mathfrak{m}),$$

which proves that $\operatorname{Tor}_i^A(M, K_N^j)$ is an $A$-module of finite length for all $i, j \in \mathbb{Z}$ too. By the Matlis duality the first part of the claim is shown. The second part follows by the same argument. $\square$

The previous result 3.1 provides the desired bounds for the length of $\operatorname{Tor}_n^A(M, N)$ and $\operatorname{Ext}_A^n(M, N)$.

**Theorem 3.2.** *Let $M, N$ be two finitely generated $A$-modules such that $M \otimes_A N$ is an $A$-module of finite length. Then*

a) $L_A(\operatorname{Ext}_A^n(M, N)) \leq \sum_{i \geq 0} L_A(\operatorname{Ext}_A^i(M, H_{\mathfrak{m}}^{n-i}(N)))$ *and*

b) $L_A(\operatorname{Tor}_n^A(M, N)) \leq \sum_{i \geq 0} L_A(\operatorname{Tor}_{n+i}^A(M, H_{\mathfrak{m}}^i(N)))$

*for all $n \in \mathbb{Z}$.*

*Proof.* First choose $\underline{x} = x_1, \ldots, x_d$, $d = \dim A$, a system of parameters of $A$. Therefore $\operatorname{Rad} \underline{x} = \mathfrak{m}$. The corresponding Čech complex $K^\cdot = K_{\underline{x}}^\cdot$ has the property that $H^n(K^\cdot \otimes_A N) \simeq H_{\mathfrak{m}}^n(N)$ for all $n \in \mathbb{Z}$, see 1.3. Furthermore choose $F^\cdot$ a minimal free resolution of $M$. In order to show the first claim consider the complex $K^\cdot \otimes_A \operatorname{Hom}_A(F^\cdot, N)$. Because of the structure of $K^i$ as the direct sum of localizations it turns out that the natural homomorphism

$$\operatorname{Hom}_A(F^\cdot, N) \to K^\cdot \otimes_A \operatorname{Hom}_A(F^\cdot, N)$$

induces an isomorphism in cohomology. Moreover

$$K^\cdot \otimes_A \operatorname{Hom}_A(F^\cdot, N) \xrightarrow{\sim} \operatorname{Hom}_A(F^\cdot, K^\cdot \otimes_A N)$$

as it is easily seen. So there is a spectral sequence

$$E_2^{i,j} = \operatorname{Ext}_A^i(M, H_{\mathfrak{m}}^j(N)) \Rightarrow E^n = \operatorname{Ext}_A^n(M, N).$$

Therefore $\operatorname{Ext}_A^n(M, N)$ possesses a finite filtration whose quotients $E_\infty^{i,n-i}$ are modules of finite length such that $L_A(E_\infty^{i,n-i}) \leq L_A(E_2^{i,n-i}) < \infty$, which proves the first bound.

In order to prove the second bound proceed by a similar argument. Consider the complex $K^\cdot \otimes_A (F^\cdot \otimes_A N)$. As above the natural map $F^\cdot \otimes_A N \to K^\cdot \otimes_A (F^\cdot \otimes_A N)$ induces an isomorphism in cohomology. In order to continue consider the spectral sequence

$$E_2^{i,j} = \operatorname{Tor}_{-i}^A(M, H_{\mathfrak{m}}^j(N)) \Rightarrow E^n = \operatorname{Tor}_{-n}^A(M, N)$$

for computing the cohomology of $K^\cdot \otimes_A (F^\cdot \otimes_A N)$. It provides – in a similar way as above – the second claim. $\square$

In the particular case of $N$ a Cohen-Macaulay module with $M \otimes_A N$ of finite length the spectral sequences in the proof of 3.2 degenerate to isomorphisms.

**Corollary 3.3.** *Let $N$ be a Cohen-Macaulay module. Then there are the following isomorphisms*

$$\operatorname{Ext}^n_A(M, N) \simeq \operatorname{Ext}^{n-d}_A(M, H^d_{\mathfrak{m}}(N)) \text{ and } \operatorname{Tor}^A_n(M, N) \simeq \operatorname{Tor}^A_{n+d}(M, H^d_{\mathfrak{m}}(N))$$

*for all $n \in \mathbb{Z}$, where $d = \dim N$.*

Under the additional assumption that $M$ is an $A$-module of finite projective dimension it is of some interest to determine the largest integer $n$ such that $\operatorname{Tor}^A_n(M, N) \neq 0$. This yields an equality of the Auslander-Buchsbaum type, shown by M. Auslander, see [1, Theorem 1.2].

**Theorem 3.4.** *Let $M, N$ be two non-zero finitely generated $A$-modules. Suppose that $\operatorname{pd}_A M$ is finite. Then*

$$\sup\{n \in \mathbb{Z} \mid \operatorname{Tor}^A_n(M, N) \neq 0\} + \operatorname{depth}_A N = \operatorname{pd}_A M$$

*provided $\operatorname{depth} \operatorname{Tor}^A_s(M, N) = 0$, where $s = \sup\{n \in \mathbb{Z} \mid \operatorname{Tor}^A_n(M, N) \neq 0\}$. In particular the equality holds whenever $M \otimes_A N$ is an $A$-module of finite length.*

*Proof.* Set $p = \operatorname{pd}_A M$ and $t = \operatorname{depth}_A N$. As in the proof of 3.2 consider the complex $C^{\cdot} := K^{\cdot} \otimes_A F^{\cdot} \otimes_A N$, where $K^{\cdot}$ resp. $F^{\cdot}$ denotes the Čech complex resp. the (finite) minimal free resolution of $M$. Then there is the following spectral sequence

$$E_2^{-i,j} = \operatorname{Tor}^A_i(M, H^j_{\mathfrak{m}}(N)) \Rightarrow E^{-i+j} = H^{-i+j}(C^{\cdot}).$$

Consider the stages $-i + j =: n \leq -p + t$. In the case $n < -p + t$ it follows that $E_2^{-i,j} = 0$. Note that whenever $j < t$, then $H^j_{\mathfrak{m}}(N) = 0$, and whenever $j \geq t$, then $i > p = \operatorname{pd}_A M$. In the case $n = -p + t$ it follows by a similar consideration that $E_2^{-i,j} = 0$ for $i \neq p$. So there is a partial degeneration to the isomorphism $H^{-p+t}(C^{\cdot}) \simeq \operatorname{Tor}^A_p(M, H^t_{\mathfrak{m}}(N))$ and the vanishing $H^n(C^{\cdot}) = 0$ for all $n < -p + t$.

