MATH 702 – SPRING 2024
FINAL EXAM.

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

You should KEEP this piece of paper.

Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. There are four problems.

1. (13 points) Let R be a commutative ring and let M and N be R-modules. Suppose that every R-submodule of M is finitely generated and every R-submodule of N is finitely generated. Prove that every R-submodule of $M \oplus N$ is finitely generated. (Please give a complete, self-contained proof.)
2. (13 points) Let $\boldsymbol{\ell}$ be a field of characteristic zero. Let t_1, t_2, t_3, t_4 be new variables, K be the field of rational functions $K = \boldsymbol{\ell}(t_1, t_2, t_3, t_4)$ and $\boldsymbol{k}$ be the subfield $\boldsymbol{\ell}(s_1, s_2, s_3, s_4)$ of K, where the s_i are the elementary symmetric polynomials:
$$s_1 = t_1 + t_2 + t_3 + t_4,$$
$$s_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4,$$
$$s_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4, \text{ and}$$
$$s_4 = t_1t_2t_3t_4.$$
   (a) Prove that $\boldsymbol{k} \subset K$ is a Galois extension.
   (b) Identify the Galois group $\operatorname{Aut}_{\boldsymbol{k}} K$.
   (c) Identify an element $d \in K \setminus \boldsymbol{k}$ with $d^2 \in \boldsymbol{k}$.
3. (12 points) Suppose $\boldsymbol{k} \subset E$ and $E \subseteq K$ are both finite dimensional Galois extensions. Does $\boldsymbol{k} \subseteq K$ have to be a Galois extension? Prove or give a counterexample.
4. (12 points) Give an example of a finite dimensional field extension $\boldsymbol{k} \subseteq K$ with an infinite number of intermediate fields. Also give an example of a finite dimensional field extension $\boldsymbol{k} \subseteq K$ with $K \neq \boldsymbol{k}[u]$ for any $u \in K$.