MATH 700
HOMEWORK 12
DUE FRIDAY DECEMBER 1, 1989
AT THE BEGINNING OF CLASS.

1. (Brown: Page 131, number 14.) Let \( A \) and \( B \) be \( n \times n \) matrices over \( \mathbb{C} \). Prove that if \( AB = BA \), then \( A \) and \( B \) have a common eigenvector. Do \( A \) and \( B \) have a common eigenvalue?

2. I want you to do page 141, number 16. My phrasing of the problem is:
   a. Suppose that \( T: V \to V \) is a linear transformation on a finite dimensional Complex vector space. Suppose further, that the Jordan canonical form of \( T \) consists of one Jordan block. Find all of the \( T \)-invariant subspaces of \( V \).
   b. Fill in the blank with some statement about the characteristic polynomial of \( T \) and/or the minimal polynomial of \( T \). If \( T: V \to V \) is a linear transformation on a finite dimensional Complex vector space, then \( V \) has only finitely many \( T \)-invariant subspaces if and only if 
   c. Prove that your assertion in (b) is correct.

3. (Brown Page 169, number 7.) Show that \( A \) is similar to \( A^t \) for any \( n \times n \) matrix \( A \) over a field \( F \).

4. Let \( T: V \to V \) be a linear transformation on a finite dimensional vector space over a field \( F \). Suppose that the characteristic polynomial of \( T \) is \( f^r \) for some irreducible polynomial \( f \in F[X] \) and some positive integer \( r \). Let \( W \) be a \( T \)-invariant subspace of \( V \).
   a. Suppose that \( W \) has a \( T \)-invariant complement. (That is, suppose that there is a \( T \)-invariant subspace \( W' \) of \( V \) with \( W \oplus W' = V \).) Prove that \( W \) has the following property:

   (*) If \( gv \in W \) for some \( g \in F[X] \) and \( v \in V \), then there exists \( w \in W \) such that \( gw = gw \).
   b. Prove the converse of (a). (That is, suppose that \( W \) is a \( T \)-invariant subspace of \( V \) with property (*). Prove that \( W \) has a \( T \)-invariant complement.)

Note. Part (a) is trivial. Part (b) requires some work. Some expositions of the theory of canonical forms use part (b) as the main lemma in the proof of the theorem about the structure of finitely generated modules over PIDs. Our approach was different. I do not want you to copy a long involved proof out of some book. Instead, I want you to deduce part (b) from the structure theorem.
5. (Brown. Page 170, number 19.) An endomorphism $T: V \rightarrow V$ is said to be *semisimple* if every $T$-invariant subspace of $V$ has a $T$-invariant complement. If the minimal polynomial of $T$ is irreducible, prove that $T$ is semisimple.

6. Compute the determinant of the Vandermonde matrix. That is, compute

$$\det \begin{bmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^n \\
1 & x_1 & x_1^2 & \cdots & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^n
\end{bmatrix}.$$ 

You are required to prove that your answer is correct. The entries in the matrix are elements of the polynomial ring

$$\mathbb{Z}[x_0, \ldots, x_n].$$