Goldbach’s conjecture states that every even integer greater than 2 is the sum of two primes. Prove that Goldbach’s conjecture is equivalent to the statement that every integer greater than 5 is the sum of three primes.

ANSWER:

Assume the original conjecture. Prove the alternate form. Let \( n \) be an integer greater than 5. If \( n \) is even, then \( n - 2 \) is an even integer greater than 2 and Goldbach’s conjecture ensures that there exist prime numbers \( p \) and \( q \) with \( p + q = n - 2 \). Thus, \( p + q + 2 = n \) and the conclusion of the alternate form holds for \( n \). If \( n \) is odd, then \( n - 3 \) is an even integer greater than 2. Once again Goldbach’s conjecture ensures that there exist prime numbers \( p \) and \( q \) with \( p + q = n - 3 \). Thus, \( p + q + 3 = n \). In any event, \( n \) is the sum of three primes.

Assume the alternate form. Prove the original conjecture. Let \( n > 2 \) be an even integer. We see that \( n + 2 \) is an arbitrary integer greater than 5. The alternate form of the conjecture ensures that there exist prime numbers \( p \), \( q \), and \( r \) with \( n + 2 = p + q + r \). We notice that at least one of the numbers \( p \), \( q \), and \( r \) must be even (because three odd numbers add up to an odd number and \( n + 2 \) is even). The only even prime number is 2. So one of the three prime numbers \( p \), \( q \) or \( r \) is equal to 2. Re-label, if necessary, in order to have \( r = 2 \). We now subtract 2 from each side of \( n + 2 = p + q + 2 \) to see that \( n = p + q \).