Next show that $H^{-p+t}(C^{\cdot}) \neq 0$. By 1.5 $H^t_{\mathfrak{m}}(N)$ is an Artinian $A$-module. Therefore it possesses a submodule which is isomorphic to $k = A/\mathfrak{m}$. The corresponding short exact sequence

$$0 \to k \to H^t_{\mathfrak{m}}(N) \to C \to 0$$

induces an injection $0 \to \operatorname{Tor}^A_p(M, k) \to \operatorname{Tor}^A_p(M, H^t_{\mathfrak{m}}(N))$. Because of $\operatorname{Tor}^A_p(M, k) \neq 0$ this shows the claim.

In order to continue with the proof consider the spectral sequence

$$E_2^{i,-j} = H^i_{\mathfrak{m}}(\operatorname{Tor}^A_j(M, N)) \Rightarrow E^{i-j} = H^{i-j}(C^{\cdot}).$$

Put $i-j =: n$. In the case of $n < -s$ it follows that $E_2^{i,-j} = 0$ by similar arguments as above in the first spectral sequence. Note that $s = \sup\{n \in \mathbb{Z} \mid \operatorname{Tor}^A_n(M, N) \neq 0\}$.

Therefore $H^n(C^\cdot) = 0$ for all $n < -s$ and $H^{-s}(C^\cdot) \simeq H^0_{\mathfrak{m}}(\operatorname{Tor}^A_s(M,N)) \neq 0$. Recall that $\operatorname{depth} \operatorname{Tor}^A_s(M,N) = 0$. This finally proves $-s = -p + t$, as required. □

Another case of describing $\sup\{n \in \mathbb{Z} \mid \operatorname{Tor}^A_n(M,N) \neq 0\}$ was investigated by M. Auslander, see [1, Theorem 1.2]. It follows in the same way as above by considering both of the spectral sequences.

As an immediate consequence of 3.4 it turns out that $\operatorname{depth}_A N \leq \operatorname{pd}_A M$ provided $M \otimes_A N$ is an $A$-module of finite length. Under these assumptions a much stronger inequality holds, namely $\dim_A N \leq \operatorname{pd}_A M$. This is the Intersection Theorem proved by C. Peskine and L. Szpiro, see [39], and M. Hochster, see [19], in the equicharacteristic case, and finally by P. Roberts, see [38], in the remaining case. For a summary of these and related results about Cohen-Macaulay rings see also the monograph [7].

In relation to that the following Cohen-Macaulay criterion could be of some interest.

**Corollary 3.5.** *Let $M, N$ be two finitely generated $A$-modules such that $M \otimes_A N$ is of finite length. Suppose that* $\operatorname{pd}_A M$ *is finite. Then $N$ is a Cohen-Macaulay module with* $\operatorname{depth}_A N = \operatorname{pd}_A M$ *if and only if* $\operatorname{Tor}^A_n(M,N) = 0$ *for all $n \geq 1$.*

*Proof.* First assume that $N$ is a Cohen-Macaulay module. Then

$$\operatorname{Tor}^A_n(M,N) \simeq \operatorname{Tor}^A_{n+d}(M, H^d_{\mathfrak{m}}(N)),\ d = \dim N,$$

as follows by 3.3. But now $d = \operatorname{pd}_A M$. Therefore the last module vanishes for all positive $n$.

For the proof of the reverse implication note that

$$0 = \operatorname{pd}_A M - \operatorname{depth}_A N \geq \operatorname{pd}_A M - \dim_A N$$

as follows by 3.4. But now $\operatorname{pd}_A M - \dim_A N \geq 0$ by view of the Intersection Theorem. This finishes the proof. □

By view of the formula of M. Auslander and D. Buchsbaum one may interpret the inequality $\operatorname{depth}_A N \leq \operatorname{pd}_A M$ in the following way

$$\operatorname{depth}_A N + \operatorname{depth}_A M \leq \operatorname{depth} A,$$

provided $M \otimes_A N$ is of finite length and $\operatorname{pd}_A M$ is finite. One might think of it as a generalization of Serre's inequality $\dim_A N + \dim_A M \leq \dim A$ in the case of $A$ a regular local ring.

In connection to 3.5 the rigidity of Tor could be of some interest. Let $n \in \mathbb{N}$. Then the conjecture says that $\operatorname{Tor}^A_{n+1}(M,N) = 0$ provided $\operatorname{Tor}^A_n(M,N) = 0$. This is true for a regular local ring $(A, \mathfrak{m})$ as shown by J. P. Serre, see [45], in the case of unramified regular local rings, and finally for any regular local ring by S. Lichtenbaum, see

[26]. There are also related results in [1]. The rigidity conjecture for a general local ring and $\operatorname{pd}_A M < \infty$ was disproved by R. Heitmann's example, see [18]. For some recent developments on the rigidity in connection to non-regular local rings, see the work of C. Huneke and R. Wiegand in [23].

Under the assumptions of 3.5 J. P. Serre, see [45], considered the Euler characteristic $\chi(M,N) = \sum_{i\geq 0}(-1)^i L_A(\operatorname{Tor}_i^A(M,N))$ as an intersection number. More generally for $n \in \mathbb{N}$ he defined the partial Euler characteristics

$$\chi_n(M,N) = \sum_{i\geq 0}(-1)^i L_A(\operatorname{Tor}_{n+i}^A(M,N)).$$

Note that $\chi(M,N) = \chi_0(M,N)$. In the case of an unramified regular local ring $(A,\mathfrak{m})$ J. P. Serre, see [45], proved the non-negativity of $\chi(M,N)$. Moreover he conjectured that this is true for any regular local ring. Recently O. Gabber, see [11], proved the non-negativity of $\chi(M,N)$ for any regular local ring. As follows by view of R. Heitmann's example $\chi_1(M,N) \geq 0$ does not hold in the case of an arbitrary local ring and $\operatorname{pd}_A M < \infty$.

The Cohen-Macaulay property of $N$ in 3.5 provides that $L_A(M\otimes_A N) = \chi_0(M,N)$. This equality is equivalent to the vanishing of $\chi_1(M,N)$. Consider the particular case of a finitely generated $A$-module $N$ and $M = A/\underline{x}A$, where $\underline{x} = x_1,\ldots,x_r$ denotes an $A$-regular sequence. Then $\operatorname{pd}_A M = r$. Suppose that $N/\underline{x}N$ is an $A$-module of finite length, i.e. $\dim N \leq r$. Then $\chi_0(M,N) = e_0(\underline{x};N)$ as follows since $\operatorname{Tor}_i^A(A/\underline{x}A,N) \simeq H_i(\underline{x};N), i \in \mathbb{N}$, and $\sum_{i\geq 0}(-1)^i L_A(H_i(\underline{x};N)) = e_0(\underline{x};N)$, where $e_0(\underline{x};N)$ denotes the multiplicity of $N$ with respect to $\underline{x}$, see [2]. So the equality $L(N/\underline{x}N) = e_0(\underline{x};N)$ says that $N$ is a Cohen-Macaulay module with $\dim N = r$.

