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Quiz for February 7, 2006

Let f_n be the n^{th} Fibonacci number; that is, $f_1 = 1$, $f_2 = 1$, and for $3 \le n$, $f_n = f_{n-2} + f_{n-1}$. Prove that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

whenever n is a positive integer.

ANSWER: We will prove the statement by induction on n.

Base case: If $n = 1$, then $f_1^2 = 1$ and $f_n f_{n+1} = 1$; so the statement holds.

Induction Hypothesis: Suppose the statement holds at $n - 1$. In other words, suppose that

$$f_1^2 + f_2^2 + \cdots + f_{n-1}^2 = f_{n-1} f_n.$$

Inductive Step: We must show that the statement holds at n. That is, we must show that:

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

holds. Well,

$$f_1^2 + f_2^2 + \cdots + f_n^2 = (f_1^2 + f_2^2 + \cdots + f_{n-1}^2) + f_n^2.$$

We use the Induction Hypothesis to see that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_{n-1} f_n + f_n^2 = f_n(f_{n-1} + f_n).$$

We use the definition of the Fibonacci numbers to see that $f_{n-1} + f_n = f_{n+1}$. We conclude that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

The inductive step is complete, and so is the proof.