Final Exam 1991

Each problem is worth 25 points.

1. Give an example of two nilpotent $4 \times 4$ matrices that have the same minimal polynomial, but are not similar. (Recall that the matrix $A$ is nilpotent if $A^n = 0$ for some integer $r$.)

2. Suppose $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ is a linear transformation which is represented in some basis by a diagonal matrix with entries $2, 2, 4, 4, 4$ on the main diagonal. What is the rational canonical form for $T$?

3. Suppose that $T : V \rightarrow V$ is a linear transformation on a finite dimensional vector space. If $T$ has rank one, then prove that $T$ is nilpotent or diagonalizable.

4. Let $M$ be an $n \times n$ matrix over the field of Complex numbers. If $r$ denotes the rank of $M$, then prove that $M$ can be written as the sum of $r$ rank one matrices.

5. Give an example of a vector space $V$, a linear transformation $T : V \rightarrow V$, and a $T$-invariant subspace $W$ of $V$ such that no complement of $W$ in $V$ is $T$-invariant. Prove that your example does what it is supposed to do. (Note: The subspace $W'$ of $V$ is a complement of $W$ in $V$ if $W \oplus W' = V$.)

6. Let $V$ and $W$ be finite dimensional vector spaces over the field $F$ and let $T : V \rightarrow W$ be a linear transformation. Fix a basis $B : v_1, \ldots, v_n$ for $V$ and a basis $C : w_1, \ldots, w_m$ for $W$. For each vector $v \in V$ and $w \in W$ define

$$[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [w]_C = \begin{bmatrix} c'_1 \\ \vdots \\ c'_m \end{bmatrix}$$

where $v = \sum_{i=1}^{n} c_i v_i$ and $w = \sum_{i=1}^{m} c'_i w_i$ for scalars $c_i$ and $c'_i$ in $F$.

(a) Define the matrix $[T]_{C \rightarrow B}$ which represents the transformation $T$ with respect to the bases $B$ and $C$.

(b) Let $T^* : W^* = \text{Hom}_F(W, F) \rightarrow V^* = \text{Hom}_F(V, F)$ be the dual of $T$. Let $B^*$ be the basis for $V^*$ which is dual to $B$ and $C^*$ be the basis for $W^*$ which is dual to $C$. How are the matrices $[T]_{C \rightarrow B}$ and $[T^*]_{B^* \rightarrow C^*}$ related?

(c) Prove your answer to part (b). Be sure to justify each step.

7. Suppose that $T : V \rightarrow V$ is a linear transformation on a finite dimensional vector space $V$ over the field $F$. Prove that $V$ is $T$-cyclic if and only if

$$\{ U \in \text{Hom}_F(V, V) : TU = UT \} = \{ f(T) : f \in F[x] \}.$$