Math 700 Homework 3 Solutions

**Question 1.** Let $T: V \to W$ and $S: W \to V$ be linear transformations of vector spaces. Suppose that the composition $ST$ is the identity map on $V$.

a. If $V$ and $W$ have the same finite dimension, then prove that $T$ is an isomorphism.

**Answer:** Let $n$ equal $\dim V = \dim W$. We see that $T$ is one-to-one. Indeed, if $v \in V$ with $T(v) = 0$, then $v = S(T(v)) = S(0) = 0$. It follows that the image of $T$ is an $n$-dimensional subspace of the $n$-dimensional vector space $W$; and therefore, $T$ is onto.

b. Give an example where $T$ is not an isomorphism, but $V$ and $W$ are both finite dimensional.

**Answer:** Let $V = F$, $W = F^2$, $T(c) = \begin{bmatrix} c \\ 0 \end{bmatrix}$, and $S\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a$. We see that $ST$ is the identity map, and that $T$ is one-to-one, but not onto.

c. Give an example where $T$ is not an isomorphism, but $V$ and $W$ have the same infinite dimension.

**Answer:** Let $V = W = \bigoplus_{i=1}^{\infty} F$. (So, the elements of $V$ look like $(c_1, c_2, c_3, \ldots)$, where each $c_i \in F$ and all but finitely many of the $c_i$ are zero.) Let $T(c_1, c_2, c_3, \ldots) = (0, c_1, c_2, c_3, \ldots)$ and $S(c_1, c_2, c_3, \ldots) = (c_2, c_3, \ldots)$. Once again, $ST$ is the identity map, $T$ is one-to-one, but $T$ is not onto.

**Question 2.** Let $V$ be a vector space over the field $F$ and let $T: V \to V$ be a linear transformation with the property that the composition $TT$ is the identity map on $V$.

a. Assume that 2 is not the zero element of $F$. Prove that there are subspaces $M$ and $N$ of $V$ which satisfy the all of the following properties: $M + N = V$, $M \cap N = 0$, $T(\alpha) = \alpha$ for all $\alpha \in M$, and $T(\alpha) = -\alpha$ for all $\alpha \in N$.

**Answer:** Let $M = \{\alpha \in V \mid T(\alpha) = \alpha\}$. Let $N = \{\alpha \in V \mid T(\alpha) = -\alpha\}$. If $\alpha \in M \cap N$, then $-\alpha = T(\alpha) = \alpha$; hence, $2\alpha = 0$; hence, $\alpha = 0$. If $v \in V$, then $v = \left(\frac{v+T(v)}{2}\right) + \left(\frac{v-T(v)}{2}\right)$. We see that $\frac{v+T(v)}{2} \in M$ and $\frac{v-T(v)}{2} \in N$.

b. Give an example which shows that part (a) is false when $F$ is the field with two elements. NOTE: Write your example up carefully! You must show exactly which property fails.

**Answer:** Let $V = F^2$ and $T$ be the linear transformation which is given by multiplication by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that $T^2$ is the identity map on $V$.

I claim that there does not exist subspaces $M$ and $N$ of $V$ with $V = M + N$, $M \cap N = 0$, $T(\alpha) = \alpha$ for all $\alpha \in M$, and $T(\alpha) = -\alpha$ for all $\alpha \in N$. Indeed, if such subspaces did exist, then $T$ would have to be the identity map because, if $v \in V$, then there exists elements $m \in M$ and $n \in N$ with $v = m + n$. So, $T(v) = T(m) + T(n) = m - n = m + n = v$, since $1 = -1$ in $F$. Well, our $T$ is not the identity map because $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is different than $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Question 3. Give an example a function $T: \mathbb{C} \to \mathbb{C}$ such that $T$ is a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{R}$, but $T$ is not a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{C}$.

Answer: Consider $T(a + bi) = a - bi$, for $a, b \in \mathbb{R}$. We see that $T$ is a linear transformation over $\mathbb{R}$:

$$T(a + bi) + T(c + di) = (a - bi) + (c - di) = (a + c) - (b + d)i = T((a + c) + (b + d)i)$$

$$= T([a + bi] + [c + di]),$$

and

$$cT(a + bi) = c(a - bi) = ca - cb = T(ca + cb) = T(c[a + bi]),$$

for all $a, b, c$ in $\mathbb{R}$. On the other hand, $T$ is not a linear transformation over $\mathbb{C}$ because

$$iT(1) = i \neq -i = T(i \cdot 1).$$