MATH 702 – SPRING 2024
FINAL EXAM.

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

You should KEEP this piece of paper.
Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. There are four problems.

1. (13 points) **Let R be a commutative ring and let M and N be R-modules. Suppose that every R-submodule of M is finitely generated and every R-submodule of N is finitely generated. Prove that every R-submodule of $M \oplus N$ is finitely generated. (Please give a complete, self-contained proof.)**

   Let X be an R-submodule of $M \oplus N$. The set

$$Y = \left\{ m \in M \,\middle|\, \exists n \in N \text{ with } \begin{bmatrix} m \\ n \end{bmatrix} \in X \right\}$$

   is a submodule of M. Every submodule of M is finitely generated. Select $x_1, \dots, x_r \in X$ such that

$$x_i = \begin{bmatrix} m_i \\ n_i \end{bmatrix}, \text{ with } m_i \in M, \text{ and } n_i \in N$$

   and $m_1, \dots, m_r$ generate Y. The set

$$Z = \left\{ n \in N \,\middle|\, \begin{bmatrix} 0 \\ n \end{bmatrix} \in X \right\}$$

   is a submodule of N. Every submodule of N is finitely generated; hence there exist $x'_1, \dots, x'_s$ in X with $x'_i = \begin{bmatrix} 0 \\ n_i \end{bmatrix}$ and $n_1, \dots, n_s$ generate Z.

   It follows that $x_1, \dots, x_r, x'_1, \dots, x'_s$ generate X.

2. (13 points) **Let $\boldsymbol{\ell}$ be a field of characteristic zero. Let t_1, t_2, t_3, t_4 be new variables, K be the field of rational functions $K = \boldsymbol{\ell}(t_1, t_2, t_3, t_4)$ and $\boldsymbol{k}$ be the subfield $\boldsymbol{\ell}(s_1, s_2, s_3, s_4)$ of K, where the s_i are the elementary symmetric polynomials:**

$$\begin{aligned} s_1 &= t_1 + t_2 + t_3 + t_4, \\ s_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4, \\ s_3 &= t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4, \ \textbf{and} \\ s_4 &= t_1t_2t_3t_4. \end{aligned}$$

   (a) Prove that $\boldsymbol{k} \subset K$ is a Galois extension.
   (b) Identify the Galois group $\operatorname{Aut}_{\boldsymbol{k}} K$.
   (c) Identify an element $d \in K \setminus \boldsymbol{k}$ with $d^2 \in \boldsymbol{k}$.

   The symmetric group S_4 acts on K by permuting the variables. It is clear that

$$\boldsymbol{k} \subseteq K^{S_4} \quad \text{and} \quad S_4 \subseteq \operatorname{Aut}_{\boldsymbol{k}} K.$$

   On the other hand, K is the splitting field of

$$f(x) = \prod_{i=1}^{4}(x - t_i) = x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 \in \boldsymbol{k}[x]$$

   over $\boldsymbol{k}$. Thus, $\boldsymbol{k} \subseteq K$ is a Galois extension and $\operatorname{Aut}_{\boldsymbol{k}} K \subseteq S_4$. We conclude that $\operatorname{Aut}_{\boldsymbol{k}} K = S_4$. (Items (a) and (b) have been established.)

   The element d will generate the subfield K^{A_4} of K, which is the unique subfield of K with $\dim_{\text{subfield}} K = 12$. So we try

$$d = (t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_2 - t_3)(t_2 - t_4)(t_3 - t_4).$$

   We see that $\sigma(d) = -d$ for each transposition $\sigma \in S_4$; so $d \in K^{A_4}$, $d \notin K^{S_4} = \boldsymbol{k}$, but $d^2 \in K^{S_4} = \boldsymbol{k}$.

3. (12 points) **Suppose $\boldsymbol{k} \subset E$ and $E \subseteq K$ are both finite dimensional Galois extensions. Does $\boldsymbol{k} \subseteq K$ have to be a Galois extension? Prove or give a counter example.**

   NO! The extensions $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{Q}[\sqrt[4]{2}]$ each have dimension two; hence each extension is Galois. Indeed, $\mathbb{Q}[\sqrt{2}]$ is the splitting field of $x^2 - 2$ over $\mathbb{Q}$ and $\mathbb{Q}[\sqrt[4]{2}]$ is the splitting field of $x^2 = \sqrt{2}$ over $\mathbb{Q}[\sqrt{2}]$. However, the extension $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt[4]{2}]$ is not Galois because some of the roots of the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$ are not in $\mathbb{Q}[\sqrt[4]{2}]$.

4. (12 points) **Give an example of a finite dimensional field extension $\boldsymbol{k} \subseteq K$ with an infinite number of intermediate fields. Also give an example of a finite dimensional field extension $\boldsymbol{k} \subseteq K$ with $K \neq \boldsymbol{k}[u]$ for any $u \in K$.**

   Let $\boldsymbol{\ell}$ be an infinite field of characteristic p for some prime integer p. (For example, $\boldsymbol{\ell}$ could be the field of rational functions in one variable over $\frac{\mathbb{Z}}{(p)}$.) Let $\boldsymbol{k} = \boldsymbol{\ell}(s^p, t^p)$

and $K = \boldsymbol{\ell}(s,t)$. (In particular, $\boldsymbol{k}$ and K are both fields of rational functions in two variables.) Observe that $x^p - s^p$ is an irreducible polynomial in $\boldsymbol{k}[x]$. [1] Thus,

$$\dim_{\boldsymbol{k}} \boldsymbol{k}(s) = p.$$

In a similar manner, $\dim_{\boldsymbol{\ell}(s,t^p)} K = p$. We conclude that $\dim_{\boldsymbol{k}} K = p^2$. We first prove that K is not equal to $\boldsymbol{k}(u)$ for any u in K. Indeed, if $u \in K$, then $u^p \in \boldsymbol{k}$ and

$$\dim_{\boldsymbol{k}} \boldsymbol{k}(u) \le p < p^2 = \dim_{\boldsymbol{k}} K.$$

Observe that for each $\alpha \in \boldsymbol{\ell}$, $\boldsymbol{k}(s+\alpha t)$ is a field with

$$\boldsymbol{k} \subseteq \boldsymbol{k}(s+\alpha t) \subseteq K.$$

Claim. All of the intermediate fields $\boldsymbol{k}(s+\alpha t)$ are distinct as α ranges over the infinite set $\boldsymbol{\ell}$.

To prove the claim, we assume α and β are distinct elements of the field $\boldsymbol{\ell}$ and

$$\boldsymbol{k}(s+\alpha t) = \boldsymbol{k}(s+\beta t).$$

We look for a contradiction. Thus

$$(\alpha-\beta)t = (s+\alpha t) - (s+\beta t) \in \boldsymbol{k}(s+\alpha t).$$

But, $\alpha - \beta$ is a unit of $\boldsymbol{k}$; hence t (and therefore s and K) are in $\boldsymbol{k}(s+\alpha t)$. We already proved that $K \neq \boldsymbol{k}(u)$ for any u. We have reached a contradiction. The Claim is established.

[1] We made this argument in class. If $x^p - s^p$ has a non-trivial factor in $\boldsymbol{k}[x]$, then g is a non-trivial factor of $x^p - s^p = (x-s)^p$ in $K[x]$. In other words, $g = (x-s)^a$ is in $\boldsymbol{k}[x]$ for some integer a, with $1 \le a \le p-1$. In particular, $s^a \in \boldsymbol{k}$. The exponents a and p are relatively prime, hence $s \in \boldsymbol{k}$, which is not true.