*Conjecture* 3.6. Let $M,N$ be two finitely generated $A$-modules such that $M\otimes_A N$ is an $A$-module of finite length and $\operatorname{pd}_A M < \infty$.

a) (Cohen-Macaulay Conjecture) Suppose that $L_A(M\otimes_A N) = \chi_0(M,N)$. Does it follows that $N$ is a Cohen-Macaulay module with $\operatorname{pd}_A M = \operatorname{depth}_A N$?
b) (Weak Rigidity Conjecture) Suppose that $\chi_n(M,N) = 0$ for a certain $n \in \mathbb{N}$. Does it follows that $\chi_{n+1}(M,N) = 0$?

Suppose that the weak rigidity conjecture is true. Then $\chi_n(M,N) = 0$ implies inductively that $\operatorname{Tor}_k^A(M,N) = 0$ for all $k \geq n$. To this end recall that $\operatorname{pd}_A M$ is finite. Let us return to this observation in the following result.

**Corollary 3.7.** *Let $\mathfrak{a},\mathfrak{b}$ be two ideals of a local ring $(A,\mathfrak{m})$. Assume that $\operatorname{pd}_A A/\mathfrak{a}$ is finite and $\mathfrak{a}+\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal.*

a) *Suppose that $A/\mathfrak{a}$ is a Cohen-Macaulay ring with* $\operatorname{depth} A/\mathfrak{a} + \operatorname{depth} A/\mathfrak{b} = \operatorname{depth} A$. *Then $\mathfrak{a}\cap\mathfrak{b} = \mathfrak{a}\mathfrak{b}$ (resp. $\chi(M,N) = L_A(A/(\mathfrak{a}+\mathfrak{b}))$.*
b) *Suppose that $A/\mathfrak{a}$ is rigid (resp. weakly rigid). Then the converse is true.*

*Proof.* The statement in a) is a consequence of 3.5. Recall that $\operatorname{Tor}_1^A(A/\mathfrak{a}, A/\mathfrak{b}) = \mathfrak{a} \cap \mathfrak{b}/\mathfrak{a}\mathfrak{b}$. So its vanishing yields the equality of the intersection with the product. The statement in b) is clear by the above discussion. $\square$

In the case of $A$ a regular local ring this says that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$ if and only if $\dim A/\mathfrak{a} + \dim A/\mathfrak{b} = \dim A$ and both $A/\mathfrak{a}$ and $A/\mathfrak{b}$ are Cohen-Macaulay rings. This was shown by J. P. Serre, see [45]. So one might think of 3.7 as a generalization to the non-regular case.

3.2. **Estimates of Betti numbers.** In the case $M = k$ the second formula shown in 3.2 provides estimates of the Betti numbers of a module in terms of Betti numbers of its local cohomology modules. This point of view is pursued in this subsection.

To this end let the local ring $(A, \mathfrak{m})$ be the quotient of a regular local ring $(B, \mathfrak{n})$ with $r = \dim B$. We are interested in the minimal free resolution of $M$ as a module over $B$. Because of the local duality, see 1.8, the local cohomology modules of $M$ are the Matlis duals of $K_M^n \simeq \operatorname{Ext}_B^{r-n}(M, B), n \in \mathbb{Z}$, the modules of deficiency of $M$. Note that $K_M = K_M^d$, $d = \dim M$, is called the canonical module of $M$. In the following let

$$\beta_n(M) = \dim_k \operatorname{Tor}_n^B(k, M),\ n \in \mathbb{Z},$$

denote the $n$-th Betti number of $M$. Here $k$ denotes the residue field of $B$.

**Theorem 3.8.** *Let $M$ denote a finitely generated $B$-module. Then*

$$\beta_n(M) \le \begin{cases} \sum_{i=0}^{r-n} \beta_{r-n-i}(K_M^i) & \text{for } n > c, \quad \text{and} \\ \sum_{i=0}^{d} \beta_{r-n-i}(K_M^i) & \text{for } n \le c, \end{cases}$$

*where $c = r - d, d = \dim M$, denotes the codimension of $M$.*

*Proof.* In order to prove the bounds note that for a $B$-module $X$ and all $n \in \mathbb{Z}$ there is an isomorphism

$$\operatorname{Tor}_n^B(k, X) \simeq H_n(\underline{x}; X),$$

where $H_n(\underline{x}; X)$ denotes the Koszul homology of $X$ with respect to $\underline{x} = x_1, \ldots, x_r$, a minimal generating set of $\mathfrak{n}$, the maximal ideal of the regular local ring $B$. This follows because $H_n(\underline{x}; B)$ provides a minimal free resolution of $B/\mathfrak{n}$ over $B$. Because of the Matlis duality it yields that

$$H_{n+i}(\underline{x}; T(K_M^i)) \simeq T(H^{n+i}(\underline{x}; K_M^i)).$$

By the self-duality of the Koszul complex it turns out that

$$H^{n+i}(\underline{x}; K_M^i) \simeq H_{r-n-i}(\underline{x}; K_M^i).$$

By counting the $k$-vector space dimension this implies $\beta_{n+i}(H_\mathfrak{m}^i(M)) = \beta_{r-n-i}(K_M^i)$ for all $i, n \in \mathbb{Z}$. Therefore the claim follows by virtue of 3.2. $\square$

In the particular case of $M$ a Cohen-Macaulay $B$-module the underlying spectral sequence degenerates, see 3.3. This proves that $\beta_n(M) = \beta_{c-n}(K_M)$, $0 \leq n \leq c$. This is well known since $\operatorname{Hom}_B(\cdot, B)$ preserves exactness of $F^{\cdot}$ in this case. Here $F^{\cdot}$ denotes the minimal free resolution of $M$.

