MATH 700
HOMEWORK 3

Due Friday, September 13, 1991 at the beginning of class

1. Let $T: V \to W$ and $S: W \to V$ be linear transformations of vector spaces. Suppose that the composition $ST$ is the identity map on $V$.
   a. If $V$ and $W$ have the same finite dimension, then prove that $T$ is an isomorphism.
   b. Give an example where $T$ is not an isomorphism, but $V$ and $W$ are both finite dimensional.
   c. Give an example where $T$ is not an isomorphism, but $V$ and $W$ have the same infinite dimension.

2. Let $V$ be a vector space over the field $F$ and let $T: V \to V$ be a linear transformation with the property that the composition $TT$ is the identity map on $V$.
   a. Assume that 2 is not the zero element of $F$. Prove that there are subspaces $M$ and $N$ of $V$ which satisfy the all of the following properties: $M + N = V$, $M \cap N = 0$, $T(\alpha) = \alpha$ for all $\alpha \in M$, and $T(\alpha) = -\alpha$ for all $\alpha \in N$.
   b. Give an example which shows that part (a) is false when $F$ is the field with two elements. NOTE: Write your example up carefully! You must show exactly which property fails.

3. Give an example a function $T: \mathbb{C} \to \mathbb{C}$ such that $T$ is a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{R}$, but $T$ is not a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{C}$.