1. (Hoffman and Kunze, page 49, number 7b.) Let $V$ be the vector space of all $2 \times 2$ matrices over the field $F$. Let $W_1$ be the subspace of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and $W_2$ be the subspace of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

Find the dimensions of $W_1$, $W_2$, $W_1 + W_2$, and $W_1 \cap W_2$.

- The matrices
  $$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

  are a basis for $W_1$; so $\dim W_1 = 3$.

- The matrices
  $$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

  are a basis for $W_2$; so $\dim W_2 = 3$.

- The matrices
  $$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

  are linearly independent and in $W_1 \cap W_2$. So

$$2 \leq \dim W_1 \cap W_2 \leq \dim W_2 = 3.$$ 

On the other hand, $W_1 \cap W_2 \neq W_2$, since

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in W_2 \setminus W_1; \text{ hence}$$

$$\dim W_1 \cap W_2 \neq 3 \text{ and therefore } \dim W_1 \cap W_2 = 2.$$

- It is clear that $W_1 + W_2 = V$ because

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
are in $W_1 + W_2$ and are a basis for $V$. Thus, $\dim W_1 + W_2 = 4$.

2. (Hoffman and Kunze, page 49, number 8.) Let $V$ be the vector space of all $2 \times 2$ matrices over the field $F$. Find a basis $A_1, A_2, A_3, A_4$ for $V$ such that $A_j^2 = A_j$ for each $j$.

- The matrices

$$
\begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
0 & 0 
\end{bmatrix}, \text{ and } \begin{bmatrix}
0 & 0 \\
1 & 1 
\end{bmatrix}
$$

form one such basis.

3. (Hoffman and Kunze, page 49, number 14.) Let $V$ be the set of real numbers. Regard $V$ as a vector space over the field of rational numbers. Prove that this vector space is not finite-dimensional.

- Every finite dimensional vector space over $\mathbb{Q}$ is countable; but $V$ is not countable; hence, $V$ is an infinite dimensional vector space over $\mathbb{Q}$.

4. Let $V$ be the the vector space of rational functions in one variable over the field of complex numbers. (A typical element of $V$ has the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials with complex coefficients in one variable.) Prove that the dimension of $V$ over $\mathbb{C}$ is not countable.

- We will show that

$$
\left\{ \frac{1}{x-a} \mid a \in \mathbb{C} \right\}
$$

is linearly independent over $\mathbb{C}$. Suppose $c_1, \ldots, c_n$ and $a_1, \ldots, a_n$ are complex numbers, with the $a_j$ distinct, and $\sum_{j=1}^{n} \frac{c_j}{x-a_j} = 0$. Multiply the most recent equation by the common denominator to see that

$$
c_1(x-a_2) \cdots (x-a_n) + \text{ a polynomial which vanishes at } a_1
$$

is equal to the zero polynomial. Plug in $a_1$ for $x$ to see that $c_1$ times a non-zero complex number is equal to zero. We conclude that $c_1 = 0$. Repeat the process to see that $c_1 = \cdots = c_n = 0$. 