Final Exam, 1989

There are 9 problems. Each problem is worth 20 points.

1. Fill in the blank and then prove the resulting statement. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. The subset $W_1 \cup W_2$ is a subspace of $V$ if and only if ________________.

2. Suppose that

$$0 \xrightarrow{f_0} V_4 \xrightarrow{f_4} V_3 \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} 0$$

is an exact sequence of linear transformations. (In other words, for each $i$, $f_i: V_i \to V_{i-1}$ is a linear transformation and $\ker f_i = \text{im} f_{i+1}$.) Suppose further, that each $V_i$ is finite dimensional.

(a) Give a formula which relates the dimensions of the $V_i$.

(b) Prove your answer to (a).

3. If $0 \to V \xrightarrow{S} W \xrightarrow{T} Z \to 0$ is an exact sequence of vector spaces over the field $F$, then prove that

$$0 \to \text{Hom}_F(Z, F) \xrightarrow{T^*} \text{Hom}_F(W, F) \xrightarrow{S^*} \text{Hom}_F(V, F) \to 0$$

is also an exact sequence of vector space. (Note: The notion of exact sequence is defined in problem 2.)

4. Let $M$ be a square matrix with entries in the arbitrary field $F$. Fill in the blank using some property of the minimal polynomial of $M$ or the characteristic polynomial of $M$. Prove that your statement is correct. The matrix $M$ is similar to a diagonal matrix over $F$ if and only if ________________.

5. Let $V$ be a finite dimension vector over the field $F$ and let $T: V \to V$ be a linear transformation. Suppose that the characteristic polynomial of $T$ is equal to $f_1^{e_1} f_2^{e_2} \ldots f_r^{e_r}$, where the $f_i$ are distinct irreducible polynomials in $F[X]$. For each $i$, let

$$W_i = \{v \in V : f_i^{e_i}v = 0\}.$$ 

Prove that $V = \bigoplus_{i=1}^r W_i$.

6. Suppose that $M$ is a matrix with entries in the field $F$. Suppose further that

- the characteristic polynomial of $M$ is $(x - 2)^{14}$,
- the minimal polynomial of $M$ is $(x - 2)^4$, and
- the dimension of the null space of $(M - 2I)$ is 5.

Describe the steps you would take in order to determine the Jordan Canonical Form of $M$.

7. Give an example of a vector space $V$, a linear transformation $T: V \to V$, and a $T$-invariant subspace $W$ of $V$ such that no complement of $W$ in $V$ is $T$-invariant. Prove that your example does what it is supposed to do. (Note: The subspace $W'$ of $V$ is a complement of $W$ in $V$ if $W \oplus W' = V$.)
8. TRUE or FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample. If you give a counterexample, then you must prove that your example does what it is supposed to do.) Let $V$ be a finite dimensional vector space over the field of Complex Numbers. If $S: V \to V$ and $T: V \to V$ are diagonalizable linear transformations, then there exists a basis $B$ for $V$ such that the matrix of $T$ with respect to $B$ and the matrix of $S$ with respect to $B$ are both diagonal.

9. TRUE or FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample. If you give a counterexample, then you must prove that your example does what it is supposed to do.) Let $(V, <>)$ be a finite dimensional Inner Product Space over the field of Complex Numbers. If $T: V \to V$ is a linear transformation, and $T(v) = cv$ for some vector $v \in V$ and some scalar $c \in \mathbb{C}$, then $T^{\text{Adj}}(v) = \overline{c}v$. 