1. Let $V_1$ and $V_2$ be subspaces of the vector space $V$. Suppose that $V_1 + V_2 = V$. Prove that there exists a subspace $W$ of $V$ such that $W \subseteq V_2$ and $V = V_1 \oplus W$.

If $V_1 = V$, then take $W = \{0\}$. Henceforth, we assume $V_1 \neq V$. Let $S_1$ be a basis for $V_1$ and $S_2$ be a basis for $V_2$. I will find $T$, a subset of $S_2$, such that $S_1 \cup T$ is a basis for $V$. At that point, I let $W$ be the span of $T$, and I am finished. I use Zorn’s Lemma to prove that $T$ exists. Let $P = \{S \subseteq S_2 \mid S_1 \cup S$ is linearly independent$. We see that $P$ is non-empty, and that $P$, together with $\subseteq$, is a poset. If $T = \{S_a \mid a \in A\}$ is a chain in $P$, then the union $U = \bigcup_{a \in A} S_a$ is an upper bound for $T$, which is in $P$. (We know that $S_1 \cup U$ is linearly independent because any relation would involve only a finite number of $S_a$’s and hence, would occur in $S_1 \cup S_{a_0}$, for some $a_0$ (since the $S_a$’s are totally ordered!). But $S_1 \cup S_{a_0}$ is linearly independent.) Zorn’s lemma tells us that $P$ has a maximal element $T$. It is clear that $S_1 \cup T$ is linearly independent. If $S_1 \cup T$ did not span $V$, then there would exist $v$ in $S_2$ such that $v \notin T$ and $S_1 \cup T \cup \{v\}$ is linearly independent. In this case, $T \cup \{v\}$ would be an element of $P$ which is properly larger than the maximal element $T$ of $P$. This is impossible. We conclude that $S_1 \cup T$ is a basis for $V$.

2. (Answer each question with a COMPLETE proof or a counterexample.) Let $X$, $Y$, and $Z$ be subspaces of the vector space $V$ with $X \oplus Y = V$ and $X \oplus Z = V$. Is $Y = Z$? Is $Y \cong Z$?

It is easy to see that $Y$ does not have to equal $Z$. Let $V = \mathbb{R}^2$, $X$ be the space spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Y$ be the space spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $Z$ be the space spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is clear that $X \oplus Y = V = X \oplus Z$ and $Y \neq Z$. On the other hand, $Y$ is isomorphic to $Z$. Define $\varphi : Y \rightarrow Z$ as follows. Each element of $V$ can be written in the form $x + z$, for some $x \in X$ and some $z \in Z$, in a unique manner! For $y \in Y$ define $\varphi(y) = z$ where $z$ is the unique element of $Z$ with $y - z \in X$. It is not difficult to see that $\varphi : Y \rightarrow Z$ is a homomorphism. Take $y$ and $y'$ from $Y$ and $c \in F$, where $F$ is the field
of scalars. Let $z$ and $z'$ be the unique elements of $Z$ with $y - z \in X$ and $y' - z' \in X$. We see that $(cy + y') - (cz + z') = c(y - z) + (y' - z') \in X$; hence, $\varphi(cy + y') = cz + z' = c\varphi(y) + \varphi(y')$. We show that $\varphi$ is one-to-one. If $y \in Y$ and $\varphi(y) = 0$, then $y \in Y \cap X = \{0\}$. We show that $\varphi$ is onto. Take $z \in Z$. We know that $z \in V = X \oplus Y$. So, $z = x + y$ for some $x \in X$ and $y \in Y$. Observe that $y - z = x \in X$; hence, $\varphi(y) = z$.

3. Suppose that $T: V \to V$ is a linear transformation on the vector space $V$ which satisfies $TT = T$. Prove that $V = \ker T \oplus \text{im } T$.

If $v \in V$, then $v = [v - T(v)] + T(v)$. We see that $v - T(v) \in \ker T$ and $T(v) \in \text{im } T$. Thus, $V = \ker T + \text{im } T$. If $v \in \ker T \cap \text{im } T$, then $v = T(v')$ for some $v' \in V$ and $0 = T(v) = TT(v') = T(v') = v$. 