Math 700  Fall 2003  Exam 2

Note! Write your answers on the blank sheets of paper provided. Use only one side of each sheet; start each problem on a new sheet of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

Each problem is worth five points.

1. Answer the question by providing a proof or a counterexample. Let $V_1$, $V_2$, $W_1$, and $W_2$ be vector spaces over the field $F$. Suppose $W_1 \cong W_2$ and $V_1 \oplus W_1 \cong V_2 \oplus W_2$. Are $V_1$ and $V_2$ isomorphic?

2. Let $I$ be a set. For each $i \in I$, let $V_i$ be a vector space. Let $W$ be a vector space. Which of the following vector spaces are isomorphic:

- $\text{Hom} \left( \bigoplus_{i \in I} V_i, W \right)$
- $\text{Hom} \left( \prod_{i \in I} V_i, W \right)$
- $\bigoplus_{i \in I} \text{Hom} (V_i, W)$
- $\prod_{i \in I} \text{Hom} (V_i, W)$
- $\text{Hom} \left( W, \bigoplus_{i \in I} V_i \right)$
- $\text{Hom} \left( W, \prod_{i \in I} V_i \right)$
- $\bigoplus_{i \in I} \text{Hom} (W, V_i)$
- $\prod_{i \in I} \text{Hom} (W, V_i)$

For each isomorphism define the isomorphism and its inverse. I do not need to see any proof.

3. Let $T: V \to V$ be a linear transformation on a finitely dimensional vector space $V$ over the field $F$. How is the matrix for $T$ related to the matrix for $T^*: \text{Hom}(V,F) \to \text{Hom}(V,F)$? (You pick the bases.) Prove your answer.

4. Let $M$ be a matrix over the field $F$ with characteristic polynomial $(x-2)^6(x-3)^2$, rank($M-3I$) = 7, rank($M-2I$) = 5, and rank($M-2I)^2$ = 3. What is the Jordan Canonical Form of $M$? (I do not need to see any work.)

5. Let $N_1$ and $N_2$ be two $n \times n$ nilpotent matrices with the same minimal polynomial and the same nullity. Find the maximal $n$ such that $N_1$ and $N_2$ must be similar. Prove that your answer is correct.

6. Let $\lambda_1, \ldots, \lambda_n$ be complex numbers which satisfy $\sum_{i=1}^{n} \lambda_i^k = 0$ for all positive integers $k$. Prove that $\lambda_i = 0$, for $1 \leq i \leq n$. 

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