**Corollary 3.9.** *Let $M$ be a finitely generated $B$-module with $\operatorname{pd}_A M = p$ and $\operatorname{depth}_B M = t$. Then $\beta_p(M) = \beta_0(K_M^t)$. That is the rank of the last module in a minimal free resolution of $M$ is given by the minimal numbers of generators of the first non-vanishing $K_M^i$.*

*Proof.* There is a partial degeneration of the spectral sequence to the isomorphism

$$\operatorname{Tor}_p^B(k, M) \simeq \operatorname{Tor}_r^B(k, H_{\mathfrak{m}}^t(M)).$$

As above $\operatorname{Tor}_r^B(k, H_{\mathfrak{m}}^t(M)) \simeq T(H^r(\underline{x}; K_M^t)) \simeq T(H_0(\underline{x}; K_M^t))$, which proves the claim. Here $r$ denotes the dimension of $B$ or – what is the same – the minimal number of generators of $\mathfrak{n}$, the maximal ideal of the regular local ring $B$. □

A case of a particular interest is the situation when $H_{\mathfrak{m}}^i(M), i < d := \dim_A M$, are finite dimensional $A/\mathfrak{m}$-vector spaces. Call a finitely generated $A$-module $M$ with this property a quasi-Buchsbaum module.

**Corollary 3.10.** *Let $M$ denote a quasi-Buchsbaum module over the local ring $(A, \mathfrak{m})$. Then*

$$\beta_n(M) \leq \sum_{i=0}^{r-n} \binom{r}{n+i} \dim_k H_{\mathfrak{m}}^i(M)$$

*for all $n > c$, where $k = A/\mathfrak{m}$ denotes the residue field.*

*Proof.* It is an immediate consequence of 3.8. Note that $H_{\mathfrak{m}}^i(M), i < d$, are finite-dimensional $k$-vector spaces. Moreover $\beta_{r-n-i}(k) = \binom{r}{n+i}$ since $B$ is a regular local ring of embedding dimension $r$. □

3.3. **Castelnuovo-Mumford regularity.** In order to obtain more precise information about the syzygies it is helpful to have additional structure, e.g. the structure of a graded $k$-algebra. So let $A = \oplus_{n \geq 0} A_n$ denote a Noetherian graded algebra with $A_0 = k$ a field and $A = A_0[A_1]$. Then $A$ is the epimorphic image of the polynomial ring $B = k[X_1, \ldots, X_r]$, where $r = \dim_k A_1$. In the following let $M$ denote a finitely generated graded $A$-module. Then one might consider it as a module over $B$. The finite dimensional $k$-vector spaces $\operatorname{Tor}_n^B(k, M)$ are graded. They reflect information about the degrees of the minimal generators of the $n$-th module of syzygies of $M$.

The Čech complex $K_{\underline{x}}^{\cdot}$ of $B$ with respect to $\underline{x} = X_1, \ldots, X_r$ is a complex of graded $B$-modules. In fact it is a flat resolution of the system of inverse polynomials. So the local cohomology modules of a graded $B$-module are also graded and therefore

$H^n_{\mathfrak{m}}(M) \simeq H^n(K^{\cdot}_{\underline{x}} \otimes_A M)$ is a homomorphism of degree zero. Here $\mathfrak{m}$ denotes the homogeneous ideal generated by all variables.

For a graded $B$-module $N$ let $e(N) = \sup\{n \in \mathbb{Z} \mid N_n \neq 0\}$, where $N_n$ denotes the $n$-th graded piece of $N$. In the case of $N$ an Artinian module it follows that $e(N) < \infty$. Recall that $e(N) = -\infty$ in the case $N = 0$.

Then define $\operatorname{reg} M$ the Castelnuovo-Mumford regularity of $M$ a finitely generated graded $B$-module by

$$\operatorname{reg} M = \max\{e(H^n_{\mathfrak{m}}(M)) + n \mid n \in \mathbb{Z}\}.$$

Note that it is well-defined by 1.5.

The basics for this construction were initiated by D. Mumford, see [31], who attributed it to Castelnuovo. The importance of the regularity lies in the following fact, a relation to the graded Betti numbers of $M$. There is the equality

$$\operatorname{reg} M = \max\{e(\operatorname{Tor}^B_n(k, M)) - n \mid n \in \mathbb{Z}\},$$

shown by D. Eisenbud and S. Goto, see [10]. In the case of $M$ a Cohen-Macaulay module it turns out that $\operatorname{reg} M = e(\operatorname{Tor}^B_c(k, M)) - c,\ c = \operatorname{codim} M$.

The following provides an improvement by showing that – just as in the Cohen-Macaulay case – the regularity is determined by the tail of the minimal free resolution of $M$.

**Theorem 3.11.** *Let $M$ denote a finitely generated graded $B$-module. Let $s \in \mathbb{N}$ be an integer. Then the following two integers coincide*

a) $\max\{e(H^i_{\mathfrak{m}}(M)) + i \mid 0 \le i \le s\}$ *and*

b) $\max\{e(\operatorname{Tor}^B_j(k, M)) - j \mid r - s \le j \le r\}$.

*In particular for $s = \dim_B M$ it follows that*

$$\operatorname{reg} M = \max\{e(\operatorname{Tor}^B_j(k, M)) - j \mid c \le j \le r\},$$

*where $c = r - \dim_B M$ denotes the codimension of $M$.*

*Proof.* The proof is based on the following spectral sequence

$$E_2^{i,j} = H_{-i}(\underline{x}; H^j_{\mathfrak{m}}(M)) \Rightarrow E^{i+j} = H_{-i-j}(\underline{x}; M)$$

as it was considered in the proof of 3.2. Here $\underline{x} = X_1, \ldots, X_r$ denotes the set of variables in $B$. Note that $\operatorname{Tor}^B_n(k, N) \simeq H_n(\underline{x}; N)$ for all $n \in \mathbb{Z}$ and any $B$-module $N$. Moreover the spectral sequence is a spectral sequence of graded modules and all the homomorphisms are homogeneous of degree zero.

First show the following claim:

*Suppose that $H_s(\underline{x}; M)_{s+t} \neq 0$ for a certain $t \in \mathbb{Z}$ and $r - i \le s \le r$. Then there exists a $j \in \mathbb{Z}$ such that $0 \le j \le i$ and $H^j_{\mathfrak{m}}(M)_{t-j} \neq 0$.*

Assume the contrary, i.e., $H^j_{\mathfrak{m}}(M)_{t-j} = 0$ for all $0 \leq j \leq i$. Then consider the spectral sequence

$$[E_2^{-s-j,j}]_{t+s} = H_{s+j}(\underline{x}; H^j_{\mathfrak{m}}(M))_{t+s} \Rightarrow [E^{-s}]_{t+s} = H_s(\underline{x}; M)_{t+s}.$$

Recall that all the homomorphisms are homogeneous of degree zero. Now the corresponding $E_2$-term is a subquotient of

$$[\oplus H^j_{\mathfrak{m}}(M)^{\binom{r}{s+j}}(-s-j)]_{t+s}.$$

Let $j \leq i$. Then this vectorspace is zero by the assumption about the local cohomology. Let $j > i$. Then $s + j > s + i \geq r$ and $\binom{r}{s+j} = 0$. Therefore the corresponding $E_2$-term $[E_2^{-s-j,j}]_{t+s}$ is zero for all $j \in \mathbb{Z}$. But then also all the subsequent stages are zero, i.e., $[E_\infty^{-s-j,j}]_{t+s} = 0$ for all $j \in \mathbb{Z}$. Whence $[E^{-s}]_{t+s} = H_s(\underline{x}; M)_{t+s} = 0$, contradicting the assumption.

