Math 574, Exam 2, Summer 2007

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 9 problems **ON TWO SIDES!**. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

You should **KEEP** this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (6 points) Simplify $\sum_{k=0}^{n} \binom{n}{k}^2$. Explain thoroughly.

2. (6 points) Simplify $\binom{50}{0} + \binom{51}{1} + \binom{52}{2} + \cdots + \binom{200}{150}$. Explain thoroughly.

3. (6 points) Simplify $\sum_{k=0}^{n} k\binom{n}{k}$. Explain thoroughly.

4. (6 points) Seven women and nine men are on the faculty in the Mathematics department at a school. How many ways are there to select a committee of five members if at least one woman must be on the committee?

5. (6 points) List the numbers from 1 to 16 in such a way that no sublist of size 5 is strictly increasing or strictly decreasing.

6. (5 points) Consider a group of 20 people where any two people are either friends or enemies. Prove that there are either four mutual friends or four mutual enemies.

NOTE: On quiz two you already proved that in any group of 10 people (where any two people are either friends or enemeies), there are either three mutual friends or four mutual enemies. The same work as on quiz two would prove that in any group of 10 people (where any two people are either friends or enemeies), there are either three mutual enemies or four mutual friends. You need not prove either of these results any further and you may use these results in today's question.

7. (5 points) In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if the bride must be next to the groom.

8. (5 points) Recall that the Fibonacci numbers are: $f_1 = 1$, $f_2 = 1$, and for $n \geq 3$ $f_n = f_{n-1} + f_{n-2}$. Prove that $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$ whenever n is a positive integer.

9. (5 points) Show that

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is taken over all nonempty subsets of the n smallest positive integers.)