MATH 700
HOMEWORK 4

Due Friday, September 19, 2003 at the beginning of class

Question 1. True or False. If true, prove it; if false, give a counterexample. If \( W_1, W_2, \text{ and } W_3 \) are subspaces of the finite dimensional vector space \( V \), then
\[
\dim (W_1 + W_2 + W_3) = \dim W_1 + \dim W_2 + \dim W_3 - \dim (W_1 \cap W_2) \\
- \dim (W_1 \cap W_3) - \dim (W_2 \cap W_3) + \dim (W_1 \cap W_2 \cap W_3).
\]

Answer. False. Let \( W_1, W_2, \text{ and } W_3 \) be three distinct one-dimensional subspaces of \( \mathbb{R}^2 \). (If you like \( W_1 \) is the \( x \)-axis, \( W_2 \) is the \( y \)-axis, and \( W_3 \) is the line \( y = x \).) All of the intersections \( W_1 \cap W_2, W_1 \cap W_3, W_2 \cap W_3, \text{ and } W_1 \cap W_2 \cap W_3 \) have dimension zero. But \( W_1 + W_2 + W_3 = \mathbb{R}^2 \). It is not true that
\[
2 = 1 + 1 + 1 - 0 - 0 - 0 + 0.
\]

Question 2. Give an example of a non-zero vector space \( V \) and a linear transformation \( T: V \to V \) with the property that the null space of \( T \) is equal to the image of \( T \).

Answer. Let \( V = \mathbb{F}^2 \) and \( T: V \to V \) be \( T(v) = Av \), where \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). Observe that the null space of \( T \) and the image of \( T \) both equal
\[
\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{F} \right\}.
\]

Question 3. Let \( T: V \to V \) be a linear transformation which is not the zero transformation and which is not an isomorphism.

(a) If \( \dim V < \infty \), then prove that there exists a linear transformation \( S: V \to V \) such that \( ST = 0 \), but \( TS \neq 0 \).

Answer. Let \( v_1, \ldots, v_r \) be a basis for the image of \( T \). I know that \( T \) is not onto because of the rank-nullity Theorem. Let \( n \) be the name of the dimension of \( V \). We have \( r < n \). Expand \( v_1, \ldots, v_r \) to become a basis \( v_1, \ldots, v_r, v_{r+1}, \ldots, v_n \) for \( V \). The hypothesis tells us that there is a vector \( w \) with \( T(w) \neq 0 \). Define \( S: V \to V \) by \( S(v_i) = 0 \) for \( 1 \leq i \leq n - 1 \) and \( S(v_n) = w \). I see that \( TS \) is not zero because \( TS(v_n) = T(w) \neq 0 \). On the other hand, \( ST = 0 \) because \( S \) sends a basis for the image of \( T \) to zero.
**Question 3 b.** Does (a) remain true if the hypothesis \( \dim V < \infty \) is removed? (Prove it or give a counterexample.)

**Answer.** No. Let \( V \) be \( \bigoplus_{i=1}^{\infty} F \). The elements of \( V \) are tuples \((a_1, a_2, \ldots)\) with all but finitely many entries zero. Addition and scalar multiplication take place coordinate-wise. Define \( T: V \to V \) by \( T(a_1, a_2, \ldots) = (a_2, a_3, \ldots) \). We see that \( T \) is not the zero transformation and \( T \) is not an isomorphism (because \((1, 0, 0, 0, \ldots)\) is not zero but is in the null space of \( T \)). On the other hand, \( T \) is onto; hence, if \( S \) is a linear transformation with \( ST = 0 \), then \( S \) must send the entire image of \( T \), which is \( V \), to zero. So \( S \) must be the zero transformation and \( TS \) must be the zero transformation.