The second partial result shows that a certain non-vanishing of $H^n_{\mathfrak{m}}(M)$ implies the existence of a certain minimal generator of a higher syzygy module. More precisely we show the following claim:

*Let $r = \dim B$ denote the number of variables of $B$. Suppose that there are integers $s, b$ such that the following conditions are satisfied:*

a) $H^i_{\mathfrak{m}}(M)_{b+1-i} = 0$ *for all* $i < s$ *and*
b) $H_r(\underline{x}; H^s_{\mathfrak{m}}(M))_{b+r-s} \neq 0$

*Then it follows that* $H_{r-s}(\underline{x}; M)_{b+r-s} \neq 0$.

Note that the condition in b) means that $H^s_{\mathfrak{m}}(M)$ possesses a socle generator in degree $b - s$. Recall that $r$ denotes the number of generators of $\mathfrak{m}$.

As above we consider the spectral sequence

$$E_2^{-r,s} = H_r(\underline{x}; H^s_{\mathfrak{m}}(M)) \Rightarrow E^{-r+s} = H_{r-s}(\underline{x}; M)$$

in degree $b + r - s$. The subsequent stages of $[E_2^{-r,s}]_{b+r-s}$ are derived by the cohomology of the following sequence

$$[E_n^{-r-n,s+n-1}]_{b+r-s} \longrightarrow [E_n^{-r,s}]_{b+r-s} \longrightarrow [E_n^{-r+n,s-n+1}]_{b+r-s}$$

for $n \geq 2$. But now $[E_n^{-r-n,s+n-1}]_{b+r-s}$ resp. $[E_n^{-r+n,s-n+1}]_{b+r-s}$ are subquotients of

$$H_{r+n}(\underline{x}; H^{s+n-1}_{\mathfrak{m}}(M))_{b+r-s} = 0 \text{ resp. } H_{r-n}(\underline{x}; H^{s-n+1}_{\mathfrak{m}}(M))_{b+r-s} = 0.$$

For the second module recall that it is a subquotient of

$$[\oplus H^{s-n+1}_{\mathfrak{m}}(M)^{\binom{r}{r-n}}(-r+n)]_{b+r-s} = 0, \quad n \geq 2.$$

Therefore $[E_2^{-r,s}]_{b+r-s} = [E_\infty^{-r,s}]_{b+r-s} \neq 0$ and

$$[E^{-r+s}]_{b+r-s} \simeq H_{r-s}(\underline{x}; M)_{b+r-s} \neq 0$$

as follows by the filtration with the corresponding $E_\infty$-terms.

Now let us prove the statement of the theorem. First of all introduce two abbreviations. Put

$$a := \max\{e(\operatorname{Tor}_j^B(k, M)) - j \mid r - s \le j \le r\}.$$

Then by the first claim it follows that $a \le b$, where

$$b := \max\{e(H^i_{\mathfrak{m}}(M)) + i \mid 0 \le i \le s\}.$$

On the other hand choose $j$ an integer $0 \le j \le s$ such that $b = e(H^j_{\mathfrak{m}}(M)) + j$. Then $H^j_{\mathfrak{m}}(M)_{b-j} \neq 0$, $H^j_{\mathfrak{m}}(M)_{c-j} = 0$ for all $c > b$, and $H^i_{\mathfrak{m}}(M)_{b+1-i} = 0$ for all $i < j$. Recall that this means that $H^j_{\mathfrak{m}}(M)$ has a socle generator in degree $b - j$. Therefore the second claim applies and $\operatorname{Tor}^B_{r-j}(K, M)_{b+r-j} \neq 0$. In other words, $b \le a$, as required. □

An easy byproduct of our investigations is the above mentioned fact that

$$\operatorname{reg} M = e(\operatorname{Tor}_c^B(K, M)) - c,\ c = r - \dim M,$$

provided $M$ is a Cohen-Macaulay module.

It is noteworthy to say that P. Jørgensen, see [24], investigated a non-commutative Castelnuovo-Mumford regularity. In fact he generalized 3.11 to the non-commutative situation by an interesting argument.

**Theorem 3.12.** *Let $M$ be a finitely generated graded $B$-module with $d = \dim_B M$. Suppose there is an integer $j \in \mathbb{Z}$ such that for all $q \in \mathbb{Z}$ either*

a) *$H^q_{\mathfrak{m}}(M)_{j-q} = 0$ or*
b) *$H^p_{\mathfrak{m}}(M)_{j+1-p} = 0$ for all $p < q$ and $H^p_{\mathfrak{m}}(M)_{j-1-p} = 0$ for all $p > q$.*

*Then for $s \in \mathbb{Z}$ it follows that*

(1) *$\operatorname{Tor}_s^B(k, M)_{s+j} \simeq \oplus_{i=0}^{r-s}(\operatorname{Tor}^B_{r-s-i}(k, K^i_M)_{s+j})^\vee$ provided $s > c$, and*
(2) *$\operatorname{Tor}_s^B(k, M)_{s+j} \simeq \oplus_{i=0}^{d-1}(\operatorname{Tor}^B_{s+i}(k, K^i_M)_{r-s-j})^\vee \oplus (\operatorname{Tor}^B_{c-s}(k, K_M)_{r-s-j})^\vee$, provided $s \le c$,*

*where $K^i_M = \operatorname{Ext}^{r-i}_B(M, B(-r))$, $0 \le i < d$, denote the module of deficiencies and $K_M$ is the canonical module of $M$.*

