Math 574, Exam 3, Summer 2007

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 5 problems. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

You should **KEEP** this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (7 points) **How many monomials of degree less than or equal to d are there in n variables. (Recall that the monomial $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ has degree equal to $e_1+e_2+\cdots+e_n$.)**

This problem is the same as counting the number of monomials of degree exactly d in $n+1$ variables and this is the same as the number of work orders with d picks and n switches:

$$\boxed{\binom{n+d}{d}.}$$

2. (7 points) **A candy store has three flavors: chocolate, vanilla, and strawberry. Every bag of candy contains 10 pieces of candy all together and at least 3 pieces of chocolate candy. How many possible bags of candy can be made?**

Put 3 pieces of Chocolate candy in the bag. Now use work orders with 7 picks and 2 switches to fill the bag:

$$\boxed{\binom{9}{2}.}$$

3. (7 points) **Twenty people have formed a club. The club has four committees: The Steering Committee has 5 people, the Issues Committee has 2 people, the Fund Raising Committee has 3 people, and the Entertainment Committee has 10 people. How many ways can the committee assignments be distributed, if every person lands on exactly one committee?**

$$\boxed{\binom{20}{5,2,3,10}.}$$

4. (7 points) **Find a recurrence relation which counts the number of strings of length n which are made out of 0's and 1's and contain at least 3 consecutive zeros.**

Let a_n equal the number of strings of length n which can be made from 0's and 1's and which contain at least 3 consecutive zeros.

We look how a given string ends. We count a_{n-1} strings that end in 1; a_{n-2} strings that end in 10, a_{n-3} strings that end in 100; and 2^{n-3} strings that end in 000. Every legal string has been counted exactly once:

$$\boxed{a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}}.$$

5. (22 points) **Find the general solution of recurrence relation**

(NHP) $$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} + 2^n.$$

We first solve the homogeneous problem. We look for r so that $a_n = r^n$ is a solution of

(HP) $$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}.$$

We look for r with
$$r^n = 5r^{n-1} - 8r^{n-2} + 4r^{n-3}.$$

Move all the terms to one side and factor out the r^{n-3}. We want to solve

$$r^{n-3}(r^3 - 5r^2 + 8r - 4) = 0.$$

Factor to get
$$r^{n-3}(r-1)(r-2)^2 = 0.$$

The general solution of the homogeneous problem is

$$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 n 2^n.$$

Now we look for a particular solution of the original non-homogeneous problem. We look for a constant β for which $a_n = \beta n^2 2^n$ is a solution of (NHP). We look for β with

$$\beta n^2 2^n = 5\beta(n-1)^2 2^{n-1} - 8\beta(n-2)^2 2^{n-2} + 4\beta(n-3)^2 2^{n-3} + 2^n$$

$$\beta n^2 2^n = 5\beta(n^2-2n+1)2^{n-1} - 8\beta(n^2-4n+4)2^{n-2} + 4\beta(n^2-6n+9)2^{n-3} + 2^n.$$

Divide both sides by 2^{n-3}:

$$\beta n^2 8 = 5\beta(n^2-2n+1)4 - 8\beta(n^2-4n+4)2 + 4\beta(n^2-6n+9) + 8.$$

The n^2 terms cancel because $8 = 20 - 16 + 4$. We solve:

$$0 = 20\beta(-2n+1) - 16\beta(-4n+4) + 4\beta(-6n+9) + 8.$$

The n terms cancel because $-40 + 64 - 24 = 0$. We solve

$$0 = 20\beta(1) - 16\beta(4) + 4\beta(9) + 8 = \beta(20 - 64 + 36) + 8 = -8\beta + 8.$$

So $\beta = 1$ and the general solution of (NHP) is

$$\boxed{a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 n 2^n + n^2 2^n.}$$