Note! Write your answers on the blank sheets of paper provided. Use only one side of each sheet; start each problem on a new sheet of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

Each problem is worth five points.

1. Let $V$ be a subspace of $\mathbb{R}^n$ and $W = \{w \in \mathbb{R}^n \mid w^Tv = 0 \text{ for all } v \in V\}$. Prove that $\mathbb{R}^n = V + W$ and $V \cap W = \{0\}$.

2. State and prove the Cayley-Hamilton Theorem.

3. Exhibit a matrix $B$ (with real entries) such that $B^2 = A$ for $A = \begin{bmatrix} 5 & -2 \\ -3 & 5 \end{bmatrix}$.

4. Let $T: V \to V$ be a linear transformation of a finite dimensional vector space over the field $F$. Suppose that $V_1$ and $V_2$ are $T$-cyclic subspace of $V$ and that the minimal polynomials of $T|_{V_1}$ and $T|_{V_2}$ are relatively prime. Prove that $V_1 + V_2$ is $T$-cyclic. (I expect a complete self-contained proof of this elementary Lemma. Do not appeal to the canonical form theorems or state that we did this in class.)

5. Let $T: V \to W$ be a linear transformation of finite dimensional vector spaces. State the formula which relates the dimension of the null space of $T$ and the dimension of the image of $T$. Prove the formula you have stated.

6. Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{C}$ with $AB = BA$. Prove that $A$ and $B$ have a common eigenvector.