*Proof.* As above consider the spectral sequence

$$E_2^{-s-i,i} = H_{s+i}(\underline{x}; H^i_{\mathfrak{m}}(M)) \Rightarrow E^{-s} = H_s(\underline{x}; M)$$

in degree $s+j$. Firstly we claim that $[E_2^{-s-i,i}]_{s+j} \simeq [E_\infty^{-s-i,i}]_{s+j}$ for all $s \in \mathbb{Z}$. Because $[E_2^{-s-i,i}]_{s+j}$ is a subquotient of

$$[\oplus H^i_{\mathfrak{m}}(M)^{\binom{r}{s+i}}(-s-i)]_{s+j}$$

the claim is true provided $H^i_{\mathfrak{m}}(M)_{j-i} = 0$. Suppose that $H^i_{\mathfrak{m}}(M)_{j-i} \neq 0$. In order to prove the claim in this case too note that $[E_{n+1}^{-s-i,i}]_{s+j}$ is the cohomology at

$$[E_n^{-s-i-n,i+n-1}]_{s+j} \to [E_n^{-s-i,i}]_{s+j} \to [E_n^{-s-i+n,i-n+1}]_{s+j}.$$

Then the module at the left resp. the right is a subquotient of

$$H_{s+i+n}(\underline{x}; H_{\mathfrak{m}}^{i+n-1}(M))_{s+j} \text{ resp. } H_{s+i-n}(\underline{x}; H_{\mathfrak{m}}^{i-n+1}(M))_{s+j}.$$

Therefore both of them vanish. But this means that the $E_2$-term coincides with the corresponding $E_\infty$-term. So the target of the spectral sequence $H_s(\underline{x}; M)_{s+j}$ admits a finite filtration whose quotients are $H_{s+i}(\underline{x}; H_{\mathfrak{m}}^i(M))_{s+j}$. Because all of these modules are finite dimensional vectorspaces it follows that

$$H_s(\underline{x}; M)_{s+j} \simeq \oplus_{i=0}^{r-s} H_{s+i}(\underline{x}; H_{\mathfrak{m}}^i(M))_{s+j}$$

for all $s \in \mathbb{Z}$.

By the Local Duality theorem there are the following isomorphisms $H_{\mathfrak{m}}^i(M) \simeq T(K_M^i),\ 0 \le i \le d$, where $T$ denotes the Matlis duality functor $\operatorname{Hom}_k(\cdot, k)$ in the case of the graded situation. Therefore we obtain the isomorphisms

$$H_{s+i}(\underline{x}; T(K_M^i))_{s+j} \simeq (T(H^{s+i}(\underline{x}; K_M^i)))_{s+j} \simeq (H_{r-s-i}(\underline{x}; K_M^i)_{r-s-j})^\vee.$$

But the last vector space is isomorphic to $(\operatorname{Tor}_{r-s-i}^B(k, K_M^i)_{r-s-j})^\vee$.

In the case of $s > c$ it is known that $r - s < d$. Hence the first part of the claim is shown to be true. In the remaining case $s \le c$ the summation is taken from $i = 0, \ldots, d$, which proves the second part of the claim. □

As an application of 3.12 we derive M. Green's duality theorem [12, Section 2].

**Corollary 3.13.** *Suppose there exists an integer $j \in \mathbb{Z}$ such that*

$$H_{\mathfrak{m}}^q(M)_{j-q} = H_{\mathfrak{m}}^q(M)_{j+1-q} = 0$$

*for all $q < \dim_B M$. Then*

$$\operatorname{Tor}_s^B(k, M)_{s+j} \simeq (\operatorname{Tor}_{c-s}^B(k, K_M)_{r-s-j})^\vee,$$

*for all $s \in \mathbb{Z}$, where $c = \operatorname{codim} M$.*

*Proof.* It follows that the assumptions of Theorem 3.12 are satisfied for $j$ because of $H_{\mathfrak{m}}^p(M)_{j-1-p} = 0$ for all $p > \dim M$. Therefore the isomorphism is a consequence of (1) and (2) in 3.12. To this end recall that

$$\operatorname{Tor}_{s+i}^B(k, H_{\mathfrak{m}}^i(M))_{s+j} \simeq H_{s+i}(\underline{x}; H_{\mathfrak{m}}^i(M))_{s+j} = 0,$$

as follows by the vanishing of $H_{\mathfrak{m}}^i(M)_{j-i}$ for all $i < \dim M$. □

M. Green's duality theorem in 3.13 relates the Betti numbers of $M$ to those of $K_M$. Because of the strong vanishing assumptions in 3.13 very often it does not give strong information about Betti numbers. Often it says just the vanishing which follows also by different arguments, e.g., the regularity of $M$.

Theorem 3.12 is more subtle. We shall illustrate its usefulness by the following example.

*Example* 3.14. Let $C \subset \mathbb{P}^n_K$ denote a reduced integral non-degenerate curve over an algebraically closed field $K$. Suppose that $C$ is non-singular and of genus $g(C) = 0$. Let $A = B/I$ denote its coordinate ring, i.e., $B = K[x_0, \ldots, x_n]$ and $I$ its homogeneous defining ideal. Then

$$\operatorname{Tor}^B_s(k, B/I)_{s+j} \simeq \operatorname{Tor}^B_{s+1}(k, H^1_{\mathfrak{m}}(B/I))_{s+j}$$

for all $s \geq 1$ and all $j \geq 3$. To this end recall that $A$ is a two-dimensional domain. Moreover it is well-known that $H^q_{\mathfrak{m}}(B/I) = 0$ for all $q \leq 0$ and $q > 2$. Furthermore it is easy to see that $H^1_{\mathfrak{m}}(B/I)_{j-1} = 0$ for all $j \leq 1$. Moreover $H^2_{\mathfrak{m}}(B/I)_{j-1-2} = 0$ for all $j \geq 3$ as follows because of $g(C) = 0$. That is, for $j \geq 3$ one might apply 3.12. In order to conclude we have to show that $\operatorname{Tor}^B_{c-s}(k, K_{B/I})_{r-s-j} = 0$ for $j \geq 3$. To this end note that

$$(H_{c-s}(\underline{x}; K_{B/I})_{r-s-j})^\vee \simeq H_{s+2}(\underline{x}; H^2_{\mathfrak{m}}(B/I))_{s+j}$$

as is shown in the proof of 3.12. But this vanishes for $j \geq 2$ as is easily seen.

3.4. **The local Green modules.** As before let $E = E_A(A/\mathfrak{m})$ denote the injective hull of the residue field of a local ring $(A, \mathfrak{m})$. Let $\underline{x} = x_1, \ldots, x_r$ denote a system of elements of $A$. Then for all $n \in \mathbb{Z}$ there are canonical isomorphisms

$$H_n(\underline{x}; T(M)) \simeq T(H^n(\underline{x}; M)) \quad \text{and} \quad H^n(\underline{x}; T(M)) \simeq T(H_n(\underline{x}; M)).$$

Here $T$ denotes the duality functor $\operatorname{Hom}_A(\cdot, E)$. In the case $(A, \mathfrak{m})$ is the factor ring of a local Gorenstein ring $B$, then use the modules of deficiency $K^n_M$ as defined in Section 1.2. In order to continue with our investigations we need a sharpening of the definition of a filter regular sequence. To this end let $M$ denote a finitely generated $A$-module.

