Exam 1, 1989

There are 6 problems worth a total of 100 points.

1. (15 points) Give an example of a linear transformation $T: V \to V$ such that $T$ is one-to-one, but $T$ is not onto.

2. (15 points) Let $F$ be a field in which 2 is a unit, and let $V$ be the vector space consisting of all $n \times n$ matrices over $F$. Let Sym be the subspace of $V$ consisting of all symmetric matrices (i.e. $M^t = M$), and let Skew be the subspace of $V$ consisting of all skew-symmetric matrices (i.e. $M^t = -M$). Prove that $V = \text{Sym} \oplus \text{Skew}$.

3. (15 points) Let $T: V \to W$ be a linear transformation of finite dimensional vector spaces. Let $\alpha$ be a basis for $V$ and $\beta$ be a basis for $W$. Let $M$ be the matrix of $T$ with respect to the bases $\alpha$ and $\beta$. Let $N$ be the matrix of $T^*$ with respect to the dual bases of $\alpha$ and $\beta$.
   (a) How are $M$ and $N$ related?
   (b) Prove your assertion from (a).

4. (15 points) State and prove the Cayley-Hamilton Theorem.

5. (20 points) Let $M$ be a square matrix with entries in the arbitrary field $F$. Fill in the blank using some property of the minimal polynomial of $M$ or the characteristic polynomial of $M$. Prove that your statement is correct. The matrix $M$ is similar to a diagonal matrix over $F$ if and only if ____________.

6. (20 points) Let $M$ be a square matrix with complex entries. Suppose that $M$ is in Jordan canonical form. A quick glance at $M$ should tell you all of the eigenvalues of $M$ and the dimension of each eigenspace. (Recall that the scalar $a$ is an eigenvalue of $M$ if $Mv = av$ for some non-zero vector $v$. The eigenspace associated to the eigenvalue $a$ is $\{v \mid Mv = av\}$.)
   (a) What are the eigenvalues of $M$.
   (b) What is the dimension of each eigenspace?
   (c) Prove that your answers to (a) and (b) are correct.