An $M$-filter regular sequence $\underline{x} = x_1, \ldots, x_r$ is called a strongly $M$-filter regular sequence provided it is filter regular with respect to $K^n_{M \otimes \widehat{A}}$ for $n = 0, 1, \ldots, \dim M$. Here $\widehat{A}$ denotes the completion of $A$.

The necessity to pass to the completion is related to the existence of $K^n_M$. In the case $A$ is the quotient of a Gorenstein ring it is enough to check the filter regularity with respect to $K^n_M$. This follows because by Cohen's Structure theorem $\widehat{A}$ is the quotient of a Gorenstein ring and $\underline{x}$ is $M$-filter regular if and only if it is $M \otimes_A \widehat{A}$-filter regular. Because $K^n_{M \otimes \widehat{A}}$ are finitely generated $\widehat{A}$-modules the existence of strongly $M$-filter regular sequences is a consequence of prime avoidance arguments.

**Lemma 3.15.** *Suppose that $\underline{x} = x_1, \ldots, x_r$ denotes a strongly $M$-filter regular sequence. Let $j \in \mathbb{Z}$ denote an integer. Then $H_i(\underline{x}; H^j_{\mathfrak{m}}(M))$ resp. $H^i(\underline{x}; H^j_{\mathfrak{m}}(M))$ are $A$-modules of finite length in the following two cases:*

a) *for all $i < r$ resp. $i > 0$, and*
b) *for all $i \in \mathbb{Z}$, provided $r \geq j$.*

*Proof.* Without loss of generality we may assume that $A$ is a complete local ring. Because of the isomorphisms

$$H_i(\underline{x}; H^j_{\mathfrak{m}}(M)) \simeq T(H^i(\underline{x}; K^j_M)),\ i, j \in \mathbb{Z},$$

it will be enough to show that $H^i(\underline{x}; K^j_M))$ is an $A$-module of finite length. By view of 1.17 is follows that this is of finite length in the case $i < r$. In the case $r > i$ we know that $\dim K^j_M \leq j$, see 1.9. Therefore the Koszul cohomology is also of finite length in the remaining case $i = r$. The rest of the statement is clear by the self-duality of the Koszul complex. □

In a certain sense the modules considered in 3.15 are local analogues to the modules studied by M. Green in [12]. For some results in the graded case see also [44]. In relation to possible further applications it would be of some interest to find interpretations of the modules $H_i(\underline{x}; H^j_{\mathfrak{m}}(M))$. One is given in the following.

**Theorem 3.16.** *Let $\underline{x} = x_1, \ldots, x_r$, $r \geq \dim M$, denote a strongly $M$-filter regular sequence. Let $n \in \mathbb{N}$ be an integer. Then there are the following bounds:*

a) $L_A(H^n(\underline{x}; M)) \leq \sum_{i\geq 0}^n L_A(H^{n-i}(\underline{x}; H^i_{\mathfrak{m}}(M)))$.

b) $L_A(H_n(\underline{x}; M)) \leq \sum_{i\geq 0} L_A(H_{n+i}(\underline{x}; H^i_{\mathfrak{m}}(M)))$.

*Proof.* It is enough to prove one of the statements as follows by self-duality of Koszul complexes. Let us prove the claim in b). To this end consider the complex

$$C^{\cdot} := K^{\cdot} \otimes_A K.(\underline{x}; M),$$

where $K^{\cdot}$ denotes the Čech complex with respect to a generating set of the maximal ideal. Now consider the spectral sequences for computing the cohomology of $C^{\cdot}$. The first of them is given by

$$H^i_{\mathfrak{m}}(H_j(\underline{x}; M)) \Rightarrow H^{i-j}(C^{\cdot}).$$

Because $H_j(\underline{x}; M)$, $j \in \mathbb{Z}$, is an $A$-module of finite length we get the vanishing of $H^i_{\mathfrak{m}}(H_j(\underline{x}, M)) = 0$ for all $j$ and $i \neq 0$ and

$$H^0_{\mathfrak{m}}(H_j(\underline{x}; M)) \simeq H_j(\underline{x}; M) \text{ for all } j \in \mathbb{Z}.$$

Therefore there is a partial degeneration of the spectral sequence to the following isomorphisms

$$H^{-n}(C^{\cdot}) \simeq H_n(\underline{x}; M) \text{ for all } n \in \mathbb{Z}.$$

On the other hand there is the spectral sequence

$$E_2^{i,-j} = H_j(\underline{x}; H^i_{\mathfrak{m}}(M)) \Rightarrow E^{i-j} = H^{i-j}(C^{\cdot}).$$

By the assumption all the initial terms $E_2^{i,-j}$ are $A$-modules of finite length for all $i, j \in \mathbb{Z}$, see 3.15. Therefore also the limit terms $E_\infty^{i,-j}$ are of finite length and

$L_A(E_2^{i,-j}) \geq L_A(E_\infty^{i,-j})$ for all $i, j \in \mathbb{Z}$. Whence $E^{-n} = H^{-n}(C^\cdot)$ admits a filtration with quotients $E_\infty^{i,-n-i}$ for $i \in \mathbb{Z}$. Therefore there is the bound

$$L_A(E^{-n}) \leq \sum_{i \geq 0} L_A(E_2^{i,-n-i}),$$

which proves the result by view of the above estimate. □

The spectral sequence considered in the proof provides also another partial degeneration. This could be helpful for different purposes.

**Corollary 3.17.** *Let $\underline{x}$ and $M$ be as in 3.16. Then there are the following canonical isomorphisms:*

a) $H^t(\underline{x}; M) \simeq H^0(\underline{x}; H_{\mathfrak{m}}^t(M))$, $t = \operatorname{depth} M$.
b) $H^n(\underline{x}; M) \simeq H^{n-d}(\underline{x}; H_{\mathfrak{m}}^d(M))$ *for all* $n \in \mathbb{Z}$, *provided* $M$ *is a* $d$*-dimensional Cohen-Macaulay module.*

In the first case of Corollary 3.17 it is possible to compute the Koszul cohomology explicitly. It turns out that $H^0(\underline{x}; H_{\mathfrak{m}}^t(M)) \simeq (x_1, \ldots, x_t)M :_M \underline{x}/(x_1, \ldots, x_t)M$. It would be of some interest to give further interpretations of some of the modules $H_i(\underline{x}; H_{\mathfrak{m}}^j(M))$.

There is one result in this direction concerning multiplicities. To this end recall the notion of a reducing system of parameters in the sense of M. Auslander and D. A. Buchsbaum, see [2]. Recall that for an arbitrary system of parameters $\underline{x} = x_1, \ldots, x_d$ of $M$ it is known that there is a strongly $M$-filter regular sequence $\underline{y} = y_1, \ldots, y_d$ such that $(x_1, \ldots, x_i)M = (y_1, \ldots, y_i)M$, $i = 1, \ldots, d = \dim M$. Note that $\underline{y}$ is a reducing system of parameters of $M$.

In the following denote by $L_A(M/\underline{x}M)$ resp. $e_0(\underline{x}; M)$ the length resp. the multiplicity of $M$ with respect to $\underline{x}$, see [2] for the details.

**Theorem 3.18.** *Let $\underline{x} = x_1, \ldots, x_d$, $d = \dim M > 1$, denote an arbitrary system of parameters. Choose $\underline{y} = y_1, \ldots, y_d$ as above. Then*

$$L_A(M/\underline{x}M) - e_0(\underline{x}; M) \leq \sum_{i=0}^{d-1} L_A(H_i(\underline{y}'; H_{\mathfrak{m}}^i(M))),$$

*where $\underline{y}' = y_1, \ldots, y_{d-1}$.*

*Proof.* Because of the previous remark one may replace $\underline{x}$ by $\underline{y}$ without loss of generality. Because $\underline{y}$ is a reducing system of parameters of $M$ it turns out that

$$L_A(M/\underline{y}M) - e_0(\underline{y}; M) = L_A(\underline{y}'M :_M y_d/\underline{y}'M),$$

see [2]. Moreover it follows that $\underline{y}'M :_M y_d/\underline{y}'M \subseteq \underline{y}'M :_M \langle \mathfrak{m} \rangle/\underline{y}'M$. Recall that $y_d$ is a parameter for the one-dimensional quotient module $M/\underline{y}'M$. But now

$$\underline{y}'M :_M \langle \mathfrak{m} \rangle/\underline{y}'M \simeq H^0_{\mathfrak{m}}(M/\underline{y}'M).$$

In order to continue with the proof let $K^{\cdot}$ denote the Čech complex with respect to a system of parameters of $(A, \mathfrak{m})$. Then consider the complex $C^{\cdot} := K^{\cdot} \otimes_A K_{\cdot}(\underline{y}'; M)$, where $K_{\cdot}(\underline{y}'; M)$ denotes the Koszul complex of $M$ with respect to $\underline{y}'$. Then use the spectral sequence

$$E_2^{i,-j} = H^i_{\mathfrak{m}}(H_j(\underline{y}'; M)) \Rightarrow E^{i-j} = H^{i-j}(C^{\cdot}).$$

Because $\underline{y}'$ is an $M$-filter regular sequence $H_j(\underline{y}'; M), j \neq 0$, is an $A$-module of finite length. That is, for $j \neq 0$ it follows that $E_2^{i,-j} = 0$ for all $i \neq 0$. So there is a partial degeneration to the isomorphism $H^0(C^{\cdot}) \simeq H^0_{\mathfrak{m}}(M/\underline{y}'M)$. On the other side there is a spectral sequence

$$E_2^{-i,j} = H_j(\underline{y}'; H^i_{\mathfrak{m}}(M)) \Rightarrow E^{i-j} = H^{i-j}(C^{\cdot}).$$

Taking into account that $E_2^{-i,i}$ is of finite length and $E_2^{-i,i} = 0$ for $i < 0$ and $i \geq d$ this provides the estimate of the statement. $\square$

In the case that $H^n_{\mathfrak{m}}(M)$, $n = 0, \ldots, d-1$, is an $A$-module of finite length the result in 3.18 specializes to the following bound

$$L_A(M/\underline{x}M) - e_0(\underline{x}; M) \leq \sum_{n=0}^{d-1} \binom{d-1}{i} L_A(H^n_{\mathfrak{m}}(M)).$$

Therefore 3.18 is a generalization of the 'classical' results about Buchsbaum and generalized Cohen-Macaulay modules to an arbitrary situation.

In this context it is noteworthy to say that there is another bound for the length $L_A(M/\underline{x}M)$ of the following type

$$L_A(M/\underline{x}M) \leq \sum_{n=0}^{d} L_A(H_n(\underline{x}; H^n_{\mathfrak{m}}(M)).$$

This follows immediately by 3.16 because of $M/\underline{x}M \simeq H^0_{\mathfrak{m}}(M/\underline{x}M)$. In the particular case that $H^n_{\mathfrak{m}}(M)$, $n = 0, \ldots, d-1$, are of finite length it implies that

$$L_A(M/\underline{x}M) \leq \sum_{i=0}^{d-1} \binom{d}{i} L_A(H^n_{\mathfrak{m}}(M)) + L_A(K_M/\underline{x}K_M).$$

To this end note that $H_d(\underline{x}; H^d_{\mathfrak{m}}(M)) \simeq T(H^d(\underline{x}; K_M))$.

Moreover in the case of a Cohen-Macaulay module it yields that $M/\underline{x}M \simeq T(K_M/\underline{x}K_M)$. This implies also the equality of the multiplicities $e_0(\underline{x}; M) = e_0(\underline{x}; K_M)$.

Let us conclude with another application of 3.18.

**Corollary 3.19.** *Let $M$ denote a finitely generated $A$-module with $\dim_A M - \operatorname{depth}_A M \leq 1$. Let $\underline{x}, \underline{y}$, and $\underline{y}'$ be as above. Suppose that $A$ is the quotient of a Gorenstein ring $B$. Then*

$$L_A(M/\underline{x}M) - e_0(\underline{x}; M) \leq L_A(K_M^{d-1}/\underline{y}' K_M^{d-1}),$$

*where $d = \dim_A M$ and $K_M^{d-1} = \operatorname{Ext}_B^{c+1}(M, B)$.*

*Proof.* The proof follows by 3.18 because of $H_{d-1}(\underline{y}'; H_{\mathfrak{m}}^{d-1}(M)) \simeq H_0(\underline{y}'; K_M^{d-1})$. Recall that $T(K_M^{d-1}) \simeq H_{\mathfrak{m}}^{d-1}(M)$ by the Local Duality Theorem. □

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